Business/Technical Mathematics

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BCCAMPUS
VICTORIA, B.C.

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- Added, removed, and reorganized questions and examples where appropriate.
- Replaced American examples and spellings with Canadian ones.

See the end of each chapter to see what content was adapted from where.

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Ebook ISBN: 978-1-77420-126-8
Print ISBN: 978-1-77420-125-1

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This book was produced with Pressbooks (https://pressbooks.com) and rendered with Prince.

Contents

4. Systems of Equations

5. Trigonometry

6. Health Option

7. Data Analysis I

8 Data Analysis 2

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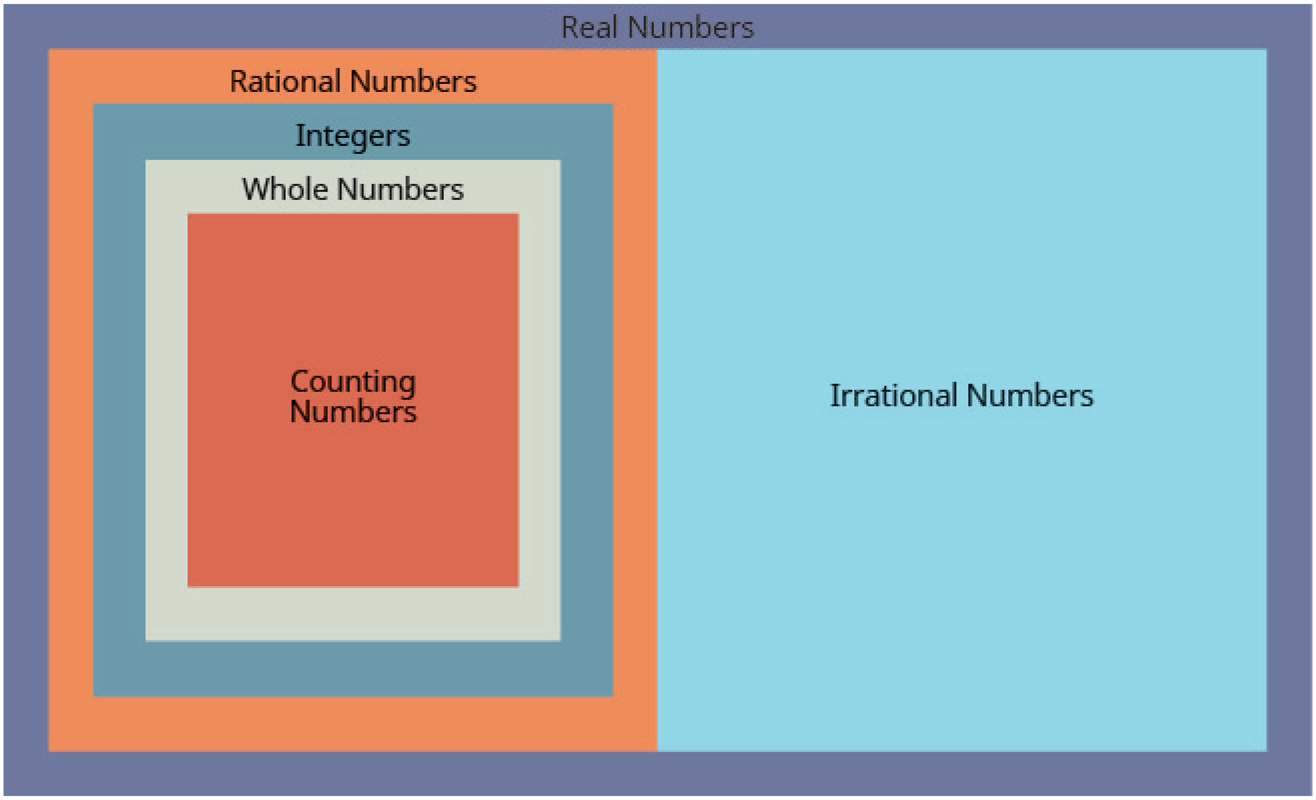
1. Operations with Real Numbers

The chart below shows us how the number sets fit together. In this chapter we will work with rational numbers, but you will be also introduced to irrational numbers. The set of rational numbers together with the set of irrational numbers make up the set of real numbers.



1.1 Algebraic Expressions

Learning Objectives

By the end of this section it is expected that you will be able to:

- Use variables and algebraic symbols
- Identify expressions and equations
- Simplify expressions with exponents
- Simplify expressions using the order of operations
- Evaluate algebraic expressions

Use Variables and Algebraic Symbols

In algebra, letters of the alphabet are used to represent variables.

Letters often used for variables are $x, y, a, b, \text{and } c$.

Variables and Constants

A variable is a letter that represents a number or quantity whose value may change.

A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Operation	**Notation**	**Say:**	**The result is…**
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a\cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Division	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

In algebra, the cross symbol, ×, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (three times y) or $3\cdot x\cdot y$ (three times x times y)? To make it clear, use • or parentheses for multiplication.When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

Equality Symbol

$a=b$ is read a is equal to b

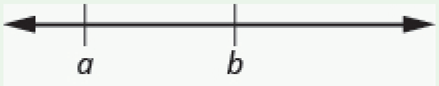
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols < and > for inequalities.

Inequality

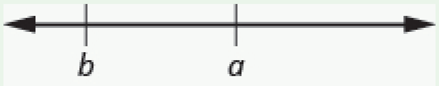
$a<b$ is read a is less than b

a is to the left of b on the number line



$a>b$ is read a is greater than b

a is to the right of b on the number line



The expressions $a<b$ and $a>b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general,

$a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent to $11 > 7$.
$a > b$ is equivalent to $b < a$. For example, $17 > 4$ is equivalent to $4 < 17$.

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$ it means not equal.

We summarize the symbols of equality and inequality in the table below.

Algebraic Notation	**Say**
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

Symbols < and >

The symbols < and > each have a smaller side and a larger side.

smaller side < larger side
larger side > smaller side

The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. The table below lists three of the most commonly used grouping symbols in algebra.

Common Grouping Symbols

Name	**Symbol**
parentheses	( )
brackets	[ ]
braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$8\left(14-8\right)$ $21-3\left[2+4\left(9-8\right)\right]$ $24\div\left\{13-2\left[1\left(6-5\right)+4\right]\right\}$

Identify Expressions and Equations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Expression	**Words**	**Phrase**
$3+5$	$3 \text{ plus } 5$	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6\cdot 7$	$6 \text{ times } 7$	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Equation	**Sentence**
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Expressions and Equations

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

An equation is made up of two expressions connected by an equal sign.

EXAMPLE 1

Determine if each is an expression or an equation:

a. $16 - 6 = 10$
b. $4 \cdot 2 + 1$
c. $x \div 25$
d. $y + 8 = 40$

Solution

a. $16 - 6 = 10$	This is an equation—two expressions are connected with an equal sign.
b. $4 \cdot 2 + 1$	This is an expression—no equal sign.
c. $x \div 25$	This is an expression—no equal sign.
d. $y + 8 = 40$	This is an equation—two expressions are connected with an equal sign.

TRY IT 1

Determine if each is an expression or an equation:

a. $23 + 6 = 29$
b. $7 \cdot 3 - 7$

Show Answer

a. equation
b. expression

Simplify Expressions with Exponents

To simplify a numerical expression means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we'd first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$4 \cdot 2 + 1$
$8 + 1$
9

Suppose we have the expression $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. We could write this more compactly using exponential notation. Exponential notation is also called **power** and is used in algebra to represent a quantity multiplied by itself several times. We write $2 \cdot 2 \cdot 2$ as 2^3 and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the base and the 3 is called the exponent. The exponent tells us how many factors of the base we have to multiply.

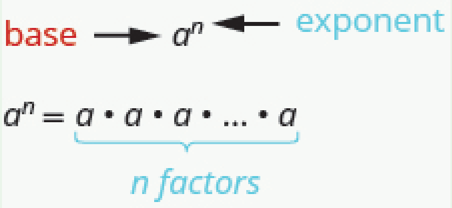
$\text{means multiply three factors of 2}$

We say 2^3 is in exponential notation and $2 \cdot 2 \cdot 2$ is in expanded notation.

Exponential Notation (Power)

For any expression a^n, a is a factor multiplied by itself n times if n is a positive integer.

$a^n \text{means multiply } n \text{ factors of } a$



The expression a^n is read a to the n^{th} power.

For powers of $n = 2$ and $n = 3$, we have special names.
a^2 is read as ”a squared”
a^3 is read as ”a cubed”

The table below lists some examples of expressions written in exponential notation.

Exponential Notation	**In Words**
7^2	7 to the second power, or 7 squared
5^3	5 to the third power, or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

EXAMPLE 2

Write each expression in exponential form:

a. $16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16$

b. $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

c. $x \cdot x \cdot x \cdot x$

d. $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

Solution

a. The base 16 is a factor 7 times.	16^7
b. The base 9 is a factor 5 times.	9^5
c. The base x is a factor 4 times.	x^4
d. The base a is a factor 8 times.	a^8

TRY IT 2

Write each expression in exponential form:

$41 \cdot 41 \cdot 41 \cdot 41 \cdot 41$

Show Answer

41^5

EXAMPLE 3

Write each exponential expression in expanded form:

a. 8^6

b. x^5

Solution

a. The base is 8 and the exponent is 6, so 8^6 means $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

b. The base is x and the exponent is 5, so x^5 means $x \cdot x \cdot x \cdot x \cdot x$

TRY IT 3

Write each exponential expression in expanded form:

a. 4^8
b. a^7

Show Answer

a. $4\cdot4\cdot4\cdot4\cdot4\cdot4\cdot4\cdot4$
b. $a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a$

To simplify an exponential expression without using a calculator, we write it in expanded form and then multiply the factors.

EXAMPLE 4

Simplify: 3^4.

Solution

	3^4
Expand the expression.	$3\cdot3\cdot3\cdot3$
Multiply left to right.	$9\cdot3\cdot3$
	$27\cdot3$
Multiply.	81

TRY IT 4

Simplify:

a. 5^3
b. 1^7

Show Answer

a. 125

b. 1

Simplify Expressions Using the Order of Operations

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression:

$4 + 3 \cdot 7$

Some students say it simplifies to 49.

	$4 + 3 \cdot 7$
Since $4 + 3$ gives 7.	$7 \cdot 7$
And $7 \cdot 7$ is 49.	49

Some students say it simplifies to 25.

	$4 + 3 \cdot 7$
Since $3 \cdot 7$ is 21.	$4 + 21$
And $21 + 4$ makes 25.	25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Order of Operations

When simplifying mathematical expressions perform the operations in the following order:

1. **P**arentheses and other Grouping Symbols
    - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. **E**xponents
    - Simplify all expressions with exponents.
3. **M**ultiplication and **D**ivision
    - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. **A**ddition and **S**ubtraction
    - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase.

Please **E**xcuse **M**y **D**ear **A**unt **S**ally.

Please	**P**arentheses
Excuse	**E**xponents
My **D**ear	**M**ultiplication and **D**ivision
Aunt **S**ally	**A**ddition and **S**ubtraction

It’s good that ‘**M**y **D**ear’ goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, ‘**A**unt **S**ally’ goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

EXAMPLE 5

Simplify the expressions:

a. $4 + 3 \cdot 7$

b. $(4 + 3) \cdot 7$

Solution

a.	
	$4 + 3 \cdot 7$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply first.	$4 + 3 \cdot 7$
Add.	$4 + 21$
	25

b.	
	$(4+3)\cdot 7$
Are there any **p**arentheses? Yes.	$(4+3)\cdot 7$
Simplify inside the parentheses.	$(7)7$
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply.	49

TRY IT 5

Simplify the expressions:

a. $12-5\cdot 2$
b. $(12-5)\cdot 2$

Show Answer

a. 2
b. 14

EXAMPLE 6

Simplify: $18\div 6+4\,(5-2)$.

Solution

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

TRY IT 6

Simplify:

$$30 \div 5 + 10\left(3 - 2\right)$$

Show Answer

16

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

EXAMPLE 7

Simplify: $5 + 2^3 + 3\left[6 - 3\left(4 - 2\right)\right]$.

Solution

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	
Simplify exponents.	$5 + 2^3 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 3[0]$
Is there any addition or subtraction? Yes.	
Add.	$5 + 8 + 0$
Add.	$13 + 0$
	13

TRY IT 7

Simplify:

$$9 + 5^3 - [4\,(9 + 3)]$$

Show Answer

86

EXAMPLE 8

Simplify: $2^3 + 3^4 \div 3 - 5^2$.

Solution

	$2^3 + 3^4 \div 3 - 5^2$
If an expression has several exponents, they may be simplified in the same step.	
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2$
Divide.	$8 + 81 \div 3 - 25$
Add.	$8 + 27 - 25$
Subtract.	$35 - 25$
	10

TRY IT 8

Simplify:

$3^2 + 2^4 \div 2 + 4^3$

Show Answer

81

Evaluate Algebraic Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

EXAMPLE 9

Evaluate $9x - 2$, when

a. $x = 5$

b. $x = 1$

Solution

Remember ab means a times b, so $9x$ means 9 times x.

a. To evaluate the expression when $x = 5$, we substitute 5 for x, and then simplify.

	$9x - 2$
Substitute 5 for x.	$9 \cdot 5 - 2$
Multiply.	$45 - 2$
Subtract.	43

b. To evaluate the expression when $x = 1$, we substitute 1 for x, and then simplify.

	$9x - 2$
Substitute 1 for x.	$9(1) - 2$
Multiply.	$9 - 2$
Subtract.	7

Notice that in part a) that we wrote $9 \cdot 5$ and in part b) we wrote $9\,(1)$. Both the dot and the parentheses tell us to multiply.

TRY IT 9

Evaluate:

$8x - 3$, when

a. $x = 2$
b. $x = 1$

Show Answer

a. 13
b. 5

EXAMPLE 10

Evaluate x^2 when $x = 10$.

Solution
We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 for x.	10^2
Use the definition of exponent.	$10 \cdot 10$
Multiply.	100

When $x = 10$, the expression x^2 has a value of 100.

TRY IT 10

Evaluate:

x^2 when $x = 8.$

Show Answer

64

EXAMPLE 11

Evaluate 2^x when $x = 5.$

Solution

In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply.	32

When $x = 5$, the expression 2^x has a value of 32.

TRY IT 11

Evaluate:

2^x when $x = 6$.

Show Answer
64

EXAMPLE 12

Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution
This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for *x* and 2 for *y*.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$
Add and subtract left to right.	32

When $x = 10$ and $y = 2$, the expression $3x + 4y - 6$ has a value of 32.

TRY IT 12

Evaluate:

$2x + 5y - 4$ when $x = 11$ and $y = 3$

Show Answer
33

EXAMPLE 13

Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution
We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2 \cdot x \cdot x$ and is different from the expression $(2x)^2$, which means $2x \cdot 2x$.

	$2x^2 + 3x + 8$
Substitute 4 for each x.	$2(4)^2 + 3(4) + 8$
Simplify 4^2.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

TRY IT 13

Evaluate:

$3x^2 + 4x + 1$ when $x = 3$.

Show Answer

40

ACCESS ADDITIONAL ONLINE RESOURCES

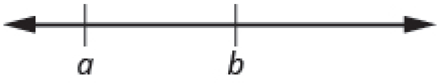
- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

Key Concepts

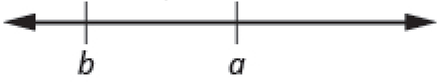
Operation	**Notation**	**Say:**	**The result is…**
Addition	$a + b$	a plus b	the sum of a and b
Multiplication	$a \cdot b, (a)\,(b)\,, (a)\,b, a\,(b)$	a times b	The product of a and b
Subtraction	$a - b$	a minus b	the difference of a and b
Division	$a \div b, a/b,\ \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - $a = b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.

- **Inequality**
    - $a < b$ is read a is less than b
    - a is to the left of b on the number line



    - $a > b$ is read a is greater than b
    - a is to the right of b on the number line



Algebraic Notation	**Say**
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

- **Exponential Notation**
    - For any expression a^n is a factor multiplied by itself n times, if n is a positive integer.
    - a^n means multiply n factors of a

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

    - The expression of a^n is read a to the $n\mathrm{th}$ power.

Order of Operations When simplifying mathematical expressions perform the operations in the following order:

- Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
- Exponents: Simplify all expressions with exponents.
- Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
- Addition and Subtraction: Perform all addition and subtraction in order from left to right.

These operations have equal priority.

Glossary

expressions
An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

equation
An equation is made up of two expressions connected by an equal sign.

1.1 Exercise Set

In the following exercises, determine if each is an expression or an equation.

1. $9 \cdot 6 = 54$
2. $5 \cdot 4 + 3$
3. $x + 7$
4. $y - 5 = 25$

In the following exercises, write in exponential form.

5. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
6. $x \cdot x \cdot x \cdot x \cdot x$

In the following exercises, write in expanded form.

7. 5^3
8. 2^8

In the following exercises, simplify.

9.
   a. $3 + 8 \cdot 5$
   b. $(3+8) \cdot 5$
10. $2^3 - 12 \div (9 - 5)$
11. $3 \cdot 8 + 5 \cdot 2$
12. $2 + 8\,(6 + 1)$
13. $4 \cdot 12/8$
14. $6 + 10/2 + 2$
15. $(6 + 10) \div (2 + 2)$
16. $20 \div 4 + 6 \cdot 5$
17. $20 \div (4 + 6) \cdot 5$
18. $4^2 + 5^2$
19. $(4 + 5)^2$
20. $3\,(1 + 9 \cdot 6) - 4^2$
21. $2\,[1 + 3\,(10 - 2)]$

In the following exercises, evaluate the expression for the given value.

22. $7x + 8$ when $x = 2$
23. $5x - 4$ when $x = 6$
24. x^2 when $x = 12$
25. x^2 when $x = 12$
26. $x^2 + 3x - 7$ when $x = 4$
27. $2x + 4y - 5$ when $x = 7, y = 8$
28. $(x - y)^2$ when $x = 10, y = 7$
29. $a^2 + b^2$ when $a = 3, b = 8$
30. $2l + 2w$ when $l = 15, w = 12$

Answers:

1. equation
2. expression
3. expression
4. equation
5. 3^7
6. x^5
7. 125
8. 256
9.
    a. 43
    b. 55
10. 5
11. 34
12. 58
13. 6
14. 13
15. 4
16. 35
17. 10
18. 41
19. 81
20. 149
21. 50
22. 22
23. 26
24. 144
25. 27
26. 21
27. 41
28. 9
29. 73
30. 54

Attributions

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1.2 Integers

Learning Objectives

By the end of this section it is expected that you will be able to:

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add ans subtract integers
- Multiply and divide integers
- Simplify Expressions with Integers

Use Negatives and Opposites

If you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See Figure 1.

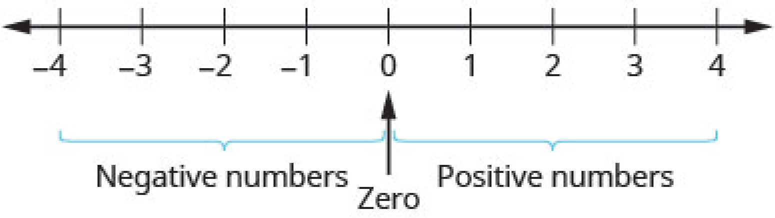


Figure 1 The number line shows the location of positive and negative numbers.

The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See Figure 2.

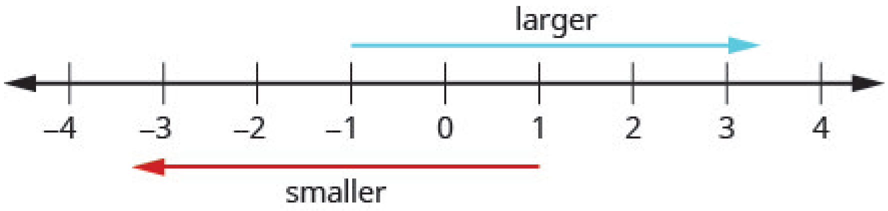


Figure 2 The numbers on a number line increase in value going from left to right and decrease in value going from right to left.

Remember that we use the notation:

$a < b$ (read "a is less than b") when a is to the left of b on the number line.

$a > b$ (read "*a* is greater than *b*") when *a* is to the right of *b* on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in Figure 3 are called the integers. The integers are the numbers $-3, -2, -1, 0, 1, 2, 3$

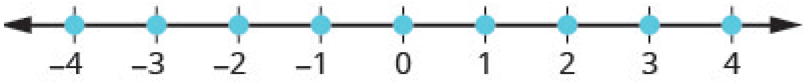
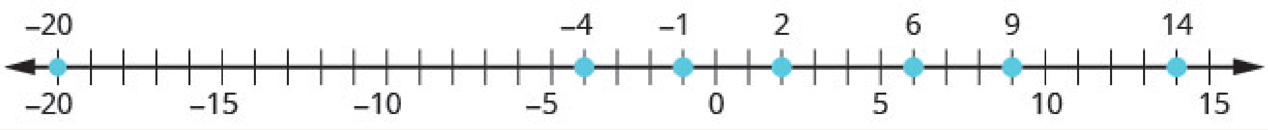


Figure 3 All the marked numbers are called integers.

EXAMPLE 1

Order each of the following pairs of numbers, using < or >: a) $14\_\_\_6$ b) $-1\_\_\_9$ c) $-1\_\_\_-4$ d) $2\_\_\_$ -20.

Solution
It may be helpful to refer to the number line shown.



a) 14 is to the right of 6 on the number line.	$14\_\_6$ $14 > 6$
b) −1 is to the left of 9 on the number line.	$-1\_\_9$ $-1 < 9$
c) −1 is to the right of −4 on the number line.	$-1\_\_-4$
d) 2 is to the right of −20 on the number line.	$2\_\_-20$ $2 > -20$

TRY IT 1

Order each of the following pairs of numbers, using < or > : a) $15\_\_\_7$ b) $-2\_\_\_5$ c) $-3\_\_\_-7$ d) 5___ $-17.$

Show answer
a) > b) < c) > d) >

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called opposite**s**. The opposite of 2 is -2, and the opposite of -2 is 2

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

(Figure 4) illustrates the definition.

The opposite of 3 is -3.

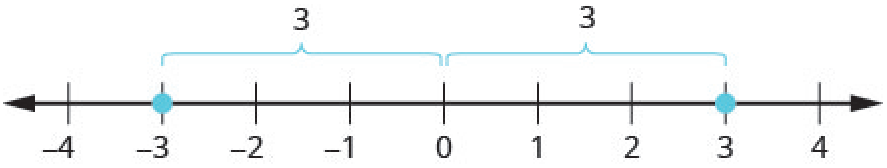


Figure 4

Opposite Notation

$-a$ means the opposite of the number *a*.

The notation $-a$ is read as “the opposite of *a*.”

The whole numbers and their opposites are called the integers. The integers are the numbers $-\ 3, -2, -1, 0, 1, 2, 3$

Integers

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$-3, -2, -1, 0, 1, 2, 3$

Simplify: Expressions with Absolute Value

We saw that numbers such as 3 and -3 are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Absolute Value

The absolute value of a number is its distance from 0 on the number line.

The absolute value of a number *n* is written as $|n|$.

For example,

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure 5 illustrates this idea.

The integers 5 and are 5 units away from 0.

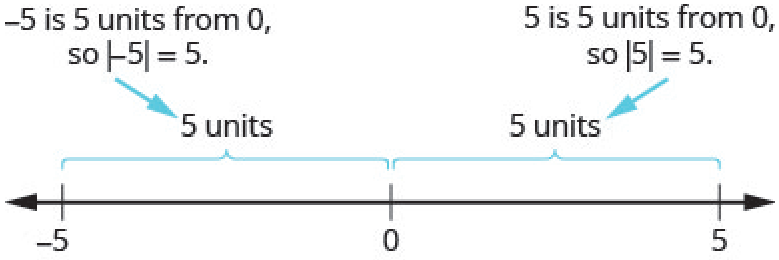


Figure 5

The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from 0 to 0 on the number line is zero units.

Property of Absolute Value

$|n| \geq 0$ for all numbers

Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, “absolute values are always non-negative.” Non-negative means greater than or equal to zero.

EXAMPLE 2

Simplify: a) $|3|$ b) $|-44|$ c) $|0|$.

Solution
The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

a) $|3|$
3

b) $|-44|$
44

c) $|0|$
0

TRY IT 2

Simplify: a) $|4|$ b) $|-28|$ c) $|0|$.

Show answer
a) 4 b) 28 c) 0

In the next example, we’ll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

EXAMPLE 3

Fill in <, >, or $=$ for each of the following pairs of numbers:

a) $|-5|$___$-|-5|$ b) 8___$-|-8|$c) -9___$-|-9|$ d) --16___$-|-16|$

Solution

	$\|-5\|$___$-\|-5\|$
a) Simplify. Order.	5 ___ -5
	5 > -5
	$\|-5\|$ > $-\|-5\|$
b) Simplify. Order.	8 ___ $-\|-8\|$
	8 ___ -8
	8 > -8
	8 > $-\|-8\|$
c) Simplify. Order.	9 ___ $-\|-9\|$ -9 ___ -9 -9 = -9 -9 = $-\|-9\|$
d) Simplify. Order.	$-(-16)$ ___ $-\|-16\|$ 16 ____ -16 16 > -16 $-(-16)$ > $-\|-16\|$

TRY IT 3

Fill in <, >, or = for each of the following pairs of numbers: a) $|-9|$___-$|-9|$ b) 2___-$|-2|$ c) -8___ $|-8|$
d) $(-9$___$-|-9|$.

Show answer
a) > b) > c) < d) >

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols

Parentheses	( )
Brackets	[ ]
Braces	{ }
Absolute value	| |

In the next example, we simplify the expressions inside absolute value bars first, just like we do with parentheses.

EXAMPLE 4

Simplify: $24-|19-3\left(6-2\right)|$.

Solution

	$24-|19-3\left(6-2\right)|$
Work inside parentheses first: subtract 2 from 6.	$24-|19-3\left(4\right)|$
Multiply 3(4).	$24-|19-12|$
Subtract inside the absolute value bars.	$24-|7|$
Take the absolute value.	$24-7$
Subtract.	17

TRY IT 4

Simplify: $19-|11-4\left(3-1\right)|$.

Show answer

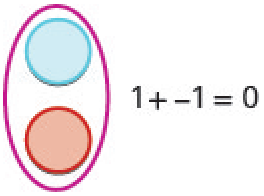
16

Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two colour counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one colour (blue) represent positive. The other colour (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



We will use the counters to show how to add the four addition facts using the numbers $5, -5$ and $3, -3$.

$5+3$ $-5+(-3)$ $-5+3$ $5+(-3)$

The first example adds 5 positives and 3 positives—both positives.

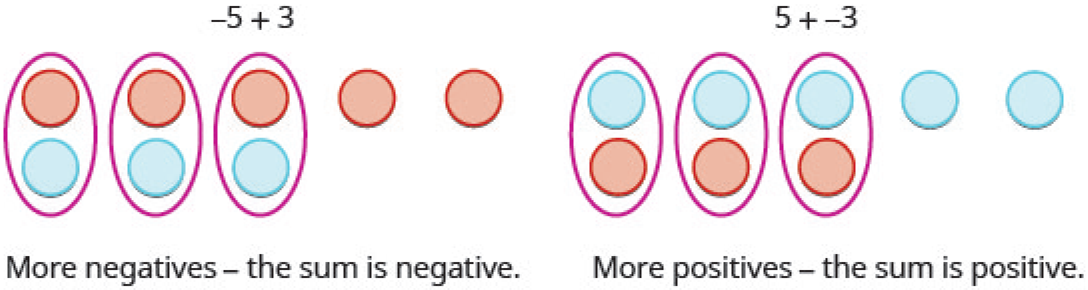
The second example adds 5 negatives and 3 negatives—both negatives.

In each case we got 8—either 8 positives or 8 negatives.

When the signs were the same, the counters were all the same color, and so we added them.

So what happens when the signs are different? Let's add $-5+3$ and $5+(-3)$.

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

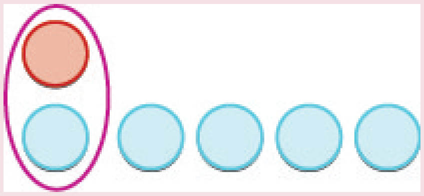


EXAMPLE 5

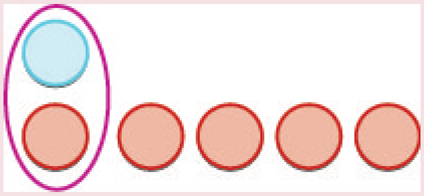
Add: a) $-1+5$ b) $1+(-5)$.

Solution

a)

	−1 + 5
	
There are more positives, so the sum is positive.	4

b)

	1 + (−5)
	
There are more negatives, so the sum is negative.	−4

TRY IT 5

Add: a) $-2+4$ b) $2+(-4)$.

Show answer

a) 2 b) -2

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37+(-53)$, you really don't want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53 - 37 = 16$, there are 16 more red counters.

Therefore, the sum of $37 + (-53)$ is -16.
$37 + (-53) = -16$

Let's try another one. We'll add $-74 + (-27)$. Again, imagine 74 red counters and 27 more red counters, so we'd have 101 red counters. This means the sum is -101.

$-74 + (-27) = -101$

Let's look again at the results of adding the different combinations of 5, -5 and 3, -3.

Addition of Positive and Negative Integers

$$\begin{array}{cc} 5+3 & -5+(-3) \\ 8 & -8 \\ \text{both positive, sum positive} & \text{both negative, sum negative} \end{array}$$

When the signs are the same, the counters would be all the same color, so add them.

$$\begin{array}{cc} -5+3 & 5+(-3) \\ -2 & 2 \\ \text{different signs, more negatives, sum negative} & \text{different signs, more positives, sum positive} \end{array}$$

When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

EXAMPLE 6

Simplify: a) $19 + (-47)$ b) $-14 + (-36)$.

Solution

a. Since the signs are different, we subtract $19 \text{ from } 47$. The answer will be negative because there are more negatives than positives.

$$\begin{array}{ll} & 19+(-47) \\ \text{Add.} & -28 \end{array}$$

b. Since the signs are the same, we add. The answer will be negative because there are only negatives.

$$\begin{array}{ll} & -14+(-36) \\ \text{Add.} & -50 \end{array}$$

TRY IT 6

Simplify: a) $-31+(-19)$ b) $15+(-32)$.

Show answer
a)-50 b)-17

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

EXAMPLE 7

Simplify: $-5+3(-2+7)$.

Solution

	$-5+3(-2+7)$
Simplify inside the parentheses.	$-5+3(5)$
Multiply.	$-5+15$
Add left to right.	10

TRY IT 7

Simplify: $-2+5(-4+7)$.

Show answer
13

Subtract Integers

We will continue to use counters to model the subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read "$5-3$" as "5 take away 3." When you use counters, you can think of subtraction the same way!

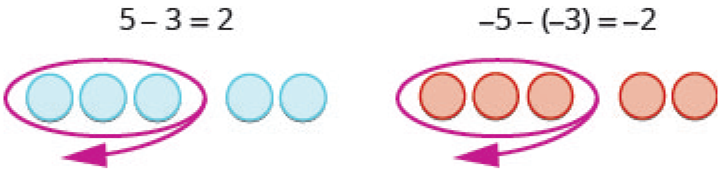
We will model the four subtraction facts using the numbers 5 and 3.

$5 - 3$ $-5 - (-3)$ $-5 - 3$ $5 - (-3)$

The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.

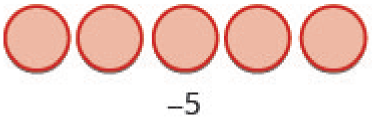
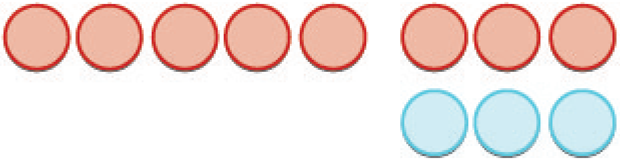
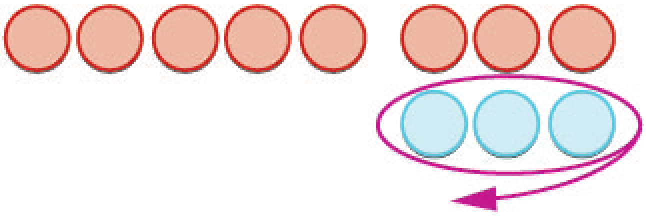


What happens when we have to subtract one positive and one negative number? We’ll need to use both white and red counters as well as some neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

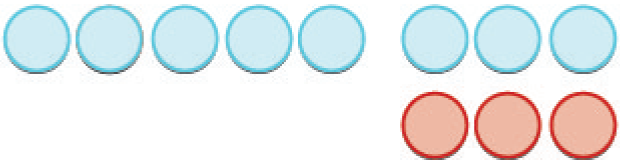
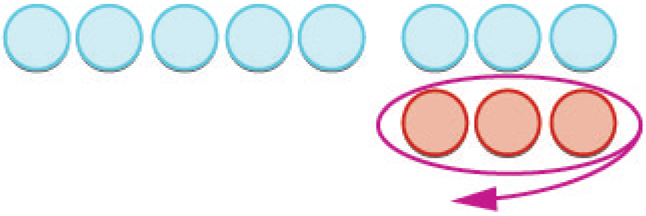
- To subtract $-5 - 3$, we restate it as -5 take away 3.

We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

	−5 − 3 means −5 take away 3.
We start with 5 negatives.	
We now add the neutrals needed to get 3 positives.	
We remove the 3 positives.	
We are left with 8 negatives.	
The difference of −5 and 3 is −8.	−5 − 3 = −8

And now, the fourth case, $5-(-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

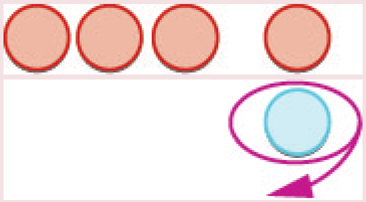
	5 − (−3) means 5 take away −3.
We start with 5 positives.	
We now add the needed neutrals pairs.	
We remove the 3 negatives.	
We are left with 8 positives.	
The difference of 5 and −3 is 8.	5 − (−3) = 8

EXAMPLE 8

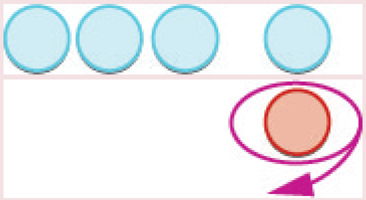
Subtract: a) $-3-1$ b) $3-(-1)$.

Solution

a)

Take 1 positive from the one added neutral pair.		−3 − 1 −4

b)

Take 1 negative from the one added neutral pair.		3 − (−1) 4

TRY IT 8

Subtract: a) $-6-4$ b) $6-(-4)$.

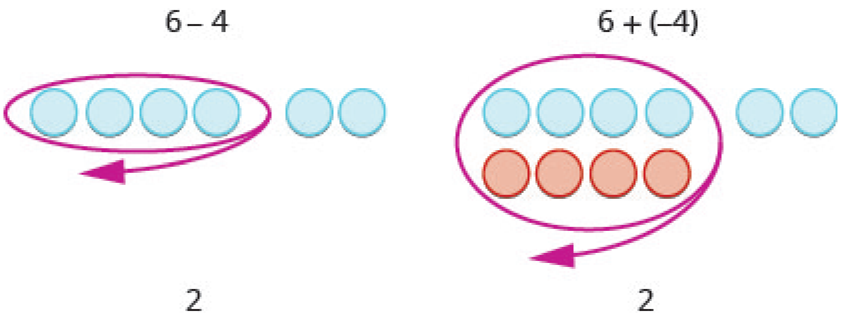
Show answer
a) -10 b) 10

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In Example 8, $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the subtraction property, written as follows:

Subtraction Property

$a-b=a+(-b)$
Subtracting a number is the same as adding its opposite.

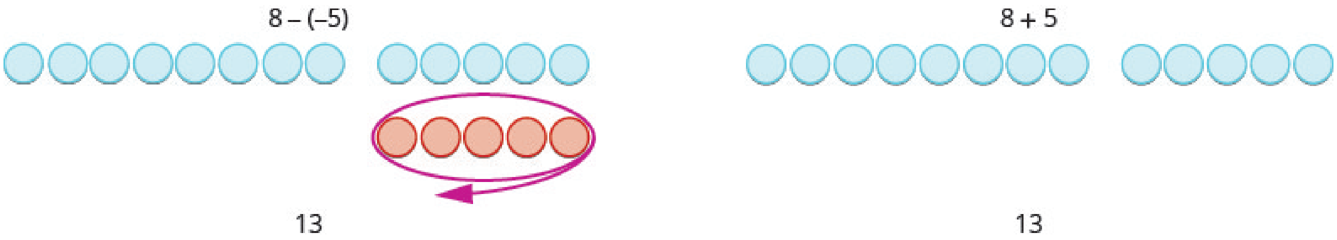
Look at these two examples.



$6-4$ gives the same answer as $6+(-4)$.

Of course, when you have a subtraction problem that has only positive numbers, like $6-4$, you just do the subtraction. You already knew how to subtract $6-4$ long ago. But *knowing* that $6-4$ gives the same answer as $6+(-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6-4$ and $6+(-4)$ give the same results!

Look at what happens when we subtract a negative.



$8-(-5)$ gives the same answer as $8+5$

Subtracting a negative number is like adding a positive!

You will often see this written as $a - (-b) = a + b$.

What happens when there are more than three integers? We just use the order of operations as usual.

EXAMPLE 9

Simplify: $7 - (-4 - 3) - 9$.

Solution

	$7 - (-4 - 3) - 9$
Simplify inside the parentheses first.	$7 - (-7) - 9$
Subtract left to right.	$14 - 9$
Subtract.	5

TRY IT 9

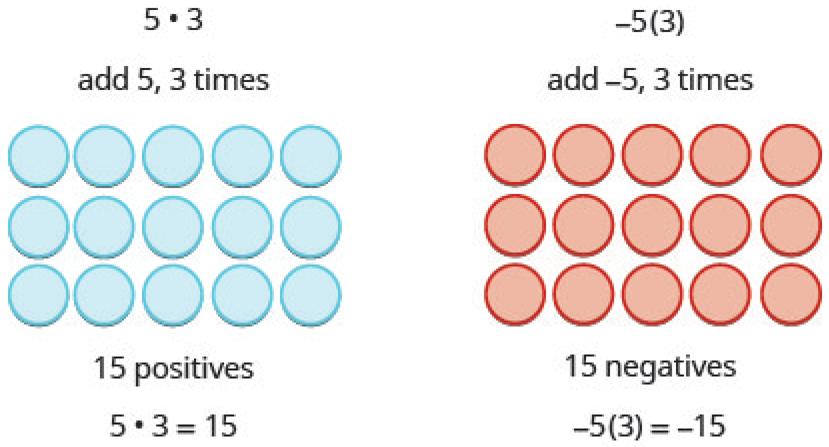
Simplify: $8 - (-3 - 1) - 9$.

Show answer
3

Multiply Integers

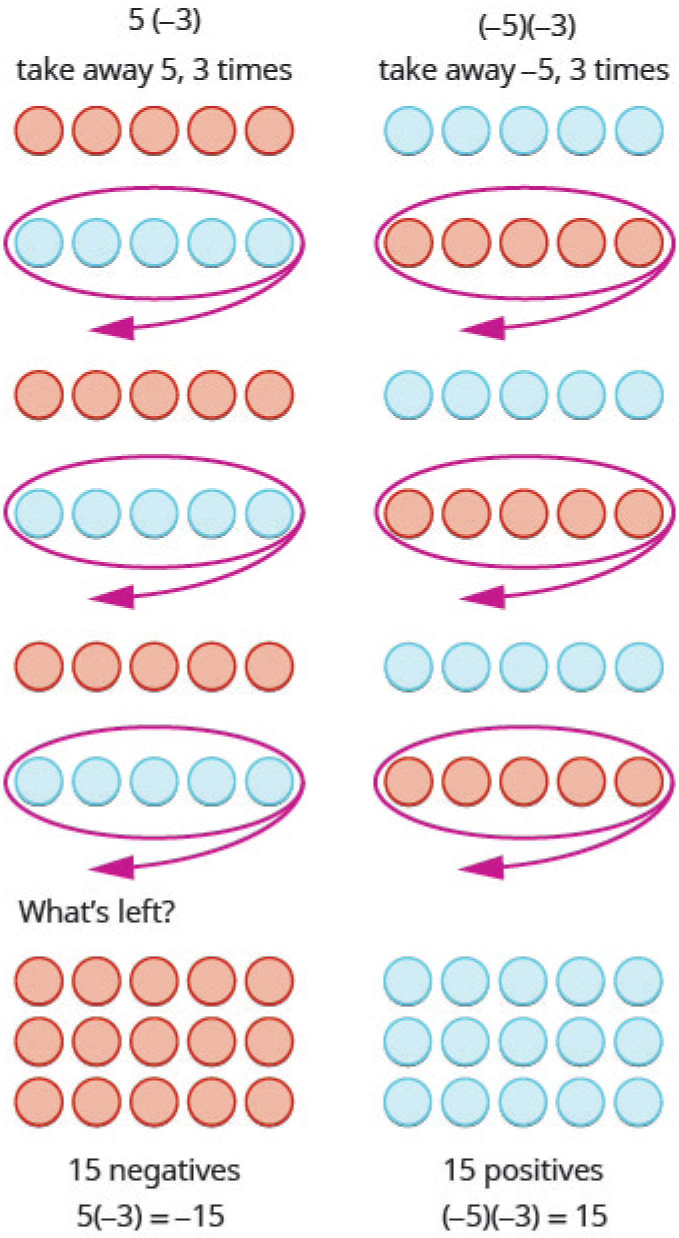
Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let’s look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \cdot b$ means add *a*, *b* times. Here, we are using the model just to help us discover the pattern.



The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as “taking away,” it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the work space. Then we take away 5 three times.



In summary:

$$\begin{array}{rclcrcl} 5 \cdot 3 & = & 15 & \quad & -5\,(3) & = & -15 \\ 5\,(-3) & = & -15 & \quad & (-5)\,(-3) & = & 15 \end{array}$$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below

Multiplication of Signed Numbers

For multiplication of two signed numbers:

Same signs	**Product**	**Example**
Two positives Two negatives	Positive Positive	$7 \cdot 4 = 28$ $-8(-6) = 48$

Different signs	**Product**	**Example**
Positive \cdot negative Negative \cdot positive	Negative Negative	$7(-9) = -63$ $-5 \cdot 10 = -50$

EXAMPLE 10

Multiply: a) $-9 \cdot 3$ b) $-2(-5)$ c) $4(-8)$ d) $7 \cdot 6$.

Solution

a) Multiply, noting that the signs are different so the product is negative.	$-9 \cdot 3$ -27
b) Multiply, noting that the signs are the same so the product is positive.	$-2(-5)$ 10
c) Multiply, with different signs.	$4(-8)$ -32
d) Multiply, with same signs.	$7 \cdot 6$ 42

TRY IT 10

Multiply: a) $-6 \cdot 8$ b) $-4(-7)$ c) $9(-7)$ d) $5 \cdot 12$.

Show answer
a)-48 b) 28 c) -63 d) 60

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

$$\begin{array}{lcc} & -1\cdot 4 & -1\,(-3) \\ \text{Multiply.} & -4 & 3 \\ & -4\,\text{is the opposite of}\,4. & 3\,\text{is the opposite of}\,-3. \end{array}$$

Each time we multiply a number by -1, we get its opposite!

Multiplication by -1

$-1a =$ -a

Multiplying a number by -1 gives its opposite.

EXAMPLE 11

Multiply: a) $-1\cdot 7$ b) $-1\,(-11)$.

Solution

a) Multiply, noting that the signs are different so the product is negative.	$-1\cdot 7$ -7 $-7\,\text{is the opposite of}\,7.$
b) Multiply, noting that the signs are the same so the product is positive.	$-1\,(-11)$ 11 $11\,\text{is the opposite of}\,-11.$

TRY IT 11

Multiply: a) $-1\cdot 9$ b) $-1\cdot(-17)$.

Show answer
a)-9 b) 17

Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \cdot 3 = 15$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.

$$\begin{array}{rclrcl} 5 \cdot 3 & = & 15 \text{ so } 15 \div 3 & = & 5 & \qquad -5\,(3) & = & -15 \text{ so } -15 \div 3 & = & -5 \\ (-5)\,(-3) & = & 15 \text{ so } 15 \div (-3) & = & -5 & \qquad 5\,(-3) & = & -15 \text{ so } -15 \div (-3) & = & 5 \end{array}$$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Multiplication and Division of Signed Numbers

For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	**Result**
Two positives	Positive
Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	**Result**
Positive and negative	Negative
Negative and positive	Negative

If the signs are different, the result is negative.

EXAMPLE 12

Divide: a) $-27 \div 3$ b) $-100 \div (-4)$.

Solution

a) Divide. With different signs, the quotient is negative.	$-27 \div 3$ -9
b) Divide. With signs that are the same, the quotient is positive.	$-100 \div (-4)$ 25

TRY IT 12

Divide: a) $-42 \div 6$ b) $-117 \div (-3)$.

Show answer
a)-7 b) 39

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let’s try some examples. We’ll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

EXAMPLE 13

Simplify: $7(-2) + 4(-7) - 6$.

Solution

	$7\left(-2\right)+4\left(-7\right)-6$
Multiply first.	$-14+\left(-28\right)-6$
Add.	$-42-6$
Subtract.	-48

TRY IT 13

Simplify: $8\left(-3\right)+5\left(-7\right)-4$.

Show answer
-63

EXAMPLE 14

Simplify: a) $\left(-2\right)^{4}$ b) -2^{4}.

Solution

a) Write in expanded form. Multiply. Multiply. Multiply.	$\left(-2\right)^{4}$ $\left(-2\right)\left(-2\right)\left(-2\right)\left(-2\right)$ $4\left(-2\right)\left(-2\right)$ $-8\left(-2\right)$ 16
b) Write in expanded form. We are asked to find the opposite of2^{4}. Multiply. Multiply. Multiply.	-2^{4} - $\left(2\cdot 2\cdot 2\cdot 2\right)$ - $\left(4\cdot 2\cdot 2\right)$ - $\left(8\cdot 2\right)$ 16

Notice the difference in parts a) and b). In part a), the exponent means to raise what is in the parentheses, the $\left(-2\right)$ to the 4^{th} power. In part b), the exponent means to raise just the 2 to the 4^{th} power and then take the opposite.

TRY IT 14

Simplify: a) $(-3)^4$ b) -3^4.

Show answer
a) 81 b) -81

The next example reminds us to simplify inside parentheses first.

EXAMPLE 15

Simplify: $12-3\,(9-12)$.

Solution

	$12-3\,(9-12)$
Subtract in parentheses first.	$12-3\,(-3)$
Multiply.	$12-(-9)$
Subtract.	21

TRY IT 15

Simplify: $17-4\,(8-11)$.

Show answer
29

EXAMPLE 16

Simplify: $8\,(-9)\div(-2)^3$.

Solution

	$8\left(-9\right) \div \left(-2\right)^3$
Exponents first.	$8\left(-9\right) \div \left(-8\right)$
Multiply.	$-72 \div \left(-8\right)$
Divide.	9

TRY IT 16

Simplify: $12\left(-9\right) \div \left(-3\right)^3$.

Show answer
4

EXAMPLE 17

Simplify: $-30 \div 2 + \left(-3\right)\left(-7\right)$.

Solution

	$-30 \div 2 + \left(-3\right)\left(-7\right)$
Multiply and divide left to right, so divide first.	$-15 + \left(-3\right)\left(-7\right)$
Multiply.	$-15 + 21$
Add.	6

TRY IT 17

Simplify: $-27 \div 3 + \left(-5\right)\left(-6\right)$.

Show answer
21

Access these online resources for additional instruction and practice with adding and subtracting integers. You will need to enable Java in your web browser to use the applications.

- Add Colored Chip
- Subtract Colored Chip

Key Concepts

- **Addition of Positive and Negative Integers**

$5+3$ 8 both positive, sum positive	$-5+(-3)$ -8 both negative, sum negative
$-5+3$ -2 different signs, more negatives sum negative	$5+(-3)$ 2 different signs, more positives sum positive

- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!
- **Subtraction of Integers**

$5-3$ 2 5 positives take away 3 positives 2 positives	$-5-(-3)$ -2 5 negatives take away 3 negatives 2 negatives
$-5-3$ -8 5 negatives, want to subtract 3 positives need neutral pairs	$5-(-3)$ 8 5 positives, want to subtract 3 negatives need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.
- **Multiplication and Division of Two Signed Numbers**
    - Same signs—Product is positive
    - Different signs—Product is negative

Glossary

absolute value
The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.

integers
The whole numbers and their opposites are called the integers: …−3, −2, −1, 0, 1, 2, 3…

opposite
The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero: $-a$ means the opposite of the number. The notation $-a$ is read "the opposite of a."

1.2 Exercise Set

In the following exercises, order each of the following pairs of numbers, using < or >.

1.
   - a. $9\_\_\_4$
   - b. $-3\_\_\_6$
   - c. $-8\_\_\_-2$
   - d. $1\_\_\_-10$

In the following exercises, simplify.

2.
   - a. $|-32|$
   - b. $|0|$
   - c. $|16|$

In the following exercises, fill in <, >, or $=$ for each of the following pairs of numbers.

3.
   - a. $-6\_\_\_|-6|$
   - b. $-|-3|\_\_\_-3$

In the following exercises, simplify.

4. $-(-5)$ and $-|-5|$
5. $8|-7|$
6. $|15-7|-|14-6|$
7. $18-|2(8-3)|$

In the following exercises, simplify each expression.

8. $-21+(-59)$
9. $48+(-16)$
10. $-14+(-12)+4$
11. $135+(-110)+83$
12. $19+2(-3+8)$

In the following exercises, simplify.

13. $8-2$
14. $-5-4$
15. $8-(-4)$
16. a. $44-28$
    b. $44+(-28)$
17. a. $27-(-18)$
    b. $27+18$

In the following exercises, simplify each expression.

18. $15-(-12)$
19. $48-87$
20. $-17-42$
21. $-103-(-52)$
22. $-45-(54)$
23. $8-3-7$
24. $-5-4+7$
25. $-14-(-27)+9$
26. $(2-7)-(3-8)\,(2)$
27. $-\,(6-8)-(2-4)$
28. $25-[10-(3-12)]$
29. $6.3-4.3-7.2$
30. 5^2-6^2

In the following exercises, multiply.

31. $-4\cdot 8$
32. $-1\cdot 6$
33. $9\,(-7)$
34. $-1\,(-14)$

In the following exercises, divide.

35. $-24\div 6$
36. $-180\div 15$
37. $-52\div(-4)$

In the following exercises, simplify each expression.

38. $5\,(-6)+7\,(-2)-3$
39. $(-2)^6$
40. -4^2
41. $-3\,(-5)\,(6)$
42. $(8-11)\,(9-12)$
43. $26-3\,(2-7)$
44. $65\div(-5)+(-28)\div(-7)$
45. $9-2\,[3-8\,(-2)]$
46. $(-3)^2-24\div(8-2)$

In the following exercises, solve.

47. **Temperature** On January 15, the high temperature in Lytton, British Columbia, was 84° . That same day, the high temperature in Fort Nelson, British Columbia was -12°. What was the difference between the temperature in Lytton and the temperature in Embarrass?
48. **Checking Account** Ester has \$124 in her checking account. She writes a check for \$152. What is the new balance in her checking account?
49. **Checking Account** Kevin has a balance of $-\$38$ in his checking account. He deposits \$225 to the account. What is the new balance?

50. **Provincial budgets** For 2019 the province of Quebec estimated it would have a budget surplus of $5.6 million. That same year, Alberta estimated it would have a budget deficit of $7.5 million.

    Use integers to write the budget of:

    a. Quebec
    b. Alberta

Answers:

1.
    a. >
    b. <
    c. <
    d. >
2.
    a. 32
    b. 0
    c. 16
3.
    a. <
    b. =
4. $5, -5$
5. 56
6. 0
7. 8
8. -80
9. 32
10. -22
11. 108
12. 29
13. 6
14. -9
15. 12
16.
    a. 16
    b. 16
17.
    a. 45
    b. 45
18. 27
19. -39
20. -59
21. -51
22. -99
23. -2
24. -2
25. 22
26. -15
27. 0
28. 6
29. -5.2
30. -11
31. -32
32. -63
33. -6
34. 14
35. -4
36. 13
37. -12
38. -47
39. 64
40. -16
41. 90
42. 9
43. 41
44. -9
45. -29
46. 5
47. 96°
48. $-\$28$
49. $187
50.
    a. $5.6 million
    b. $-\$7.5$ million

Attributions

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1.3 Fractions

Learning Objectives

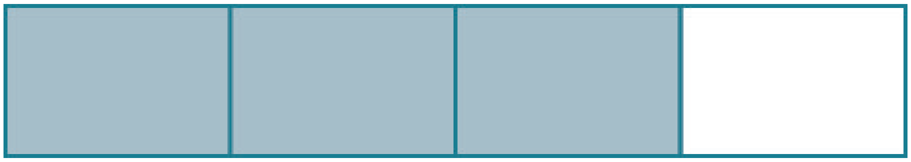
By the end of this section it is expected that you will be able to:

- Multiply and divide fractions
- Simplifying expressions with fraction bar
- Add or subtract fractions with a common denominator
- Add or subtract fractions with different denominators
- Use the order of operations to simplify complex fractions

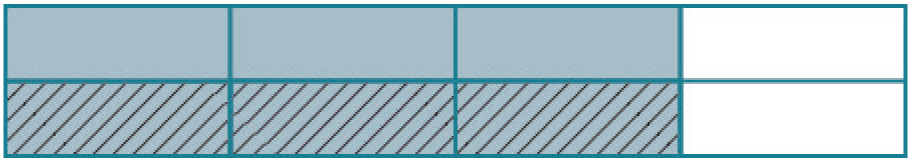
Multiply Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions. So we will start with fraction multiplication.

We'll use a model to show you how to multiply two fractions and to help you remember the procedure. Let's start with $\frac{3}{4}$.



Now we'll take $\frac{1}{2}$ of $\frac{3}{4}$.



Notice that now, the whole is divided into 8 equal parts. So $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

To multiply fractions, we multiply the numerators and multiply the denominators.

Fraction Multiplication

If a, b, c and d are numbers where $b \neq 0$ and $d \neq 0$, then
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In Example 1, we will multiply negative and a positive, so the product will be negative.

EXAMPLE 1

Multiply: $-\frac{11}{12} \cdot \frac{5}{7}$.

Solution
The first step is to find the sign of the product. Since the signs are the different, the product is negative.

	$-\frac{11}{12} \cdot \frac{5}{7}$
Determine the sign of the product; multiply.	$-\frac{11 \cdot 5}{12 \cdot 7}$
Are there any common factors in the numerator and the demoninator? No.	$-\frac{55}{84}$

TRY IT 1

Multiply: $-\frac{10}{28} \cdot \frac{8}{15}$.

Show answer
$-\frac{4}{21}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, *a*, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

EXAMPLE 2

Multiply: $-\frac{12}{5}\cdot(-20x)$.

Solution

Determine the sign of the product. The signs are the same, so the product is positive.

	$-\frac{12}{5}\cdot(-20x)$
Write $20x$ as a fraction.	$\frac{12}{5}\cdot\left(\frac{20x}{1}\right)$
Multiply.	
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12\cdot 4\cdot \cancel{5}x}{\cancel{5}\cdot 1}$
Simplify.	$48x$

TRY IT 2

Multiply: $\frac{11}{3}\cdot(-9a)$.

Show answer

$-33a$

Divide Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, that we need some vocabulary.

The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Notice that $\frac{2}{3}\cdot\frac{3}{2}=1$. A number and its reciprocal multiply to 1.

To get a product of positive 1 when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

The reciprocal of $-\frac{10}{7}$ is $-\frac{7}{10}$, since $-\frac{10}{7}\cdot\left(-\frac{7}{10}\right)=1$.

Reciprocal

The **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$.

A number and its reciprocal multiply to one $\frac{a}{b} \cdot \frac{b}{a} = 1$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division

If a, b, c and d are numbers where $b \neq 0, c \neq 0$ and $d \neq 0$, then

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero!

EXAMPLE 3

Find the quotient: $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$.

Solution

	$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18} \cdot -\frac{27}{14}$
Determine the sign of the product, and then multiply..	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

TRY IT 3

Find the quotient: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$.

Show answer

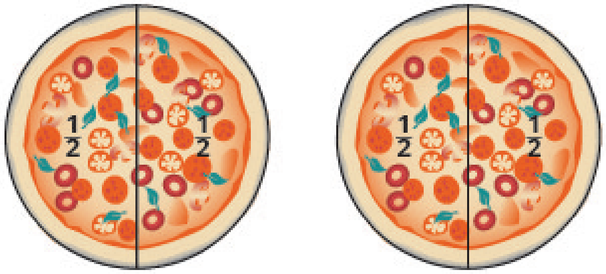
$\frac{4}{15}$

There are several ways to remember which steps to take to multiply or divide fractions. One way is to repeat the call outs to yourself. If you do this each time you do an exercise, you will have the steps memorized.

- “To multiply fractions, multiply the numerators and multiply the denominators.”
- “To divide fractions, multiply the first fraction by the reciprocal of the second.”

Another way is to keep two examples in mind:

One fourth of two pizzas is one half of a pizza.



$$2 \cdot \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{1}{4}$$

$$\frac{2}{4}$$

$$\frac{1}{2}$$

There are eight quarters in $2.00.

$$2 \div \frac{1}{4}$$

$$\frac{2}{1} \div \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{4}{1}$$

$$8$$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Complex Fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$\frac{\frac{6}{7}}{3}$ $\frac{\frac{3}{4}}{\frac{5}{8}}$ $\frac{\frac{x}{2}}{\frac{5}{6}}$

To simplify a complex fraction, we remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

EXAMPLE 4

Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$.

Solution

	$\frac{\frac{3}{4}}{\frac{5}{8}}$
Rewrite as division.	$\frac{3}{4} \div \frac{5}{8}$
Multiply the first fraction by the reciprocal of the second.	$\frac{3}{4} \cdot \frac{8}{5}$
Multiply.	$\frac{3 \cdot 8}{4 \cdot 5}$
Look for common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 5}$
Divide out common factors and simplify.	$\frac{6}{5}$

TRY IT 4

Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$.

Show answer

$\frac{4}{5}$

Simplify Expressions with a Fraction Bar

The line that separates the numerator from the denominator in a fraction is called a fraction bar. A fraction bar acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

To simplify the expression $\frac{5-3}{7+1}$, we first simplify the numerator and the denominator separately. Then we divide.

$\frac{5-3}{7+1}$

$\frac{2}{8}$

$\frac{1}{4}$

HOW TO: Simplify an Expression with a Fraction Bar

1. Simplify the expression in the numerator. Simplify the expression in the denominator.
2. Simplify the fraction.

EXAMPLE 5

Simplify: $\frac{4-2(3)}{2^2+2}$.

Solution

	$\frac{4-2(3)}{2^2+2}$
Use the order of operations to simpliy the numerator and the denominator.	$\frac{4-6}{4+2}$
Simplify the numerator and the denominator.	$\frac{-2}{6}$
Simplify. A negative divided by a positive is negative.	$-\frac{1}{3}$

TRY IT 5

Simplify: $\frac{6-3(5)}{3^2+3}$.

Show answer
$-\frac{3}{4}$

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$\frac{-1}{3} = -\frac{1}{3}$ $\frac{\text{negative}}{\text{positive}} = \text{negative}$

$\frac{1}{-3} = -\frac{1}{3}$ $\frac{\text{positive}}{\text{negative}} = \text{negative}$

Placement of Negative Sign in a Fraction

For any positive numbers *a* and *b*,

$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

EXAMPLE 6

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

TRY IT 6

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Show answer
4

Add or Subtract Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Fraction Addition and Subtraction

If $a, b, \text{ and } c$ are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

EXAMPLE 7

Find the sum: $\frac{3}{7} + \frac{2}{7}$.

Solution

	$\frac{3}{7} + \frac{2}{7}$
Add the numerators and place the sum over the common denominator. Simplify.	$\frac{3+2}{7}$ $\frac{5}{7}$

TRY IT 7

Find the sum: $\frac{5}{9}+\frac{2}{9}$.

Show answer
$\frac{7}{9}$

EXAMPLE 8

Find the difference: $-\frac{23}{24}-\frac{13}{24}$.

Solution

	$-\frac{23}{24}-\frac{13}{24}$
Subtract the numerators and place the difference over the common denominator.	$\frac{-23-13}{24}$
Simplify.	$\frac{-36}{24}$
Simplify. Remember, $-\frac{a}{b}=\frac{-a}{b}$.	$-\frac{3}{2}$

TRY IT 8

Find the difference: $-\frac{19}{28}-\frac{7}{28}$.

Show answer
$-\frac{26}{28}$

Now we will do an example that has both addition and subtraction.

EXAMPLE 9

Simplify: $\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$.

Solution

Add and subtract fractions—do they have a common denominator? Yes.	$\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$
Add and subtract the numerators and place the difference over the common denominator.	$\frac{3+(-5)-1}{8}$
Simplify left to right.	$\frac{-2-1}{8}$
Simplify.	$-\frac{3}{8}$

TRY IT 9

Simplify: $\frac{2}{5}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Show answer
-1

Add or Subtract Fractions with Different Denominators

As we have seen, to add or subtract fractions, their denominators must be the same. The least common denominator (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Least Common Denominator

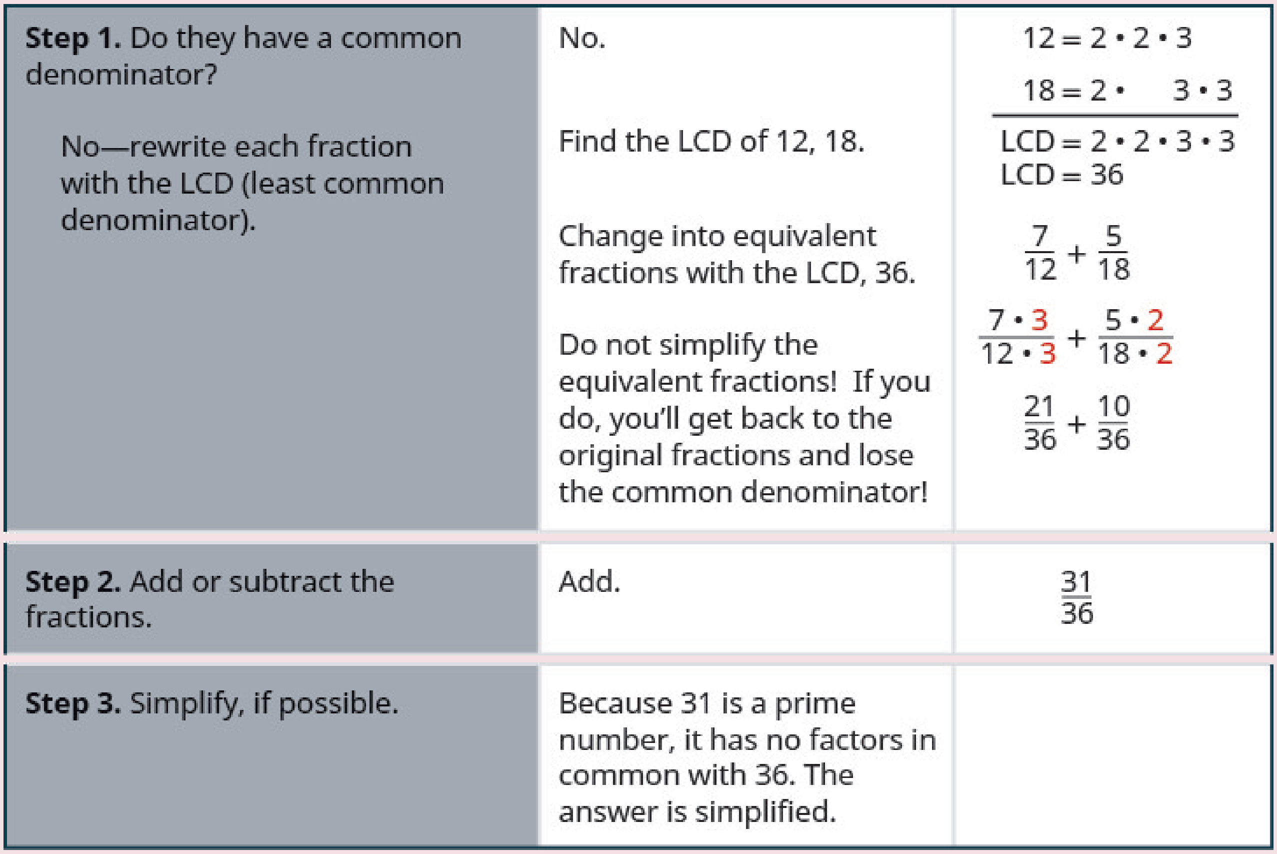
The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

EXAMPLE 10

Add: $\frac{7}{12}+\frac{5}{18}$.

Solution



Step 1. Do they have a common denominator?	No.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
No—rewrite each fraction with the LCD (least common denominator).	Find the LCD of 12, 18.	$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$
	Change into equivalent fractions with the LCD, 36.	$\frac{7}{12} + \frac{5}{18}$
	Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.	

TRY IT 10

Add: $\frac{7}{12} + \frac{11}{15}$.

Show answer

$\frac{79}{60}$

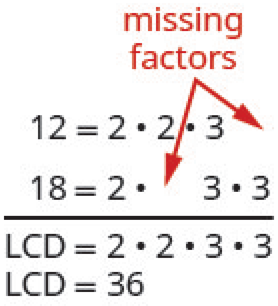
HOW TO: Add or Subtract Fractions

1. Do they have a common denominator?
    - Yes—go to step 2.
    - No—rewrite each fraction with the LCD (least common denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
2. Add or subtract the fractions.

3. Simplify, if possible.

When finding the equivalent fractions needed to create the common denominators, there is a quick way to find the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The “missing” factors of each denominator are the numbers we need.



In (Example 10), the LCD, 36, has two factors of 2 and two factors of 3.

The numerator 12 has two factors of 2 but only one of 3—so it is “missing” one 3—we multiply the numerator and denominator by 3

The numerator 18 is missing one factor of 2—so we multiply the numerator and denominator by 2

We will apply this method as we subtract the fractions in the next example.

EXAMPLE 11

Subtract: $\frac{7}{15} - \frac{19}{24}$.

Solution

Do the fractions have a common denominator? No, so we need to find the LCD.

Find the LCD. $\frac{7}{15} - \frac{19}{24}$ $15 = 3 \cdot 5$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ $\text{LCD} = 120$	
Notice, 15 is “missing” three factors of 2 and 24 is “missing” the 5 from the factors of the LCD. So we multiply 8 in the first fraction and 5 in the second fraction to get the LCD.	
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 8}{15 \cdot 8} - \frac{19 \cdot 5}{24 \cdot 5}$
Simplify.	$\frac{56}{120} - \frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Check to see if the answer can be simplified.	$-\frac{13 \cdot 3}{40 \cdot 3}$
Both 39 and 120 have a factor of 3.	
Simplify.	$-\frac{13}{40}$

Do not simplify the equivalent fractions! If you do, you’ll get back to the original fractions and lose the common denominator!

TRY IT 11

Subtract: $\frac{13}{24} - \frac{17}{32}$.

Show answer

$\frac{1}{96}$

We now have all four operations for fractions. The table below summarizes fraction operations.

Summary of Fraction Operations

Fraction Operation	Sample Equation	What to Do
Fraction multiplication	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	Multiply the numerators and multiply the denominators
Fraction division	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	Multiply the first fraction by the reciprocal of the second.
Fraction addition	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	Add the numerators and place the sum over the common denominator.
Fraction subtraction	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$	Subtract the numerators and place the difference over the common denominator.

To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.

Use the Order of Operations to Simplify Complex Fractions

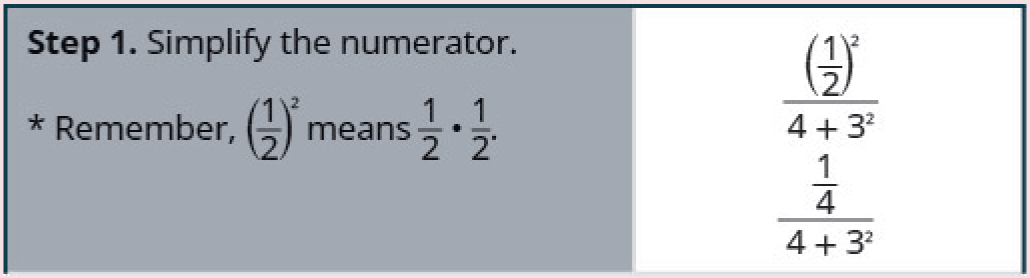
We have seen that a complex fraction is a fraction in which the numerator or denominator contains a fraction. The fraction bar indicates division. We simplified the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ by dividing $\frac{3}{4}$ by $\frac{5}{8}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator.

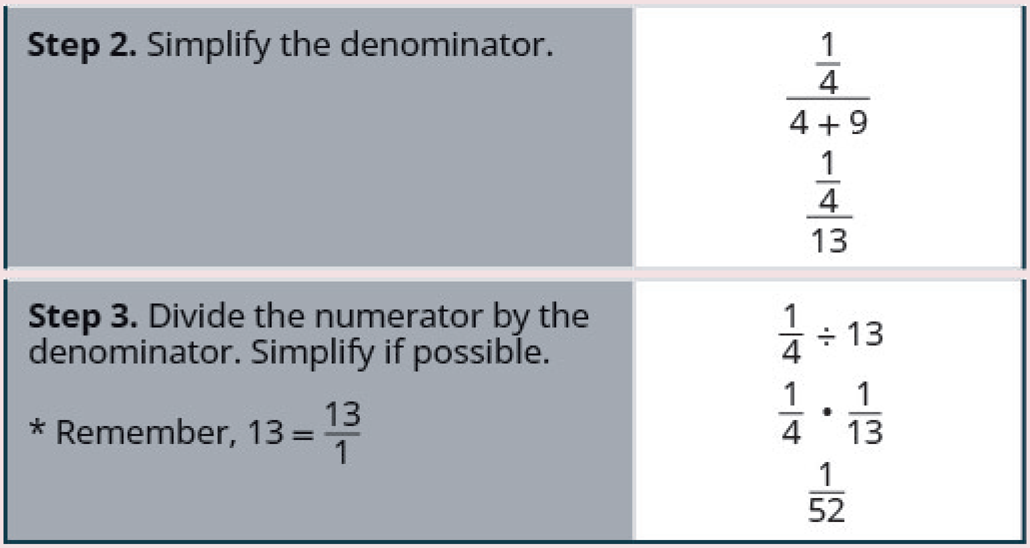
EXAMPLE 12

Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution



Step 1. Simplify the numerator. * Remember, $\left(\frac{1}{2}\right)^2$ means $\frac{1}{2} \cdot \frac{1}{2}$.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$



Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible. * Remember, $13 = \frac{13}{1}$	$\frac{1}{4} \div 13$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

TRY IT 12

Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Show answer

$\frac{1}{90}$

HOW TO: Simplify Complex Fractions

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator. Simplify if possible.

EXAMPLE 13

Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution

It may help to put parentheses around the numerator and the denominator.

	$\frac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator (LCD = 6) and simplify the denominator (LCD = 12).	$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\frac{7 \cdot 6 \cdot 2}{6 \cdot 7}$
Simplify.	2

TRY IT 13

Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Show answer
2

Key Concepts

- **Fraction Division:** If a, b, c and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. To divide fractions, multiply the first fraction by the reciprocal of the second.
- **Fraction Multiplication:** If a, b, c and d are numbers where $b \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. To multiply fractions, multiply the numerators and multiply the denominators.
- **Placement of Negative Sign in a Fraction:** For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Fraction Addition and Subtraction:** If $a, b,$ and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **Strategy for Adding or Subtracting Fractions**

1. Do they have a common denominator?
   Yes—go to step 2.
   No—Rewrite each fraction with the LCD (Least Common Denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
2. Add or subtract the fractions.
3. Simplify, if possible. To multiply or divide fractions, an LCD IS NOT needed. To add or subtract fractions, an LCD IS needed.

- **Simplify Complex Fractions**
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator. Simplify if possible.

Glossary

least common denominator
The least common denominator (LCD) of two fractions is the Least common multiple (LCM) of their denominators.

1.3 Exercise Set

In the following exercises, multiply.

1. $\frac{3}{4} \cdot \frac{9}{10}$
2. $-\frac{2}{3} \cdot \left(-\frac{3}{8}\right)$
3. $-\frac{5}{9} \cdot \frac{3}{10}$
4. $\left(-\frac{14}{15}\right) \cdot \left(\frac{9}{20}\right)$
5. $\left(-\frac{63}{84}\right) \cdot \left(-\frac{44}{90}\right)$
6. $4 \cdot \frac{5}{11}$
7. $-8 \cdot \left(\frac{17}{4}\right)$

In the following exercises, divide.

8. $\frac{3}{4} \div \frac{2}{3}$
9. $-\frac{7}{9} \div \left(-\frac{7}{4}\right)$
10. $\frac{5}{18} \div \left(-\frac{15}{24}\right)$
11. $-5 \div \frac{1}{2}$
12. $\frac{3}{4} \div (-12)$

In the following exercises, simplify.

13. $\frac{-\frac{8}{21}}{\frac{12}{35}}$
14. $\frac{-\frac{4}{5}}{2}$

15. $\frac{22+3}{10}$
16. $\frac{48}{24-15}$
17. $\frac{-6+6}{8+4}$
18. $\frac{4\cdot3}{6\cdot6}$
19. $\frac{4^2-1}{25}$
20. $\frac{8\cdot3+2\cdot9}{14+3}$
21. $\frac{5\cdot6-3\cdot4}{4\cdot5-2\cdot3}$
22. $\frac{5^2-3^2}{3-5}$
23. $\frac{7\cdot4-2(8-5)}{9\cdot3-3\cdot5}$
24. $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$

In the following exercises, add.

25. $\frac{6}{13}+\frac{5}{13}$
26. $-\frac{3}{16}+\left(-\frac{7}{16}\right)$
27. $-\frac{8}{17}+\frac{15}{17}$
28. $\frac{6}{13}+\left(-\frac{10}{13}\right)+\left(-\frac{12}{13}\right)$

In the following exercises, subtract.

29. $\frac{11}{15}-\frac{7}{15}$
30. $\frac{11}{12}-\frac{5}{12}$
31. $\frac{19}{21}-\frac{4}{21}$
32. $-\frac{3}{5}-\left(-\frac{4}{5}\right)$
33. $-\frac{7}{9}-\left(-\frac{5}{9}\right)$

In the following exercises, add or subtract.

34. $\frac{1}{2}+\frac{1}{7}$
35. $\frac{1}{3}-\left(-\frac{1}{9}\right)$
36. $\frac{7}{12}+\frac{5}{8}$
37. $\frac{7}{12}-\frac{9}{16}$
38. $\frac{2}{3}-\frac{3}{8}$
39. $-\frac{11}{30}+\frac{27}{40}$
40. $-\frac{13}{30}+\frac{25}{42}$
41. $-\frac{39}{56}-\frac{22}{35}$
42. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$
43. $1+\frac{7}{8}$

In the following exercises, simplify.

44. $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$
45. $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$
46. $\frac{2}{\frac{1}{3}+\frac{1}{5}}$
47. $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$
48. $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}1-\frac{3}{5}\div\frac{1}{10}$
49. $1-\frac{3}{5}\div\frac{1}{10}$
50. $\frac{2}{3}+\frac{1}{6}+\frac{3}{4}$
51. $\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$
52. $12\left(\frac{9}{20}-\frac{4}{15}\right)$
53. $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$
54. $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$

55. **Decorating.** Kayla is making covers for the throw pillows on her sofa. For each pillow cover, she needs $\frac{1}{2}$ yard of print fabric and $\frac{3}{8}$ yard of solid fabric. What is the total amount of fabric Kayla needs for each pillow cover?

56. **Baking.** A recipe for chocolate chip cookies calls for $\frac{3}{4}$ cup brown sugar. Leona wants to double the recipe. a) How much brown sugar will Leona need? Show your calculation. b) Measuring cups usually come in sets of $\frac{1}{4}, \frac{1}{3}, \frac{1}{2},$ and 1 cup. Draw a diagram to show two different ways that Leona could measure the brown sugar needed to double the cookie recipe.
57. **Portions.** Regin purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?

Answers:

1. $\frac{27}{40}$
2. $\frac{1}{4}$
3. $-\frac{1}{6}$
4. $-\frac{21}{50}$
5. $\frac{11}{30}$
6. $\frac{20}{11}$
7. -34
8. $\frac{9}{8}$
9. $\frac{4}{9}.$
10. $-\frac{4}{9}$
11. -10
12. $-\frac{1}{16}$
13. $-\frac{10}{9}$
14. $-\frac{2}{5}$
15. $\frac{5}{2}$
16. $\frac{16}{3}$
17. 0
18. $\frac{1}{3}$
19. $\frac{3}{5}$
20. $2\frac{8}{17}$
21. $1\frac{2}{7}$
22. -8
23. $\frac{11}{6}$
24. $\frac{5}{2}$
25. $\frac{11}{13}$
26. $-\frac{5}{8}$
27. $\frac{7}{17}$
28. $-\frac{16}{13}$
29. $\frac{4}{15}$
30. $\frac{1}{2}$
31. $\frac{5}{7}$
32. $\frac{1}{5}$
33. $-\frac{2}{9}$
34. $\frac{9}{14}$
35. $\frac{4}{9}$
36. $\frac{29}{24}$
37. $\frac{1}{48}$
38. $\frac{7}{24}$
39. $\frac{37}{120}$
40. $\frac{17}{105}$
41. $-\frac{53}{40}$
42. $\frac{1}{12}$
43. $\frac{15}{8}$
44. $\frac{4x+3}{12}$
45. $\frac{49}{25}$
46. $\frac{15}{4}$
47. $\frac{5}{21}$
48. $\frac{7}{9}$

49. -5
50. $\frac{19}{12}$
51. $\frac{23}{24}$
52. $\frac{11}{5}$
53. 1
54. $\frac{13}{3}$
55. $\frac{7}{8}$ yard
56. a) $1\frac{1}{2}$ cups b) answers will vary
57. 20 bags

Attributions

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1.4 Decimals

Learning Objectives

By the end of this section it is expected that you will be able to:

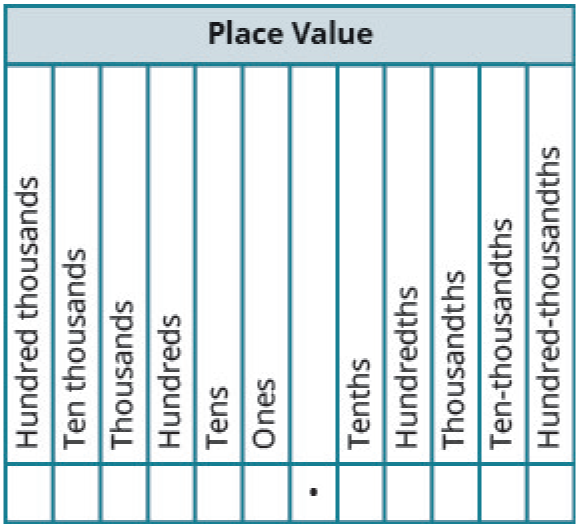
- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percent

Decimals are another way of writing fractions whose denominators are powers of 10.

$$\begin{aligned} 0.1 &= \frac{1}{10} && 0.1 \text{ is "one tenth"} \\ 0.01 &= \frac{1}{100} && 0.01 \text{ is "one hundredth"} \\ 0.001 &= \frac{1}{1{,}000} && 0.001 \text{ is "one thousandth"} \\ 0.0001 &= \frac{1}{10{,}000} && 0.0001 \text{ is "one ten-thousandth"} \end{aligned}$$

Notice that “ten thousand” is a number larger than one, but “one ten-thousand**th**” is a number smaller than one. The “th” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the decimal places correspond to their fraction values. Figure 1 shows the names of the place values to the left and right of the decimal point.
Place value of decimal numbers are shown to the left and right of the decimal point.



Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

Figure 1

Round Decimals

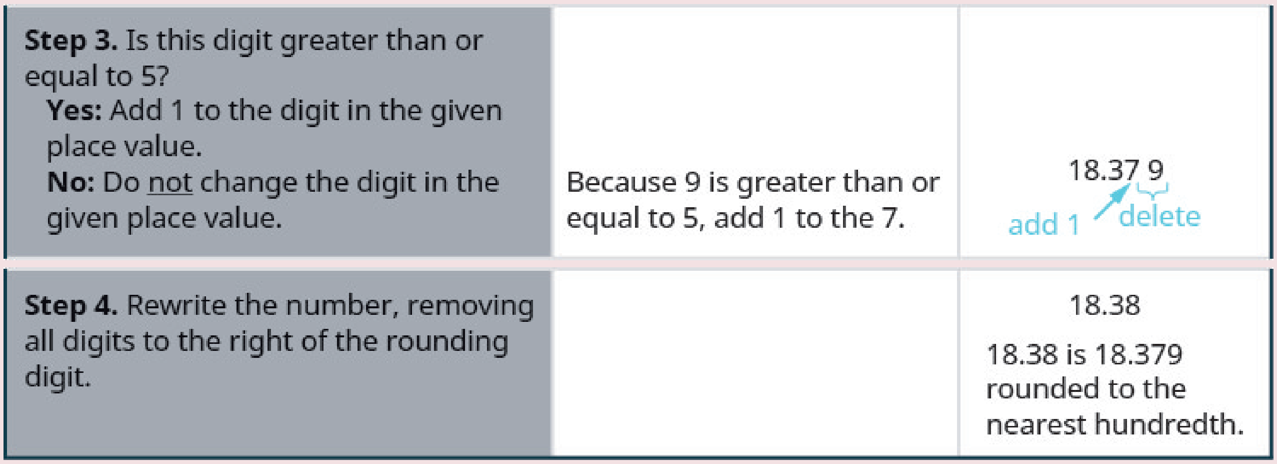
Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

EXAMPLE 1

Round 18.379 to the nearest hundredth.

Solution

Step 1. Locate the given place value and mark it with an arrow.		
Step 2. Underline the digit to the right of the given place value.		



Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do not change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	$18.3\underset{\text{add 1}}{7}\,\underbrace{9}_{\text{delete}}$
Step 4. Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

TRY IT 1

Round to the nearest hundredth: 1.047.

Show answer
1.05

We summarize the steps for rounding a decimal here.

HOW TO: Round Decimals

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
   - Yes—add 1 to the digit in the given place value.
   - No—do not change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

EXAMPLE 2

Round 18.379 to the nearest a) tenth b) whole number.

Solution

Round 18.379

a) to the nearest tenth

Locate the tenths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

b) to the nearest whole number

Locate the ones place with an arrow.	
Underline the digit to the right of the given place value.	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

TRY IT 2

Round 6.582 to the nearest a) hundredth b) tenth c) whole number.

Show answer
a) 6.58 b) 6.6 c) 7

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

HOW TO: Add or Subtract Decimals

1. Write the numbers so the decimal points line up vertically.
2. Use zeros as place holders, as needed.
3. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

EXAMPLE 3

Add: $23.5 + 41.38$.

Solution

Write the numbers so the decimal points line up vertically.	$\begin{array}{r} 23.5 \\ \underline{+41.38} \end{array}$
Put 0 as a placeholder after the 5 in 23.5. Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.	$\begin{array}{r} 23.50 \\ \underline{+41.38} \end{array}$
Add the numbers as if they were whole numbers. Then place the decimal point in the sum.	$\begin{array}{r} 23.50 \\ \underline{+41.38} \\ 64.88 \end{array}$

TRY IT 3

Add: $4.8 + 11.69$.

Show answer
16.49

EXAMPLE 4

Subtract: $20 - 14.65$.

Solution

	$20 - 14.65$
Write the numbers so the decimal points line up vertically. Remember, 20 is a whole number, so place the decimal point after the 0.	$\begin{array}{r} 20. \\ -14.65 \\ \hline \end{array}$
Put in zeros to the right as placeholders.	$\begin{array}{r} 20.00 \\ \underline{-14.65} \end{array}$
Subtract and place the decimal point in the answer.	$\begin{array}{r} \overset{1}{\cancel{2}}\overset{\overset{9}{\cancel{10}}}{\cancel{0}}\,.\,\overset{\overset{9}{\cancel{10}}}{\cancel{0}}\overset{10}{\cancel{0}} \\ -1\,4\,.\,6\,5 \\ \hline 5\,.\,3\,5 \end{array}$

TRY IT 4

Subtract: $10 - 9.58$.

Show answer
0.42

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let’s see what we would get as the product of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

	$\underbrace{(0.3)}_{\text{1 place}} \quad \underbrace{(0.7)}_{\text{1 place}} \qquad\qquad \underbrace{(0.2)}_{\text{1 place}} \quad \underbrace{(0.46)}_{\text{2 places}}$
Convert to fractions.	$\frac{3}{10} \cdot \frac{7}{10} \qquad\qquad \frac{2}{10} \cdot \frac{46}{100}$
Multiply.	$\frac{21}{100} \qquad\qquad \frac{92}{1000}$
Convert to decimals.	$0.\underbrace{21}_{\text{2 places}} \qquad\qquad 0.\underbrace{092}_{\text{3 places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

HOW TO: Multiply Decimals

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

EXAMPLE 5

Multiply: $(-3.9)\,(4.075)$.

Solution

	(−3.9)(4.075)
The signs are different. The product will be negative.	
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline \end{array}$
Multiply.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Add the number of decimal places in the factors (1 + 3). $\underbrace{(-3.9)}_{\text{1 place}} \quad \underbrace{(4.075)}_{\text{3 places}}$ Place the decimal point 4 places from the right.	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 15.\underbrace{8925}_{\text{4 places}} \end{array}$
The signs are different, so the product is negative.	(−3.9)(4.075) = −15.8925

TRY IT 5

Multiply: $-4.5\,(6.107)$.

Show answer

-27.4815

In many of your other classes, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc.). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

HOW TO: Multiply a Decimal by a Power of Ten

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

EXAMPLE 6

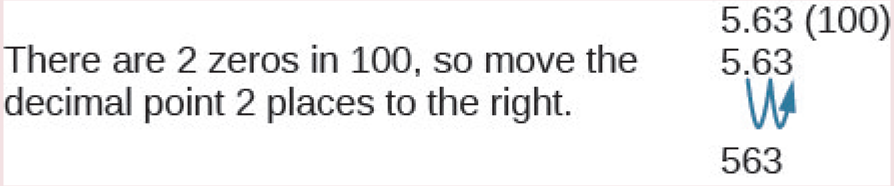
Multiply 5.63 a) by 10 b) by 100 c) by 1,000.

Solution
By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

a)

	5.63(10)
There is 1 zero in 10, so move the decimal point 1 place to the right.	

b)

	5.63(100)
There are 2 zeros in 100, so move the decimal point 2 places to the right.	 There are 2 zeros in 100, so move the decimal point 2 places to the right. 5.63 (100) 5.63 563

c)

	5.63(1,000)
There are 3 zeros in 1,000, so move the decimal point 3 places to the right.	
A zero must be added at the end.	5,630

TRY IT 6

Multiply 2.58 a) by 10 b) by 100 c) by 1,000.

Show answer
a) 25.8 b) 258 c) 2,580

Just as with multiplication, division of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10. Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:

$$\frac{0.8}{0.4}$$
$$\frac{0.8(10)}{0.4(10)}$$
$$\frac{8}{4}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{\text{c}} \qquad\qquad \underset{\text{divisor}}{b}\ \overset{\overset{\text{c}}{\text{quotient}}}{\overline{\big)\ \underset{\text{dividend}}{a}}}$$

We'll write the steps to take when dividing decimals, for easy reference.

HOW TO: Divide Decimals

1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

EXAMPLE 7

Divide: $-25.56 \div (-0.06)$.

Solution

Remember, you can “move” the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by “moving” the decimal point all the way to the right.	
“Move” the decimal point in the dividend the same number of places.	$0.\underrightarrow{06}\overline{)25.\underrightarrow{65}}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 427.5 \\ 006.\overline{)2565.0} \\ \underline{-24} \\ 16 \\ \underline{-12} \\ 45 \\ \underline{-42} \\ 30 \\ \underline{30} \end{array}$
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

TRY IT 7

Divide: $-23.492 \div (-0.04)$.

Show answer

687.3

A common application of dividing whole numbers into decimals is when we want to find the price of one item that is sold as part of a multi-pack. For example, suppose a case of 24 water bottles costs \$3.99. To find the price of one water bottle, we would divide \$3.99 by 24. We show this division in the next example. In calculations with money, we will round the answer to the nearest cent (hundredth).

EXAMPLE 8

Divide: $\$3.99 \div 24$.

Solution

	$\$3.99 \div 24$
Place the decimal point in the quotient above the decimal point in the dividend.	
Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth.) To do this, we must carry the division to the thousandths place.	$\begin{array}{r} 0.166 \\ 24\overline{)3.990} \\ \underline{24} \\ 159 \\ \underline{144} \\ 150 \\ \underline{144} \\ 6 \end{array}$
Round to the nearest cent.	$\$0.166 \approx \$0.17$ $\$3.99 \div 24 \approx \$0.17$

TRY IT 8

Divide: $\$6.99 \div 36$.

Show answer

$0.19

Convert Decimals and Fractions

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03

$$0\,0.03 = \frac{3}{100}$$

Notice, when the number to the left of the decimal is zero, we get a fraction whose numerator is less than its denominator. Fractions like this are called proper fractions.

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

HOW TO: Covert a Decimal to a Proper Fraction

1. Determine the place value of the final digit.
2. Write the fraction.
    - numerator—the “numbers” to the right of the decimal point
    - denominator—the place value corresponding to the final digit

EXAMPLE 12

Write 0.374 as a fraction.

Solution

	0.374
Determine the place value of the final digit.	
Write the fraction for 0.374: • The numerator is 374. • The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$ so, $0.374 = \frac{187}{500}$

Did you notice that the number of zeros in the denominator of $\frac{374}{1,000}$ is the same as the number of decimal places in 0.374?

TRY IT 12

Write 0.234 as a fraction.

Show answer

$\frac{117}{500}$

We've learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar means division. So $\frac{4}{5}$ can be written $4 \div 5$ or $5\overline{)4}$. This leads to the following method for converting a fraction to a decimal.

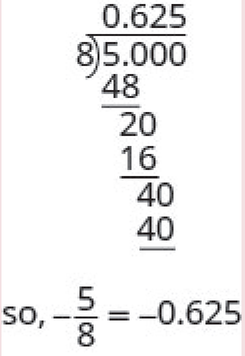
HOW TO: Covert a Fraction to a Decimal

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

EXAMPLE 13

Write $-\frac{5}{8}$ as a decimal.

Solution
Since a fraction bar means division, we begin by writing $\frac{5}{8}$ as $8\overline{)5}$. Now divide.



so, $-\frac{5}{8} = -0.625$

TRY IT 13

Write $-\frac{7}{8}$ as a decimal.

Show answer
-0.875

When we divide, we will not always get a zero remainder. Sometimes the quotient ends up with a decimal that repeats. A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly. A bar is placed over the repeating block of digits to indicate it repeats.

Repeating Decimal

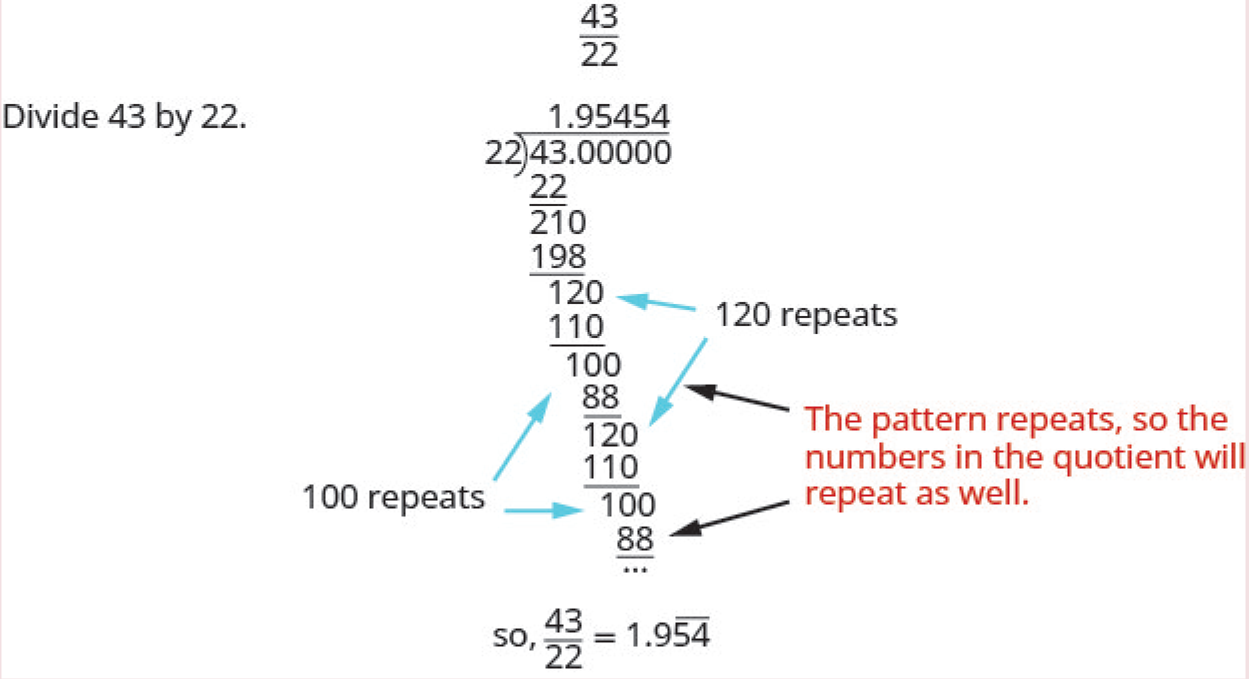
A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

A bar is placed over the repeating block of digits to indicate it repeats.

EXAMPLE 14

Write $\frac{43}{22}$ as a decimal.

Solution



$\frac{43}{22}$

Divide 43 by 22.

The pattern repeats, so the numbers in the quotient will repeat as well.

so, $\frac{43}{22} = 1.9\overline{54}$

TRY IT 14

Write $\frac{27}{11}$ as a decimal.

Show answer

$2.\ 45$

Sometimes we may have to simplify expressions with fractions and decimals together.

EXAMPLE 15

Simplify: $\frac{7}{8} + 6.4$.

Solution
First we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Usually it is easier to change the fraction to a decimal.

		$\frac{7}{8} + 6.4$
Change $\frac{7}{8}$ to a decimal.	$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$	
Add.		$0.875 + 6.4$
		7.275
		So, $\frac{7}{8} + 6.4 = 7.275$

TRY IT 15

Simplify: $\frac{3}{8} + 4.9$.

Show answer
5.275

Key Concepts

- **Round a Decimal**
    1. Locate the given place value and mark it with an arrow.
    2. Underline the digit to the right of the place value.
    3. Is this digit greater than or equal to 5? Yes—add 1 to the digit in the given place value. No—do not change the digit in the given place value.

4. Rewrite the number, deleting all digits to the right of the rounding digit.

- **Add or Subtract Decimals**
    1. Write the numbers so the decimal points line up vertically.
    2. Use zeros as place holders, as needed.
    3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.
- **Multiply Decimals**
    1. Determine the sign of the product.
    2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
    3. Place the decimal point. The number of decimal places in the product is the sum of the decimal places in the factors.
    4. Write the product with the appropriate sign.
- **Multiply a Decimal by a Power of Ten**
    1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
    2. Add zeros at the end of the number as needed.
- **Divide Decimals**
    1. Determine the sign of the quotient.
    2. Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places – adding zeros as needed.
    3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
    4. Write the quotient with the appropriate sign.
- **Convert a Decimal to a Proper Fraction**
    1. Determine the place value of the final digit.
    2. Write the fraction: numerator—the ‘numbers’ to the right of the decimal point; denominator—the place value corresponding to the final digit.
- **Convert a Fraction to a Decimal** Divide the numerator of the fraction by the denominator.

1.4 Exercise Set

In the following exercises, round each number to the nearest tenth.

1. 0.67
2. 2.84

In the following exercises, round each number to the nearest hundredth.

3. 0.845
4. 0.299
5. 4.098

In the following exercises, round each number to the nearest a) hundredth b) tenth c) whole number.

6. 5.781
7. 63.479

In the following exercises, add or subtract.

8. $16.92 + 7.56$
9. $21.76 - 30.99$
10. $-16.53 - 24.38$
11. $-38.69 + 31.47$
12. $72.5 - 100$
13. $15 + 0.73$
14. $91.95 - (-10.462)$
15. $55.01 - 3.7$
16. $2.51 - 7.4$

In the following exercises, multiply.

17. $(0.24)(0.6)$
18. $(5.9)(7.12)$
19. $(-4.3)(2.71)$
20. $(-5.18)(-65.23)$
21. $(0.06)(21.75)$
22. $(9.24)(10)$
23. $(55.2)(1000)$

In the following exercises, divide.

24. $4.75 \div 25$
25. $\$117.25 \div 48$
26. $0.6 \div 0.2$
27. $1.44 \div (-0.3)$
28. $-1.75 \div (-0.05)$
29. $5.2 \div 2.5$
30. $11 \div 0.55$

In the following exercises, write each decimal as a fraction.

31. 0.04
32. 0.52
33. 1.25
34. 0.375
35. 0.095

In the following exercises, convert each fraction to a decimal.

36. $\frac{17}{20}$
37. $\frac{11}{4}$

38. $-\frac{310}{25}$
39. $\frac{15}{11}$
40. $\frac{15}{111}$
41. $2.4+\frac{5}{8}$

42. **Salary Increase.** Marta got a raise and now makes $58,965.95 a year. Round this number to the nearest
a) dollar
b) thousand dollars
c) ten thousand dollars.

43. **Sales Tax.** Hyo Jin lives in Vancouver. She bought a refrigerator for $1,624.99 and when the clerk calculated the sales tax it came out to exactly $142.186625. Round the sales tax to the nearest
a) penny and
b) dollar.

44. **Paycheck.** Annie has two jobs. She gets paid $14.04 per hour for tutoring at Community College and $8.75 per hour at a coffee shop. Last week she tutored for 8 hours and worked at the coffee shop for 15 hours.
a) How much did she earn?
b) If she had worked all 23 hours as a tutor instead of working both jobs, how much more would she have earned?

Glossary

decimal
A decimal is another way of writing a fraction whose denominator is a power of ten.
repeating decimal
A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly.

Answers:

1. 0.7
2. 2.8
3. 0.85
4. 0.30
5. 4.10
6. a) 5.78 b) 5.8 c) 6
7. a) 63.48 b) 63.5 c) 63
8. 24.48
9. -9.23
10. -40.91
11. -7.22
12. -27.5
13. 15.73
14. 102.212
15. 51.31
16. -4.89
17. 0.144
18. 42.008
19. -11.653
20. 337.8914
21. 1.305
22. 92.4
23. 55,200
24. 0.19

25. $2.44
26. 3
27. -4.8
28. 35
29. 2.08
30. 20
31. $\frac{1}{25}$
32. $\frac{13}{25}$
33. $\frac{5}{4}$
34. $\frac{3}{8}$
35. $\frac{19}{200}$
36. 0.85
37. 2.75
38. -12.4
39. $1.\ 36$
40. $0.\ 135$
41. 3.025
42. a) $58,966 b) $59,000 c) $60,000
43. a) $142.19; b) $142
44. a) $243.57 b) $79.35

Attributions

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1.5. Exponents and Scientific Notation

Learning Objectives

By the end of this section it is expected that you will be able to:

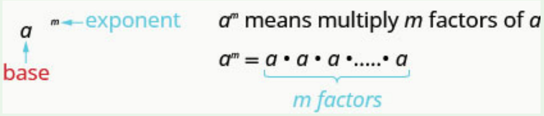
- Simplify expressions with exponents
- Simplify expressions with zero exponents
- Use the definition of a negative exponent
- Use formulas with exponents in applications
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means 2 · 2 · 2 · 2

Let's review the vocabulary for expressions with exponents.

Exponential Notation (Power)



This is read a to the m^{th} power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.

$$4^3 \qquad\qquad (-9)^5$$

$$\underbrace{4 \cdot 4 \cdot 4}_{\text{3 factors}} \qquad\qquad \underbrace{(-9)(-9)(-9)(-9)(-9)}_{\text{5 factors}}$$

Before we begin working with expressions containing exponents, let's simplify a few expressions involving only numbers.

EXAMPLE 1

Simplify: a) 4^3 b) 7^1 c) $\left(\frac{5}{6}\right)^2$ d) $(0.63)^2$.

Solution

a)	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64
b)	7^1
Multiply one factor of 7.	7
c)	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$
d)	$(0.63)^2$
Multiply two factors.	$(0.63)\,(0.63)$
Simplify.	0.3969

TRY IT 1

Simplify: a) 6^3 b) 15^1 c) $\left(\frac{3}{7}\right)^2$ d) $(0.43)^2$.

Show answer

a) 216 b) 15 c) $\frac{9}{49}$ d) 0.1849

EXAMPLE 2

Simplify: a) $(-5)^4$ b) -5^4.

Solution

a)	$(-5)^4$
Multiply four factors of -5.	$(-5)(-5)(-5)(-5)$
Simplify.	625
b)	-5^4
Multiply four factors of 5.	-(5 · 5 · 5 · 5)
Simplify.	-625

TRY IT 2

Simplify: a) $(-3)^4$ b) -3^4.

Show answer

a) 81 b) -81

Notice the similarities and differences in (Example 2) a) and (Example 2) b)! Why are the answers different? As we follow the order of operations in part a) the parentheses tell us to raise the (-5) to the 4th power. In part b) we raise just the 5 to the 4th power and then take the opposite.

When simplifying with exponents instead of multiplying the same factors, we can use scientific calculator and a key labelled $\boxed{y^x}$ **or** $\boxed{x^y}$.

For example, to find $(0.7)^5$, press: 0.7 $\boxed{y^x}$ 5 $\boxed{=}$ .You should get 0.16807.

Simplify Expressions with an Exponent of Zero

When simplifying expressions with exponents we very often use the **Product Property and the Quotient Property.**

Product Property for Exponents

If a is a real number, and m and n are counting numbers, then
$a^m \cdot a^n = a^{m+n}$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} 2^2 \cdot 2^3 &\overset{?}{=} 2^{2+3} \\ 4 \cdot 8 &\overset{?}{=} 2^5 \\ 32 &= 32\checkmark \end{aligned}$$

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then
$\dfrac{a^m}{a^n} = a^{m-n}, m > n$ and $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}, n > m$

A couple of examples with numbers may help to verify this property.

$$\begin{aligned} \frac{3^4}{3^2} &= 3^{4-2} & \frac{5^2}{5^3} &= \frac{1}{5^{3-2}} \\ \frac{81}{9} &= 3^2 & \frac{25}{125} &= \frac{1}{5^1} \\ 9 &= 9\checkmark & \frac{1}{5} &= \frac{1}{5}\checkmark \end{aligned}$$

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

$$\frac{2}{2}=1 \qquad \frac{17}{17}=1 \qquad \frac{-43}{-43}=1$$

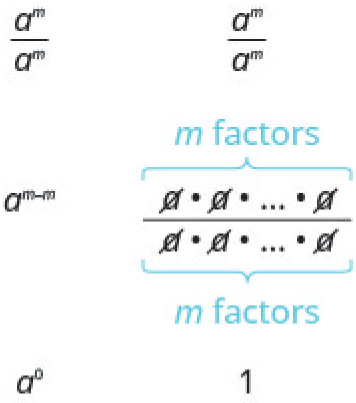
In words, a number divided by itself is 1. So, $\frac{x}{x}=1$, for any x $(x \neq 0)$, since any number divided by itself is 1

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1

	$\frac{8}{8}=1$
Write 8 as 2^3.	$\frac{2^3}{2^3}=1$
Subtract exponents.	$2^{3-3}=1$
Simplify.	$2^0=1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:



We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Zero Exponent

If a is a non-zero number, then $a^0 = 1$.

Any nonzero number raised to the zero power is 1

EXAMPLE 3

Simplify: -9^0.

Solution

The definition says any non-zero number raised to the zero power is 1

Use the definition of the zero exponent.	9^0 1

TRY IT 3

Simplify: -1.5^0

Show answer

1

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

Use the Definition of a Negative Exponent

Now, let’s use the definition of a *negative exponent* to simplify expressions.

Negative Exponent

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

EXAMPLE 4

Simplify: a) 4^{-2} b) 10^{-3}.

Solution

a)	4^{-2}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$
b)	10^{-3}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

TRY IT 4

Simplify: a) 2^{-3} b) 10^{-7}.

Show answer

a) $\frac{1}{8}$ b) $\frac{1}{10^7}$

In (Example 4) we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^n}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Property of Negative Exponents

If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$.

EXAMPLE 5

Simplify: $\frac{1}{3^{-2}}$.

Solution

	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

TRY IT 5

Simplify: $\frac{1}{4^{-3}}$.

Show answer

64

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.	
This tells us that:	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0, b \neq 0$, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

EXAMPLE 6

Simplify: $\left(\frac{5}{7}\right)^{-2}$.

Solution

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^2$
Simplify.	$\frac{49}{25}$

TRY IT 6

Simplify: $\left(\frac{2}{3}\right)^{-4}$.

Show answer
$\frac{81}{16}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

EXAMPLE 7

Simplify: a) $(-3)^{-2}$ b) -3^{-2} c) $\left(-\frac{1}{3}\right)^{-2}$ d) $-\left(\frac{1}{3}\right)^{-2}$.

Solution

a) Here the exponent applies to the base -3.	$(-3)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^{-2}}$
Simplify.	$\frac{1}{9}$
b) The expression -3^{-2} means "find the opposite of 3^{-2}." Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.	-3^{-2}
Rewrite as a product with -1.	$-1\cdot 3^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1\cdot\frac{1}{3^2}$
Simplify.	$-\frac{1}{9}$
c) Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9
d) The expression $-\left(\frac{1}{3}\right)^{-2}$ means "find the opposite of $\left(\frac{1}{3}\right)^{-2}$." Here the exponent applies to the base $\left(\frac{1}{3}\right)$.	
Rewrite as a product with -1.	$-1\cdot\left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1\cdot\left(\frac{3}{1}\right)^{2}$
Simplify.	-9

TRY IT 7

Simplify: a) $(-5)^{-2}$ b) -5^{-2} c) $\left(-\frac{1}{5}\right)^{-2}$ d) $-\left(\frac{1}{5}\right)^{-2}$.

Show answer
a) $\frac{1}{25}$ b) $-\frac{1}{25}$ c) 25 d) -25

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

EXAMPLE 8

Simplify: a) $4 \cdot 2^{-1}$ b) $(4 \cdot 2)^{-1}$.

Solution

a) Do exponents before multiplication.	$4 \cdot 2^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$4 \cdot \frac{1}{2^1}$
Simplify.	2
b)	$(4 \cdot 2)^{-1}$
Simplify inside the parentheses first.	$(8)^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

TRY IT 8

Simplify: a) $6 \cdot 3^{-1}$ b) $(6 \cdot 3)^{-1}$.

Show answer

a) 2 b) $\frac{1}{18}$

Use Formulas with Exponents in Applications

In this section, we will use geometry formulas that contain exponents to solve problems. Since we will be solving applications, we will once again show our Problem-Solving Strategy for Geometry Applications.

Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.

3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

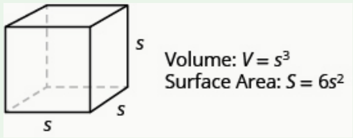
A cube is a rectangular solid whose length, width, and height are equal. See Volume and Surface Area of a Cube, below. Substituting, *s* for the length, width and height into the formulas for volume and surface area of a rectangular solid, we get:

$$\begin{array}{ll} V = LWH & S = 2LH + 2LW + 2WH \\ V = \mathrm{s} \cdot \mathrm{s} \cdot \mathrm{s} & S = 2\mathrm{s} \cdot \mathrm{s} + 2\mathrm{s} \cdot \mathrm{s} + 2\mathrm{s} \cdot \mathrm{s} \\ V = \mathrm{s}^3 & S = 2s^2 + 2s^2 + 2s^2 \\ & S = 6s^2 \end{array}$$

So for a cube, the formulas for volume and surface area are $V = s^3$ and $S = 6s^2$.

Volume and Surface Area of a Cube

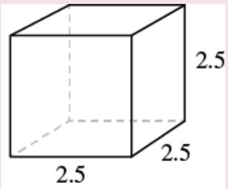
For any cube with sides of length s,



EXAMPLE 3

A cube is 2.5 inches on each side. Find its a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cube
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula.	$V = s^3$
Step 5. **Solve.** Substitute and solve.	$V = (2.5)^3$ $V = 15.625$
Step 6. **Check:** Check your work.	
Step 7. **Answer** the question.	The volume is 15.625 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula.	$S = 6s^2$
Step 5. **Solve.** Substitute and solve.	$S = 6 \cdot (2.5)^2$ $S = 37.5$
Step 6. **Check:** The check is left to you.	
Step 7. **Answer** the question.	The surface area is 37.5 square inches.

TRY IT 3

For a cube with side 4.5 metres, find the a) volume and b) surface area of the cube.

Show answer

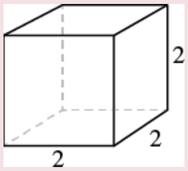
a. 91.125 cu. m

b. 121.5 sq. m

EXAMPLE 4

A notepad cube measures 2 inches on each side. Find its a) volume and b) surface area.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the cube
Step 3. **Name.** Choose a variable to represent it.	let *V* = volume
Step 4. **Translate.** Write the appropriate formula.	$V = s^3$
Step 5. **Solve** the equation.	$V = 2^3$ $V = 8$
Step 6. **Check:** Check that you did the calculations correctly.	
Step 7. **Answer** the question.	The volume is 8 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let *S* = surface area
Step 4. **Translate.** Write the appropriate formula.	$S = 6s^2$
Step 5. **Solve** the equation.	$S = 6 \cdot 2^2$ $S = 24$
Step 6. **Check:** The check is left to you.	
Step 7. **Answer** the question.	The surface area is 24 square inches.

TRY IT 4

A packing box is a cube measuring 4 feet on each side. Find its a) volume and b) surface area.

Show answer

a. 64 cu. ft
b. 96 sq. ft

A sphere is the shape of a basketball, like a three-dimensional circle. Just like a circle, the size of a sphere is determined by its radius, which is the distance from the center of the sphere to any point on its surface. The formulas for the volume and surface area of a sphere are given below.

Showing where these formulas come from, like we did for a rectangular solid, is beyond the scope of this course. We will approximate π with 3.14.

Volume and Surface Area of a Sphere

For a sphere with radius r:

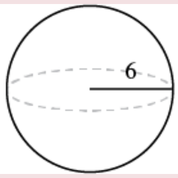
Volume: $V = \frac{4}{3}\pi r^3$

Surface Area: $S = 4\pi r^2$

EXAMPLE 5

A sphere has a radius 6 inches. Find its a) volume and b) surface area.

Solution
Step 1 is the same for both a) and b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the sphere
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula.	$V = \frac{4}{3}\pi r^3$
Step 5. **Solve.**	$V \approx \frac{4}{3}(3.14)\,6^3$ $V \approx 904.32 \text{ cubic inches}$
Step 6. **Check:** Double-check your math on a calculator.	
Step 7. **Answer** the question.	The volume is approximately 904.32 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the cube
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula.	$S = 4\pi r^2$
Step 5. **Solve.**	$S \approx 4(3.14)\,6^2$ $S \approx 452.16$
Step 6. **Check:** Double-check your math on a calculator	
Step 7. **Answer** the question.	The surface area is approximately 452.16 square inches.

TRY IT 5

Find the a) volume and b) surface area of a sphere with radius 3 centimetres.

Show answer

a. 113.04 cu. cm
b. 113.04 sq. cm

EXAMPLE 6

A globe of Earth is in the shape of a sphere with radius 14 centimetres. Find its a) volume and b) surface area. Round the answer to the nearest hundredth.

Solution

Step 1. **Read** the problem. Draw a figure with the given information and label it.	

a)	
Step 2. **Identify** what you are looking for.	the volume of the sphere
Step 3. **Name.** Choose a variable to represent it.	let V = volume
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \frac{4}{3}\pi r^3$ $V \approx \frac{4}{3}(3.14)\,14^3$
Step 5. **Solve.**	$V \approx 11,488.21$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume is approximately 11,488.21 cubic inches.

b)	
Step 2. **Identify** what you are looking for.	the surface area of the sphere
Step 3. **Name.** Choose a variable to represent it.	let S = surface area
Step 4. **Translate.** Write the appropriate formula. Substitute. (Use 3.14 for π)	$S = 4\pi r^2$ $S \approx 4\,(3.14)\,14^2$
Step 5. **Solve.**	$S \approx 2461.76$
Step 6. **Check:** We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is approximately 2461.76 square inches.

TRY IT 6

A beach ball is in the shape of a sphere with radius of 9 inches. Find its a) volume and b) surface area.

Show answer

a. 3052.08 cu. in.

b. 1017.36 sq. in.

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1,000$ and 0.004 means $4 \times \frac{1}{1,000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

$4,000$	0.004
$4 \times 1,000$	$4 \times \frac{1}{1,000}$
4×10^3	$4 \times \frac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

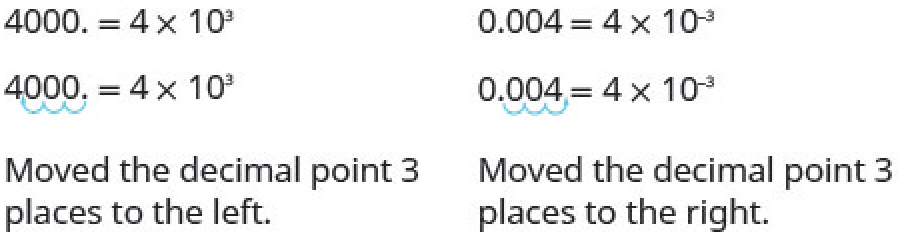
Scientific Notation

A number is expressed in scientific notation when it is of the form

$a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$

It is customary in scientific notation to use as the × multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.



$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10

The power of 10 is positive when the number is larger than 1: $4,000 = 4 \times 10^3$
The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

EXAMPLE 9

How to Convert from Decimal Notation to Scientific Notation

Write in scientific notation: 37,000.

Solution

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	 37000.
Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37,000 = 3.7 \times 10^4$

TRY IT 9

Write in scientific notation: $96,000$.

Show answer
9.6×10^4

HOW TO: Convert from decimal notation to scientific notation

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
2. Count the number of decimal places, n, that the decimal point was moved.
3. Write the number as a product with a power of 10.
   If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.
- Check.

EXAMPLE 10

Write in scientific notation: 0.0052.

Solution
The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10

	 0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	 0.0052
Count the number of decimal places the point was moved.	 3 places
Write as a product with a power of 10.	 5.2 x 10⁻³
Check.	
5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001	
0.0052	

TRY IT 10

Write in scientific notation: 0.0078.

Show answer

7.8×10^{-3}

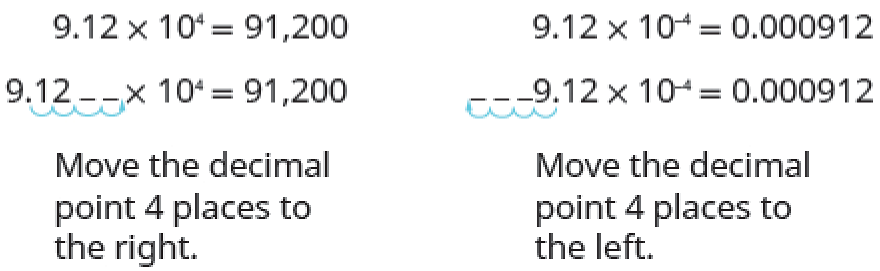
Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

$$\begin{array}{cc} 9.12 \times 10^4 & 9.12 \times 10^{-4} \\ 9.12 \times 10,000 & 9.12 \times 0.0001 \\ 91,200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$$9.12 \times 10^4 = 91,200 \qquad 9.12 \times 10^{-4} = 0.000912$$



In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

EXAMPLE 11

How to Convert Scientific Notation to Decimal Form

Convert to decimal form: 6.2×10^3.

Solution

Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^3
Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	6.200.  6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

TRY IT 11

Convert to decimal form: 1.3×10^3.

Show answer
1,300

The steps are summarized below.

HOW TO: Convert scientific notation to decimal form.

To convert scientific notation to decimal form:

1. Determine the exponent, n, on the factor 10.
2. Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.
    - If the exponent is negative, move the decimal point $|n|$ places to the left.
3. Check.

EXAMPLE 12

Convert to decimal form: 8.9×10^{-2}.

Solution

	8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is −2.
Since the exponent is negative, move the decimal point 2 places to the left.	 8 . 9
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

TRY IT 12

Convert to decimal form: 1.2×10^{-4}.

Show answer
0.00012

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

EXAMPLE 13

Multiply. Write answers in decimal form: $\left(4 \times 10^5\right)\left(2 \times 10^{-7}\right)$.

Solution

	$\left(4 \times 10^5\right)\left(2 \times 10^{-7}\right)$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

TRY IT 13

Multiply $\left(3 \times 10^6\right)\left(2 \times 10^{-8}\right)$. Write answers in decimal form.

Show answer
0.06

EXAMPLE 14

Multiply. Write answer in scientific notation: $\left(3.2 \times 10^{-3}\right)\left(6.3 \times 10^9\right)$.

Solution

	$\left(3.2 \times 10^{-3}\right)\left(6.3 \times 10^9\right)$
Use the Commutative Property to rearrange the factors.	$3.2 \cdot 6.3 \cdot 10^{-3} \cdot 10^9$
Multiply.	20.16×10^6
Write the answer in scientific notation.	2.016×10^7

TRY IT 14

Multiply. Write answer in scientific notation: $\left(2.7 \times 10^{-6}\right)\left(7.9 \times 10^{3}\right)$.

Show answer

\(2.133\times{10}^{-2}\)

EXAMPLE 15

Divide. Write answers in decimal form: $\frac{9 \times 10^{3}}{3 \times 10^{-2}}$.

Solution

	$\frac{9\times10^{3}}{3\times10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3} \times \frac{10^{3}}{10^{-2}}$
Divide.	3×10^{5}
Change to decimal form by moving the decimal five places right.	$300,000$

TRY IT 15

Divide $\frac{8 \times 10^{4}}{2 \times 10^{-1}}$. Write answers in decimal form.

Show answer

400,000

EXAMPLE 16

Divide. Write answer in scientific notation: $\frac{3.2 \times 10^{4}}{8 \times 10^{-5}}$.

Solution

	$\frac{3.2\times10^4}{8times10^{-5}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{3.2}{8}\times\frac{10^4}{10^{-5}}$
Divide.	0.4×10^3
Write answer in scientific notation	4×10^2

TRY IT 16

Divide $\frac{2.585\times10^5}{3.8\times10^{-1}}$. Write answer in scientific notation.

Show answer

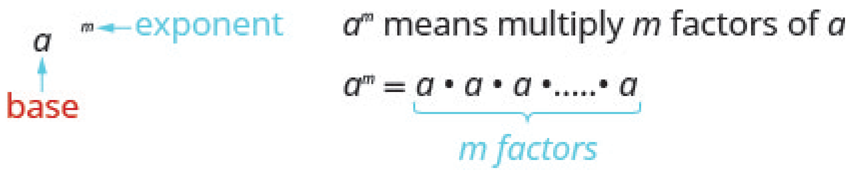
6.8×10^3

Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- Negative Exponents
- Scientific Notation
- Scientific Notation 2

Key Concepts

- **Exponential Notation**



- **Product Property of Exponents**
  - If a, b are real numbers and m, n are whole numbers, then $a^m \cdot a^n = a^{m+n}$
- **Quotient Property for Exponents:**
  - If a is a real number, $a \neq 0$, and m, n are whole numbers, then:

$$\frac{a^m}{a^n} = a^{m-n}, m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$$

- **Zero Exponent**
    - If a is a non-zero number, then $a^0 = 1$.
- **Property of Negative Exponents**
    - If n is a positive integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$
- **Quotient to a Negative Exponent**
    - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- **To convert a decimal to scientific notation:**
    1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
    2. Count the number of decimal places, n, that the decimal point was moved.
    3. Write the number as a product with a power of 10. If the original number is:
        - greater than 1, the power of 10 will be 10^n
        - between 0 and 1, the power of 10 will be 10^{-n}
    4. Check.
- **To convert scientific notation to decimal form:**
    1. Determine the exponent, n, on the factor 10.
    2. Move the decimal $n$places, adding zeros if needed.
        - If the exponent is positive, move the decimal point n places to the right.
        - If the exponent is negative, move the decimal point $|n|$ places to the left.
    3. Check

1.5 Exercise Set

In the following exercises, simplify each expression with exponents.

1.
    a. 3^5
    b. 9^1
    c. $\left(\frac{1}{3}\right)^2$
    d. $(0.2)^4$
2.
    a. 2^6
    b. 14^1
    c. $\left(\frac{2}{5}\right)^3$
    d. $(0.7)^2$
3.
    a. $(-6)^4$
    b. -6^4
4.
    a. $-\left(\frac{2}{3}\right)^2$

b. $\left(-\frac{2}{3}\right)^2$

5\.
a. -0.5^2
b. $(-0.5)^2$

In the following exercises, simplify.

6\.
a. 20^0
b. b^0

7\.
a. $(-27)^0$
b. $-\left(27^0\right)$

In the following exercises, simplify.

8\.
a. 3^{-4}
b. 10^{-2}

9\.
a. 2^{-8}
b. 10^{-2}

10\.
a. $\frac{1}{5^{-2}}$
b. $\frac{1}{10^{-4}}$
c. $\left(\frac{3}{10}\right)^{-2}$
d. $\left(\frac{7}{2}\right)^{-3}$

11\.
a. $(-7)^{-2}$
b. -7^{-2}
c. $\left(-\frac{1}{7}\right)^{-2}$
d. $-\left(\frac{1}{7}\right)^{-2}$

12\.
a. -5^{-3}
b. $\left(-\frac{1}{5}\right)^{-3}$
c. $-\left(\frac{1}{5}\right)^{-3}$
d. $(-5)^{-3}$

13\.
a. $2 \cdot 5^{-1}$
b. $(2 \cdot 5)^{-1}$

14\.
a. $3 \cdot 4^{-2}$
b. $(3 \cdot 4)^{-2}$

In the following exercises, find a) the volume and b) the surface area of the cube with the given side length.

18\. 5 centimetres

19\. 10.4 feet

In the following exercises, solve.

20\. **Museum** A cube-shaped museum has sides 64 metres long. Find its a) volume and b) surface area.

21\. **Base of statue** The base of a statue is a cube with sides 2.8 metres long. Find its a) volume and b) surface area.

In the following exercises, find a) the volume and b) the surface area of the sphere with the given radius. Round answers to the nearest hundredth.

22\. 3 centimetres

23\. 7.5 feet

In the following exercises, solve. Round answers to the nearest hundredth.

24\. **Exercise ball** An exercise ball has a radius of 15 inches. Find its a) volume and b) surface area.

25. **Golf ball** A golf ball has a radius of 4.5 centimetres. Find its a) volume and b) surface area.

In the following exercises, write each number in scientific notation.

26. 340,000
27. 1,290,000
28. 0.041
29. 0.00000103

In the following exercises, convert each number to decimal form.

30. 8.3×10^2
31. 1.6×10^{10}
32. 3.8×10^{-2}
33. 1.93×10^{-5}

In the following exercises, multiply. Write your answer in decimal form.

34. $\left(2 \times 10^2\right)\left(1 \times 10^{-4}\right)$
35. $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

In the following exercises, divide. Write your answer in decimal form.

36. $\frac{5 \times 10^{-2}}{1 \times 10^{-10}}$
37. $\frac{8 \times 10^6}{4 \times 10^{-1}}$
38. The population of the world on July 1, 2010 was more than 6,850,000,000. Write the number in scientific notation
39. The probability of winning the lottery was about 0.0000000057. Write the number in scientific notation.
40. The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.
41. **Coin production** In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.
42. **Debt** At the end of fiscal year 2019 the gross Canadian federal government debt was estimated to be approximately $685,450,000,000 ($685.45 billion), according to the Federal Budget. The population of Canada was approximately 37,590,000 people at the end of fiscal year 2019

    a) Write the debt in scientific notation.

    b) Write the population in scientific notation.

    c) Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Answers:

1. a. 243 b. 9

- c. $\frac{1}{9}$
- d. 0.0016

2.
- a. 64
- b. 14
- c. $\frac{8}{125}$
- d. 0.49

3.
- a. 1296
- b. -1296

4.
- a. $-\frac{4}{9}$
- b. $\frac{4}{9}$

5.
- a. -0.25
- b. 0.25

6.
- a. 1
- b. 1

7.
- a. 1
- b. -1

8.
- a. $\frac{1}{81}$
- b. $\frac{1}{100}$

9.
- a. $\frac{1}{256}$
- b. $\frac{1}{100}$

10. 25
11. 10000
12. $\frac{100}{9}$
13. $\frac{8}{343}$

14.
- a. $\frac{1}{49}$
- b. $-\frac{1}{49}$
- c. 49
- d. -49

15.
- a. $-\frac{1}{125}$
- b. -125
- c. -125
- d. $-\frac{1}{125}$

16.
- a. $\frac{2}{5}$
- b. $\frac{1}{10}$

17.
- a. $\frac{3}{16}$
- b. $\frac{1}{144}$

18.
- a. 125 cu. cm
- b. 150 sq. cm

19.
- a. 1124.864 cu. ft.
- b. 648.96 sq. ft

20.
- a. 262,144 cu. ft
- b. 24,576 sq. ft

21.
- a. 21.952 cu. m
- b. 47.04 sq. m

22.
- a. 113.04 cu. cm
- b. 113.04 sq. cm

23.
- a. 1,766.25 cu. ft
- b. 706.5 sq. ft

24.
- a. 14,130 cu. in.
- b. 2,826 sq. in.

25.
- a. 381.51 cu. cm
- b. 254.34 sq. cm

26. 3.4×10^5
27. 1.29×10^6
28. 4.1×10^{-2}
29. 1.03×10^{-6}
30. 830
31. 16,000,000,000
32. 0.038
33. 0.0000193
34. 0.02
35. 5.6×10^{-6}
36. 500,000,000
37. 20,000,000
38. 6.85×10^9.
39. 5.7×10^{-10}
40. 0.00001
41. 1.545×10^8

42.
- a. 1.86×10^{13}
- b. 3×10^8
- c. 6.2×10^4

Attributions

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1.6 Roots and Radicals

Learning Objectives

By the end of this section it is expected that you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Use radicals in applications

Simplify Expressions with Square Roots

Remember that when a number *n* is multiplied by itself, we write n^2 and read it “n squared.” The result is called the square of *n*. For example,

8^2	read ’8 squared’
64	64 is called the square of 8.

Similarly, 121 is the square of 11, because 11^2 is 121.

Square of a Number

If $n^2 = m$, then *m* is the **square** of *n*.

Complete the following table to show the squares of the counting numbers 1 through 15.

Number	*n*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2								64			121				

The numbers in the second row are called perfect square numbers. It will be helpful to learn to recognize the perfect square numbers.

The squares of the counting numbers are positive numbers. What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$

Did you notice that these squares are the same as the squares of the positive numbers?

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is m is called a square root of *m*.

Square Root of a Number

If $n^2 = m$, then *n* is a **square root** of *m*.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The radical sign, $\sqrt{m}$, denotes the positive square root. The positive square root is called the principal square root. When we use the radical sign that always means we want the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

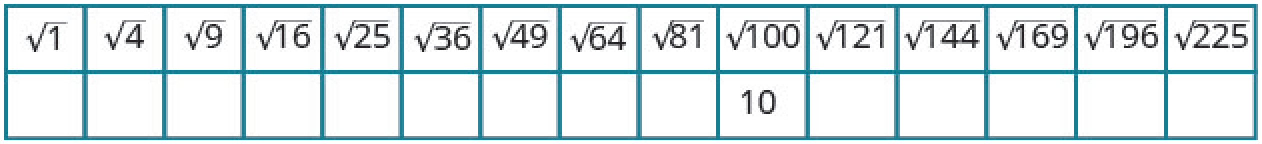
Square Root Notation

$\sqrt{m}$ is read "the square root of *m*"

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.

Since 10 is the principal square root of 100, we write $\sqrt{100} = 10$. You may want to complete the following table to help you recognize square roots.



$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
									10					

EXAMPLE 1

Simplify: a) $\sqrt{25}$ b) $\sqrt{121}$.

Solution

a) Since $5^2 = 25$	$\sqrt{25}$ 5
b) Since $11^2 = 121$	$\sqrt{121}$ 11

TRY IT 1

Simplify: a) $\sqrt{36}$ b) $\sqrt{169}$.

Show answer

a) 6 b) 13

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, -$\sqrt{100} = -10$. We read -$\sqrt{100}$ as "the opposite of the square root of 100."

EXAMPLE 2

Simplify: a) $-\sqrt{49}$ b) $-\sqrt{144}$.

Solution

a) The negative is in front of the radical sign.	$-\sqrt{49}$ -7
b) The negative is in front of the radical sign.	$-\sqrt{144}$ -12

TRY IT 2

Simplify: a) $-\sqrt{16}$ b) $-\sqrt{225}$.

Show answer

a) -4 b) -15

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\ )^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

EXAMPLE 3

Simplify: a) $\sqrt{-196}$ b) $-\sqrt{64}$.

Solution

a)

	$\sqrt{-196}$
There is no real number whose square is -196.	$\sqrt{-196}$.

b)

	$-\sqrt{64}$
The negative is in front of the radical.	-8

TRY IT 3

Simplify: a) $\sqrt{-169}$ b) $-\sqrt{81}$.

Show answer

a) not a real number b) -9

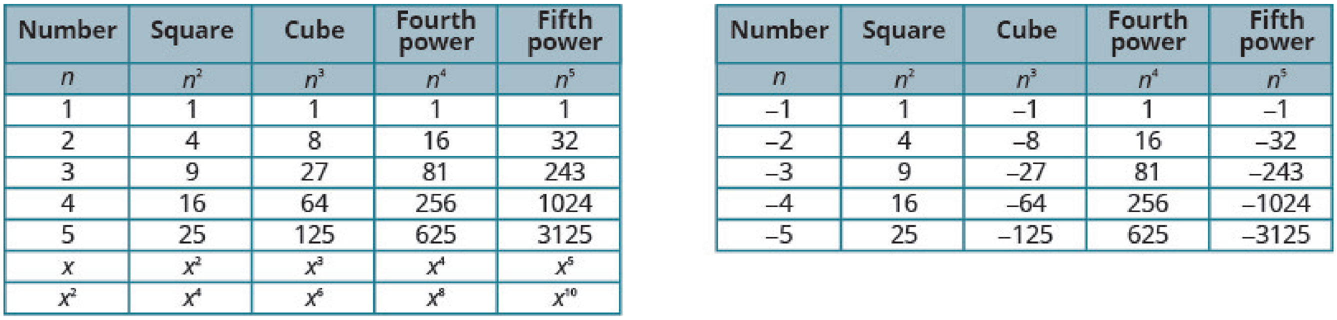
So far we have only talked about squares and square roots. Let’s now extend our work to include higher powers and higher roots.

Let’s review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms ‘squared’ and ‘cubed’ come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from −5 to 5. See (Table 1).

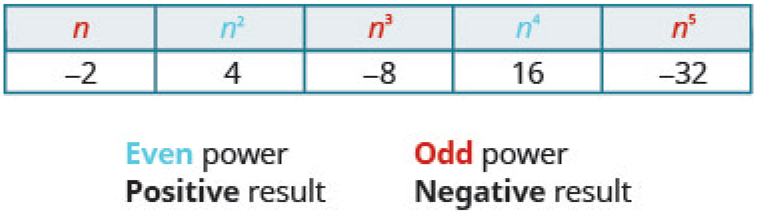


Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Table 1

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We’ll copy the row with the powers of −2 to help you see this.



n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

We will now extend the square root definition to higher roots.

n^{th} Root of a Number

$$\text{If } b^n = a, \text{ then } b \text{ is an } n^{th} \text{ root of } a.$$
$$\text{The principal } n^{th} \text{ root of } a \text{ is written } \sqrt[n]{a}.$$
$$n \text{ is called the index of the radical.}$$

Just like we use the word ‘cubed’ for b^3, we use the term ‘cube root’ for $\sqrt[3]{a}$.

We can refer to (Table 1) to help find higher roots.

$$\begin{aligned} 4^3 &= 64 & \sqrt[3]{64} &= 4 \\ 3^4 &= 81 & \sqrt[4]{81} &= 3 \\ (-2)^5 &= -32 & \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When *n* is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When *n* is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.

We will apply these properties in the next two examples.

EXAMPLE 4

Simplify: a) $\sqrt[3]{64}$ b) $\sqrt[4]{81}$ c) $\sqrt[5]{32}$.

Solution

a)

	$\sqrt[3]{64}$
Since $4^3 = 64$.	4

b)

	$\sqrt[4]{81}$
Since $(3)^4 = 81$	3

c)

	$\sqrt[5]{32}$
Since $(2)^5 = 32$	2

TRY IT 4

Simplify: a) $\sqrt[3]{27}$ b) $\sqrt[4]{256}$ c) $\sqrt[5]{243}$.

Show answer
a) 3 b) 4 c) 3

In this example be alert for the negative signs as well as even and odd powers.

EXAMPLE 5

Simplify: a) $\sqrt[3]{-125}$ b) $\sqrt[4]{-16}$ c) $\sqrt[5]{-243}$.

Solution

a)

	$\sqrt[3]{-125}$
Since $(-5)^3 = -125$	-5

b)

	$\sqrt[4]{-16}$
Think, $(?)^4 = -16$. No real number raised to the fourth power is negative.	Not a real number.

c)

	$\sqrt[5]{-243}$
Since $left(-3^5 = -243.$	-3

TRY IT 5

Simplify: a) $\sqrt[3]{-27}$ b) $\sqrt[4]{-256}$ c) $\sqrt[5]{-32}$.

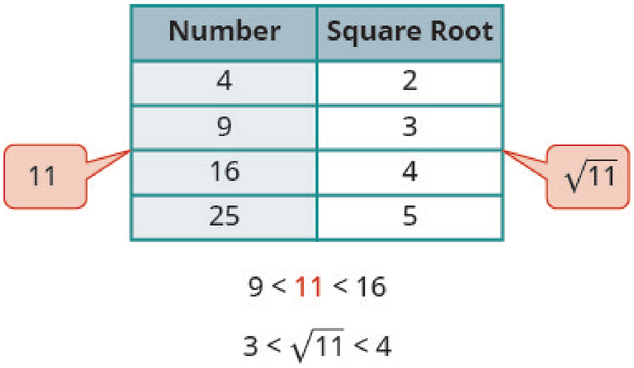
Show answer

a) -3 b) not real c) -2

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4}=2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

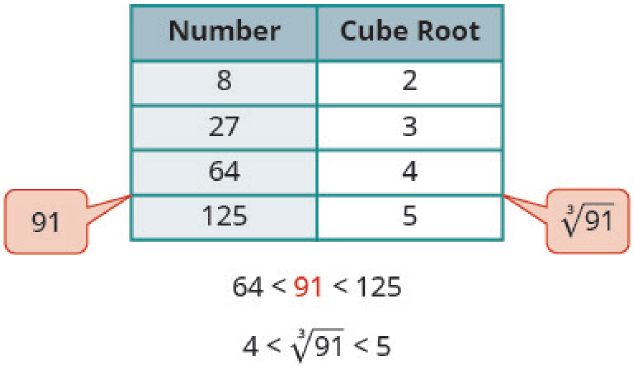
To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.



Number	Square Root
4	2
9	3
16	4
25	5

$9 < 11 < 16$

$3 < \sqrt{11} < 4$



Number	Cube Root
8	2
27	3
64	4
125	5

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

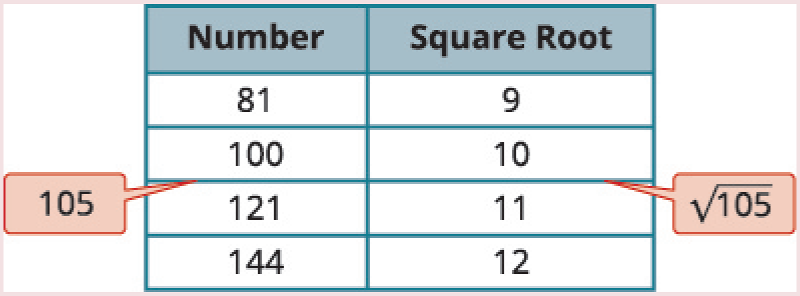
Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

EXAMPLE 6

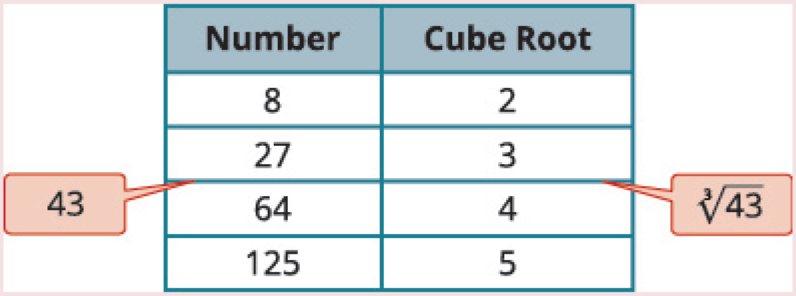
Estimate each root between two consecutive whole numbers: a) $\sqrt{105}$ b) $\sqrt[3]{43}$.

Solution

a) Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

<table>
<tr><td></td><td>$\sqrt{105}$</td></tr>
<tr><td></td><td>
<table>
<tr><th>Number</th><th>Square Root</th></tr>
<tr><td>81</td><td>9</td></tr>
<tr><td>100</td><td>10</td></tr>
<tr><td>121</td><td>11</td></tr>
<tr><td>144</td><td>12</td></tr>
</table>
</td></tr>
<tr><td>Locate 105 between two consecutive perfect squares.</td><td>$100 < 105 < 121$</td></tr>
<tr><td>$\sqrt{105}$ is between their square roots.</td><td>$10 < \sqrt{105} < 11$</td></tr>
</table>

b) Similarly we locate 43 between two perfect cube numbers.

<table>
<tr><td></td><td>$\sqrt[3]{43}$</td></tr>
<tr><td></td><td>
<table>
<tr><th>Number</th><th>Cube Root</th></tr>
<tr><td>8</td><td>2</td></tr>
<tr><td>27</td><td>3</td></tr>
<tr><td>64</td><td>4</td></tr>
<tr><td>125</td><td>5</td></tr>
</table>
</td></tr>
<tr><td>Locate 43 between two consecutive perfect cubes.</td><td>$27 < 43 < 64$</td></tr>
<tr><td>$\sqrt[3]{43}$ is between their cube roots.</td><td>$3 < \sqrt[3]{43} < 4$</td></tr>
</table>

TRY IT 6

Estimate each root between two consecutive whole numbers:

a) $\sqrt{38}$b) $\sqrt[3]{93}$

Show answer

a) $6 < \sqrt{38} < 7$

b) $4 < \sqrt[3]{93} < 5$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 & (3.105422799)^4 &= 92.999999991 \\ (2.24)^2 &= 5.0176 & (3.11)^4 &= 93.54951841 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

EXAMPLE 7

Round to two decimal places: a) $\sqrt{17}$ b) $\sqrt[3]{49}$ c) $\sqrt[4]{51}$.

Solution

a)

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626...$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

b)

	$\sqrt[3]{49}$
Use the calculator $\sqrt[y]{x}$ key.	$3.659305710...$
Round to two decimal places.	3.66
	$\sqrt[3]{49} \approx 3.66$

c)

	$\sqrt[4]{51}$
Use the calculator $\sqrt[y]{x}$ key.	2.6723451177
Round to two decimal places.	2.67
	$\sqrt[4]{51} \approx 2.67$

TRY IT 7

Round to two decimal places:

a) $\sqrt{11}$ b) $\sqrt[3]{71}$ c) $\sqrt[4]{127}$.

Show answer

a) ≈ 3.32 b) ≈ 4.14

c) ≈ 3.36

Use Radicals in Applications

As you progress through your college or university courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a problem solving strategy for applications with formulas.

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula

$t = \frac{\sqrt{h}}{4}$.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 8

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let t = time.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5. Solve the equation.	$t = \frac{20}{4}$
	$t = 5$
Step 6. Check the answer in the problem and make sure it makes sense.	$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5\ \checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

TRY IT 8

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Show answer
9 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

EXAMPLE 9

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what we are looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5. Solve the equation.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

TRY IT 9

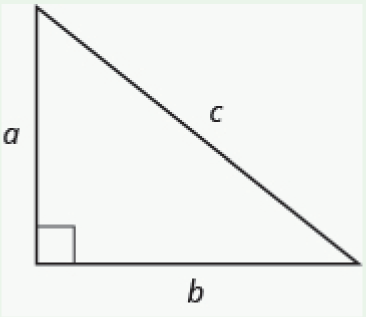
An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Show answer
42.7 feet

The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

The Pythagorean Theorem

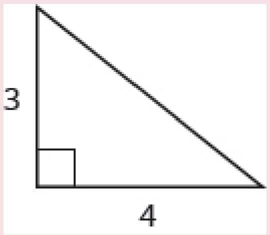
In any right triangle ΔABC,
$a^2 + b^2 = c^2$
where c is the length of the hypotenuse a and b are the lengths of the legs.



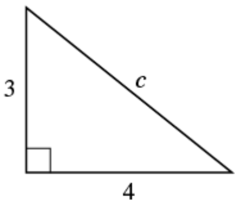
We will use this definition of square roots to solve for the length of a side in a right triangle.

EXAMPLE 10

Use the Pythagorean Theorem to find the length of the hypotenuse.

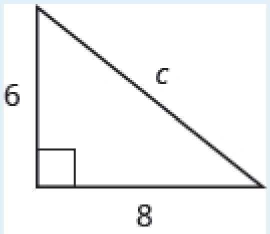


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length of the hypotenuse of the triangle
Step 3. **Name.** Choose a variable to represent it.	Let $c = \text{the length of the hypotenuse}$ 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $3^2 + 4^2 = c^2$
Step 5. **Solve** the equation.	$9 + 16 = c^2$ $25 = c^2$ $\sqrt{25} = c^2$ $5 = c$
Step 6. **Check:** $3^2 + 4^2 = 5^2$ $9 + 16 \overset{?}{=} 25$ $25 = 25$ ✓	
Step 7. **Answer** the question.	The length of the hypotenuse is 5.

TRY IT 10

Use the Pythagorean Theorem to find the length of the hypotenuse.



Show answer
10

Access these online resources for additional instruction and practice with solving radical equations.

- Radical Equation Application

Glossary

square root notation

- $\sqrt{m}$ is read ‘the square root of *m*’
- If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

- The square root of *m*, $\sqrt{m}$, is a positive number whose square is *m*.

n^{th} root of a number

- If $b^n = a$, then *b* is an n^{th} root of *a*.
- The principal n^{th} root of *a* is written $\sqrt[n]{a}$.
- *n* is called the *index* of the radical.

1.6 Exercise Set

In the following exercises, simplify.

1\. a. $\sqrt{64}$
b. $-\sqrt{81}$

2\. a. $\sqrt{196}$
b. $-\sqrt{1}$

3\. a. $\sqrt{\frac{4}{9}}$
b. $-\sqrt{0.01}$

4\. a. $\sqrt{-121}$
b. $-\sqrt{289}$

5\. a. $-\sqrt{225}$
b. $\sqrt{-9}$

6\. a. $\sqrt[3]{216}$
b. $\sqrt[4]{256}$

7\. a. $\sqrt[3]{512}$
b. $\sqrt[4]{81}$
c. $\sqrt[5]{1}$

8\. a. $\sqrt[3]{-8}$
b. $\sqrt[4]{-81}$
c. $\sqrt[5]{-32}$

9\. a. $\sqrt[3]{-125}$
b. $\sqrt[4]{-1296}$
c. $\sqrt[5]{-1024}$

In the following exercises, estimate each root between two consecutive whole numbers.

10\. a. $\sqrt{70}$ b. $\sqrt[3]{71}$

11. a. $\sqrt{200}$ b. $\sqrt[3]{137}$

In the following exercises, approximate each root and round to two decimal places.

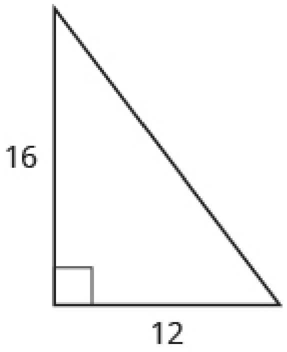
12. a. $\sqrt{19}$
b. $\sqrt[3]{89}$
c. $\sqrt[4]{97}$

13. a. $\sqrt{53}$
b. $\sqrt[3]{147}$
c. $\sqrt[4]{452}$

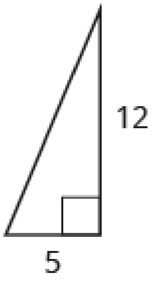
In the following exercises, solve. Round approximations to one decimal place.

14. **Landscaping.** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
15. **Gravity.** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
16. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.



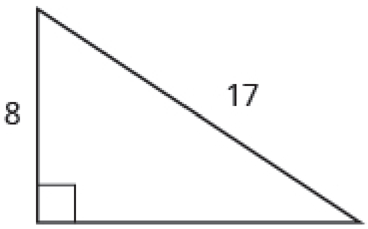
17.



18.

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

19.



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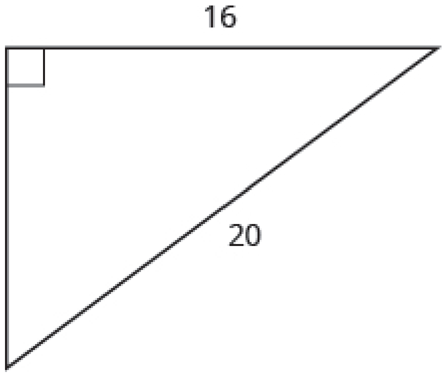

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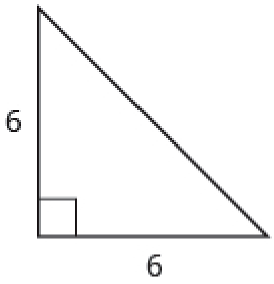

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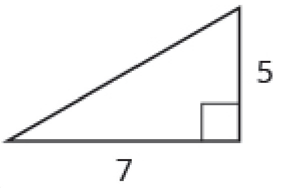

Answers:

1.
 a. 8
 b. -9

2.
 a. 14
 b. -1

3.
 a. $\frac{2}{3}$
 b. -0.1

4.
 a. not real number
 b. -17

5.
 a. -15
 b. not real number

6.
 a. 6
 b. 4

7.
 a. 8
 b. 3
 c. 1

8.
 a. -2
 b. not real
 c. -2

9.
 a. -5
 b. not real
 c. -4

10.
 a. $8 < \sqrt{70} < 9$
 b. $4 < \sqrt[3]{71} < 5$

11.
 a. $14 < \sqrt{200} < 15$
 b. $5 < \sqrt[3]{137} < 6$

12.
 a. 4.36
 b. ≈ 4.46
 c. ≈ 3.14

13.
 a. 7.28
 b. ≈ 5.28
 c. ≈ 4.61

14. 8.7 feet

15. 4.7 seconds

16. 72 feet

17. 20

18. 13

19. 15

20. 12

21. 8.5

22. 8.6

Attributions (imported from my Introductory Algebra)

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1.7 The Real Numbers

Learning Objectives

By the end of this section it is expected that you will be able to:

- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions on the number line
- Locate decimals on the number line

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers?

$$\begin{array}{ll}
\text{Counting numbers} & 1, 2, 3, 4, \\
\text{Whole numbers} & 0, 1, 2, 3, 4, \\
\text{Integers} & -3, -2, -1, 0, 1, 2, 3,
\end{array}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A rational number is a number that can be written as a ratio of two integers.

Rational Number

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

A rational number can be written as the ratio of two integers.

All signed fractions, such as $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, -\frac{20}{3}$ are rational numbers. Each numerator and each denominator is an integer.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. Each integer can be written as a ratio of integers in many ways. For example, 3 is equivalent to $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}$

An easy way to write an integer as a ratio of integers is to write it as a fraction with denominator one.

$3 = \frac{3}{1}$ $\quad -8 = -\frac{8}{1}$ $\quad 0 = \frac{0}{1}$

Since any integer can be written as the ratio of two integers, *all integers are rational numbers*! Remember that the counting numbers and the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers.

We've already seen that integers are rational numbers. The integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $\frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684 is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction.

EXAMPLE 1

Write as the ratio of two integers: a) -27 b) 7.31

Solution

a) Write it as a fraction with denominator 1.	-27 $\frac{-27}{1}$
b) Write it as a mixed number. Remember, 7 is the whole number and the decimal part, 0.31, indicates hundredths. Convert to an improper fraction.	7.31 $7\frac{31}{100}$ $\frac{731}{100}$

So we see that -27 and 7.31 are both rational numbers, since they can be written as the ratio of two integers.

TRY IT 1

Write as the ratio of two integers: a) -24 b) 3.57

Show answer

a) $\frac{-24}{1}$ b) $\frac{357}{100}$

Let's look at the decimal form of the numbers we know are rational.

We have seen that *every integer is a rational number*, since $a = \frac{a}{1}$ for any integer, *a*. We can also change any integer to a decimal by adding a decimal point and a zero.

Integer	-2	-1	0	1	2	3
Decimal form	-2.0	-1.0	0.0	1.0	2.0	3.0

These decimal numbers stop.

We have also seen that *every fraction is a rational number*. Look at the decimal form of the fractions we considered above.

Ratio of integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
The decimal form	0.8	-0.875	3.25	$-6.666..$

These decimals either stop or repeat.

What do these examples tell us?

Every rational number can be written both as a ratio of integers, $\frac{p}{q}$*,where p and q are integers and* $q \neq 0$*,and as a decimal that either stops or repeats.*

Here are the numbers we looked at above expressed as a ratio of integers and as a decimal:

Fractions

Number	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
Ratio of Integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
Decimal Form	0.8	-0.875	3.25	$-6.\overline{6}$

Integers

Number	-2	-1	0	1	2	3
Ratio of Integers	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$
Decimal Form	-2.0	-1.0	0.0	1.0	2.0	3.0

Rational Number

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes!

The number π (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654...$$

We can even create a decimal pattern that does not stop or repeat, such as

$$2.01001000100001\ldots$$

Numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call these numbers irrational. More on irrational numbers later on is this course.

Irrational Number

An irrational number is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

Rational or Irrational?

If the decimal form of a number

- *repeats or stops*, the number is **rational**.
- *does not repeat and does not stop*, the number is irrational

EXAMPLE 2

Given the numbers $0.58\ \overline{3}, 0.47, 3.605551275...$ list the a) rational numbers b) irrational numbers.

Solution

a) Look for decimals that repeat or stop.	The 3 repeats in $0.58\ \overline{3}$. The decimal 0.47 stops after the 7. So $0.58\ \overline{3}$ and 0.47 are rational.
b) Look for decimals that neither stop nor repeat.	3.605551275 has no repeating block of digits and it does not stop. So 3.605551275 is irrational.

TRY IT 2

For the given numbers list the a) rational numbers b) irrational numbers: $0.29, 0.81\ \overline{6}, 2.515115111\ldots$.

Show answer

a) $0.29, 0.81\ \overline{6}$ b) 2.515115111

EXAMPLE 3

For each number given, identify whether it is rational or irrational: a) $\sqrt{36}$ b) $\sqrt{44}$.

Solution

a) Recognize that 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$, therefore $\sqrt{36}$ is rational.

b) Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square. Therefore, the decimal form of $\sqrt{44}$ will never repeat and never stop, so $\sqrt{44}$ is irrational.

TRY IT 3

For each number given, identify whether it is rational or irrational: a) $\sqrt{81}$ b) $\sqrt{17}$.

Show answer

a) rational b) irrational

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of real numbers.

Real Number

A **real number** is a number that is either rational or irrational.

All the numbers we use in algebra are real numbers. Figure 1 illustrates how the number sets we've discussed in this section fit together.

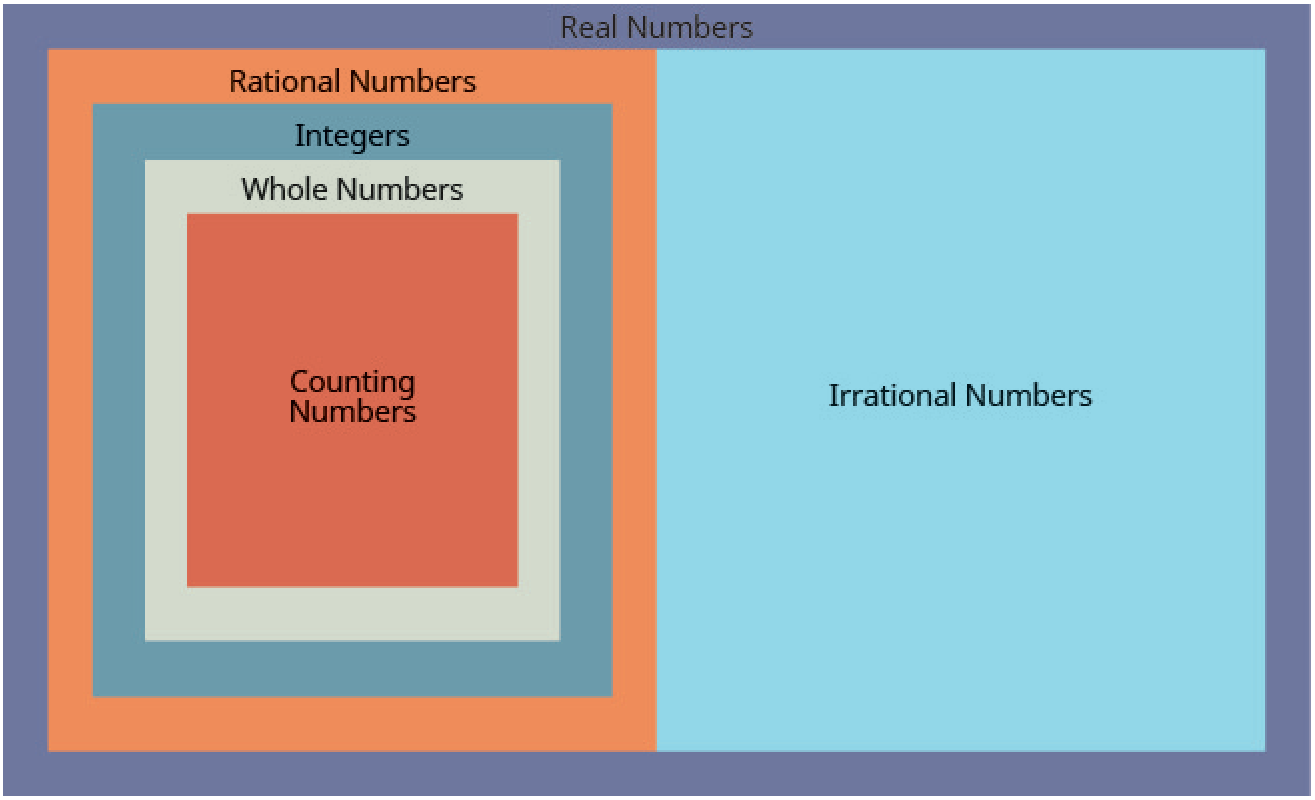


Figure 1 This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

Do you remember that the square root of a negative number was not a real number?

EXAMPLE 4

For each number given, identify whether it is a real number or not a real number: ? $\sqrt{-169}$? $\sqrt{64}$.

Solution

a) There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.

b) Since the negative is in front of the radical, $-\sqrt{64}$ is -8, Since -8 is a real number, $-\sqrt{64}$ is a real number.

TRY IT 4

For each number given, identify whether it is a real number or not a real number: a) $\sqrt{-196}$ b) $-\sqrt{81}$.

Show answer

a) not a real number b) real number

EXAMPLE 5

Given the numbers -7, $\frac{14}{5}$, 8, $\sqrt{5}$, 5.9, -$\sqrt{64}$, list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers.

Solution

a) Remember, the whole numbers are 0, 1, 2, 3, … and 8 is the only whole number given.

b) The integers are the whole numbers, their opposites, and 0. So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7, 8, -\sqrt{64}$.

c) Since all integers are rational, then $-7, 8, -\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is $-7, \frac{14}{5}, 8, 5.9, -\sqrt{64}$.

d) Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

e) All the numbers listed are real numbers.

TRY IT 5

For the given numbers, list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers: $-3, -\sqrt{2}, 0.\bar{3}, \frac{9}{5}, 4, \sqrt{49}$.

Show answer

a) $4, \sqrt{49}$ b) $-3, 4, \sqrt{49}$ c) $-3, 0.\ \bar{3}, \frac{9}{5}, 4, \sqrt{49}$ d) -$\sqrt{2}$ e) $-3,$ -$\sqrt{2}, 0.\ \bar{3}, \frac{9}{5}, 4, \sqrt{49}$

Locate Fractions on the Number Line

The last time we looked at the number line, it only had positive and negative integers on it. We now want to include fractions and decimals on it.

Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5,$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5. because they are the easiest to plot. See Figure 2.

The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$. See Figure 2.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$. See Figure 2.

Finally, look at the improper fractions $\frac{7}{4}, -\frac{9}{2}, \frac{8}{3}$. These are fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change each of them to a mixed number. See Figure 2.

$\frac{7}{4} = 1\frac{3}{4}$ $\qquad -\frac{9}{2} = -4\frac{1}{2}$ $\qquad \frac{8}{3} = 2\frac{2}{3}$

Figure 2 shows the number line with all the points plotted.

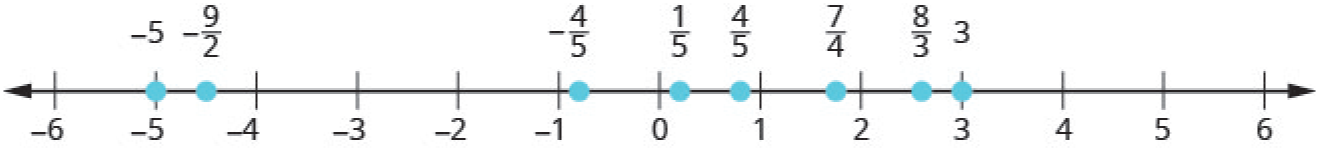


Figure 2

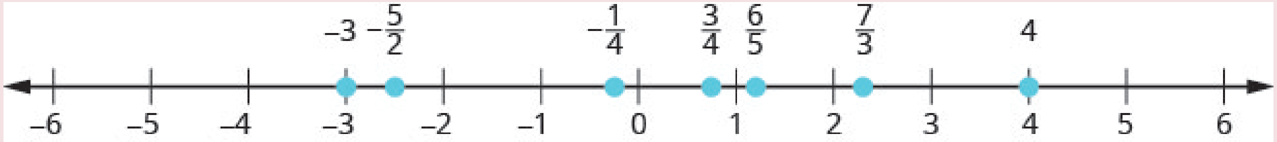
EXAMPLE 6

Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2},$ and $\frac{7}{3}$.

Solution
Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts then, we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

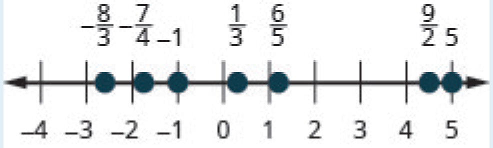
Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}, \frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



TRY IT 6

Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

Show answer



In Example 5, we'll use the inequality symbols to order fractions. In previous chapters we used the number line to order numbers.

- $a < b$ "*a* is less than *b*" when *a* is to the left of *b* on the number line
- $a > b$ "*a* is greater than *b*" when *a* is to the right of *b* on the number line

As we move from left to right on a number line, the values increase.

EXAMPLE 7

Order each of the following pairs of numbers, using < or >. It may be helpful to refer Figure 3.

a) -$\frac{2}{3}$____-1 b) -3$\frac{1}{2}$____-3 c) $\frac{3}{4}$____ -$\frac{1}{4}$ d) -2____ -$\frac{8}{3}$

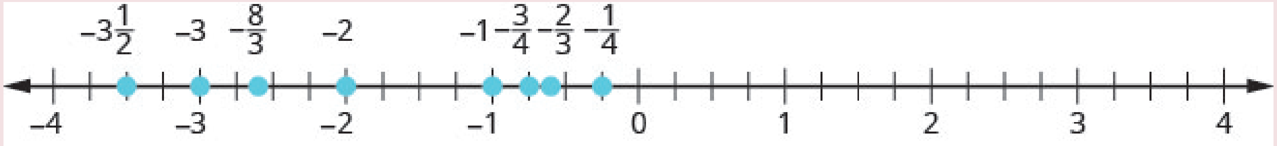


Figure 3

-3$\frac{1}{2}$_____ -3

$-3\frac{1}{2} < -3$

Solution

a) $-\frac{2}{3}$ is to the right of -1 on the number line.	$-\frac{2}{3}$___-1 $-\frac{2}{3}$ > -1
b) $-3\frac{1}{2}$ is to the right of -3 on the number line.	$-3\frac{1}{2}\_\_\_\_\_ -3$ $-3\frac{1}{2} < -3$
c) $-\frac{3}{4}$ is to the right of $-\frac{1}{4}$ on the number line.	$-\frac{3}{4}\_\_\_\_\_ -\frac{1}{4}$ $-\frac{3}{4} < -\frac{1}{4}$
d) -2 is to the right of $-\frac{8}{3}$ on the number line.	$-2\_\_\_\_\_ -\frac{8}{3}$ $-2 > -\frac{8}{3}$

TRY IT 7

Order each of the following pairs of numbers, using < or >:

a) $-\frac{1}{3}\_\_\_\_\_ -1$ b) $-1\frac{1}{2}\_\_\_\_\_ -2$ c) $-\frac{2}{3}\_\_\_\_\_ -\frac{1}{3}$ d) $-3\_\_\_\_\_ -\frac{7}{3}$.

Show answer

a) > b) > c) < d) <

Locate Decimals on the Number Line

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

EXAMPLE 8

Locate 0.4 on the number line.

Solution

A proper fraction has value less than one. The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 and 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line. See Figure 4.

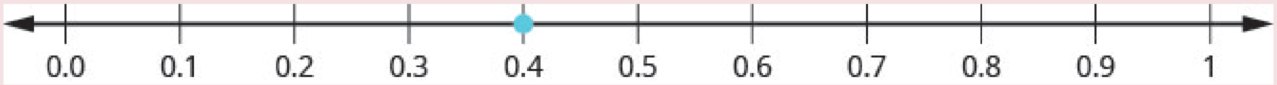
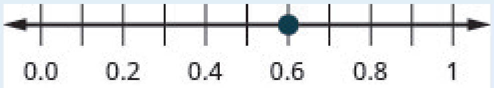


Figure 4

TRY IT 8

Locate on the number line: 0.6

Show answer



EXAMPLE 9

Locate -0.74 on the number line.

Solution

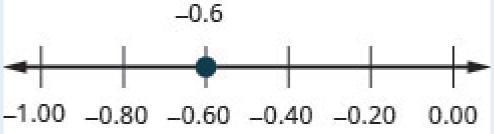
The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and -1. On a number line, mark off and label the hundredths in the interval between 0 and -1. See Figure 5.

Figure 5

TRY IT 9

Locate on the number line: -0.6.

Show answer



Which is larger, 0.04 or 0.40? If you think of this as money, you know that ?0.40 (forty cents) is greater than ?0.04 (four cents). So,

$0.40 > 0.04$

Again, we can use the number line to order numbers.

- $a < b$ "*a* is less than *b*" when *a* is to the left of *b* on the number line
- $a > b$ "*a* is greater than *b*" when *a* is to the right of *b* on the number line

Where are 0.04 and 0.40 located on the number line? See Figure 6.

Figure 6

We see that 0.40 is to the right of 0.04 on the number line. This is another way to demonstrate that 0.40 > 0.04

How does 0.31 compare to 0.308? This doesn't translate into money to make it easy to compare. But if we convert 0.31 and 0.308 into fractions, we can tell which is larger.

	0.31	0.308
Convert to fractions.	$\frac{31}{100}$	$\frac{308}{1000}$
We need a common denominator to compare them.	$\frac{31 \cdot 10}{100 \cdot 10}$	$\frac{308}{1000}$
	$\frac{310}{1000}$	$\frac{308}{1000}$

Because 310 > 308, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, 0.31 > 0.308

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$. Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does not change its value!

$$\frac{31}{100} = \frac{310}{1000} \quad \text{and} \quad 0.31 = 0.310$$

We say 0.31 and 0.310 are equivalent decimals.

Equivalent Decimals

Two decimals are equivalent if they convert to equivalent fractions.

We use equivalent decimals when we order decimals.

The steps we take to order decimals are summarized here.

HOW TO: Order Decimals.

1. Write the numbers one under the other, lining up the decimal points.
2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
3. Compare the numbers as if they were whole numbers.
4. Order the numbers using the appropriate inequality sign.

EXAMPLE 10

Order $0.64\_\_\_\_0.6$ using < or >.

Solution

Write the numbers one under the other, lining up the decimal points.	0.64 0.6
Add a zero to 0.6 to make it a decimal with 2 decimal places. Now they are both hundredths.	0.64 0.60
64 is greater than 60.	$64 > 60$
64 hundredths is greater than 60 hundredths.	$0.64 > 0.60$
	$0.64 > 0.6$

TRY IT 10

Order each of the following pairs of numbers, using $<$ or $>$: $0.42\_\_\_\_0.4$.

Show answer
>

EXAMPLE 11

Order $0.83\underline{\quad}0.803$ using < or >.

Solution

	$0.83\underline{\quad}0.803$
Write the numbers one under the other, lining up the decimals.	0.83 0.803
They do not have the same number of digits. Write one zero at the end of 0.83.	0.830 0.803
Since $830 > 803$, 830 thousandths is greater than 803 thousandths.	$0.830 > 0.803$
	$0.83 > 0.803$

TRY IT 11

Order the following pair of numbers, using $<$ or $>$: $0.76\underline{\quad}0.706$.

Show answer
>

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because -2 lies to the right of -3 on the number line, we know that $-2 > -3$. Similarly, smaller numbers lie to the left on the number line. For example, because -9 lies to the left of -6 on the number line, we know that $-9 < -6$. See Figure 7.

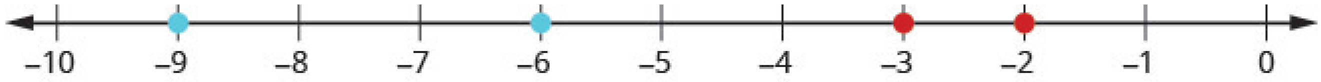


Figure 7

If we zoomed in on the interval between 0 and -1, as shown in Example 10, we would see in the same way that $-0.2 > -0.3$ and $-0.9 < -0.6$.

EXAMPLE 12

Use < or > to order $-0.1\underline{\quad} - 0.8$.

Solution

	$-0.1\_\_\_\_ - 0.8$
Write the numbers one under the other, lining up the decimal points. They have the same number of digits.	-0.1 -0.8
Since $-1 > -8$, −1 tenth is greater than −8 tenths.	$-0.1 > -0.8$

TRY IT 12

Order the following pair of numbers, using < or >: $-0.3\_\_\_\_ - 0.5$.

Show answer
>

Key Concepts

- **Order Decimals**
    1. Write the numbers one under the other, lining up the decimal points.
    2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
    3. Compare the numbers as if they were whole numbers.
    4. Order the numbers using the appropriate inequality sign.

Glossary

equivalent decimals
Two decimals are equivalent if they convert to equivalent fractions.

irrational number
An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

rational number
A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. A rational number can be written as the ratio of two integers. Its decimal form stops or repeats.

real number
A real number is a number that is either rational or irrational

1.7 Exercise Set

In the following exercises, write as the ratio of two integers.

1.
   a. 5
   b. 3.19

2.
   a. -12
   b. 9.279

In the following exercises, list the a) rational numbers, b) irrational numbers

3. $0.75, 0.22\overline{3}, 1.39174$
4. $0.4\overline{5}, 1.919293\ldots, 3.59$

In the following exercises, list the a) whole numbers, b) integers, c) rational numbers, d) irrational numbers, e) real numbers for each set of numbers.

5. $-8, 0, \sqrt[5]{-32}, 1.95286\ldots, \frac{12}{5}, \sqrt[2]{-9}, \sqrt[3]{9}$
6. $-7, \sqrt[3]{512}, -\frac{8}{3}, -1, \sqrt[4]{-75}, 0.77, 3\frac{1}{4}$

In the following exercises, locate the numbers on a number line.

7. $\frac{3}{4}, \frac{8}{5}, \frac{10}{3}$
8. $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$
9. $\frac{2}{5}, -\frac{2}{5}$
10. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

In the following exercises, order each of the pairs of numbers, using < or >.

11. $-1\_\_\_\_ -\frac{1}{4}$
12. $-2\frac{1}{2}\_\_\_\_ -3$
13. $-\frac{5}{12}\_\_\_\_ -\frac{7}{12}$
14. $-3\_\_\_\_ -\frac{13}{5}$

Locate Decimals on the Number Line In the following exercises, locate the number on the number line.

15. 0.8
16. -1.6

In the following exercises, order each pair of numbers, using < or >.

17. $0.37\_\_\_\_0.63$
18. $0.91\_\_\_\_0.901$
19. $-0.5\_\_\_\_ -0.3$
20. $-0.62\_\_\_\_ -0.619$

21. **Child care.** Serena wants to open a licensed child care center. Her state requires there be no more than 12 children for each teacher. She would like her child care centre to serve 40 children.
    a. How many teachers will be needed?
    b. Why must the answer be a whole number?
    c. Why shouldn't you round the answer the usual way, by choosing the whole number closest to the exact answer?

Answers:

1.
   a. $\frac{5}{1}$

b. $\frac{319}{100}$

2. a. $\frac{-12}{1}$

   b. $\frac{9297}{1000}$

3. a. $0.75, 0.22\overline{3}$

   b. 1.39174

4. a. $0.4\overline{5}, 3.59$

   b. 1.919293

5. a. 0

   b. $-8, 0, \sqrt[5]{-32}$

   c. $-8, 0, \sqrt[5]{-32}, \frac{12}{5}$

   d. $1.95286\ldots, \sqrt[3]{9}$

   e. $-8, 0, \sqrt[5]{-32}, 1.95286\ldots, \frac{12}{5}, \sqrt[3]{9}$

6. a. $\sqrt[3]{512}$

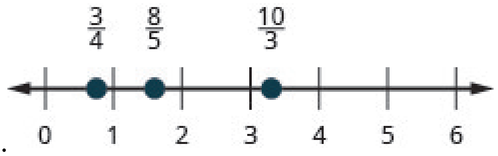
   b. $-7, -1, \sqrt[3]{512}$

   c. -7, $-\frac{8}{3}, -1, 0.77, 3\frac{1}{4}, \sqrt[3]{512}$

   d. none

   e. $-7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}, \sqrt[3]{512}$

7.



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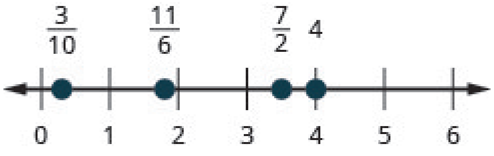

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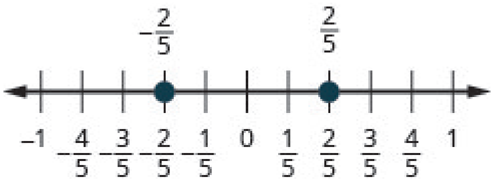
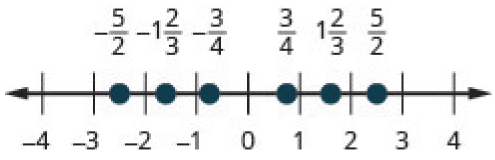


10.

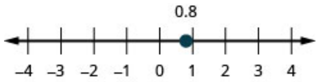


11. <

12. >

13. >

14. <

15. 	16. 

17. <
18. >
19. <
20. <
21.
    a. 4 buses
    b. answers may vary
    c. answers may vary

Attributions

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2. Solving Linear Equations and Inequalities

The rocks in this formation must remain perfectly balanced around the centre for the formation to hold its shape.

If we carefully placed more rocks of equal weight on both sides of this formation, it would still balance. Similarly, the expressions in an equation remain balanced when we add the same quantity to both sides of the equation. In this chapter, we will solve equations, remembering that what we do to one side of the equation, we must also do to the other side.

Attributions

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2.1 Solve Linear Equations

Izabela Mazur, Lynn Marecek, and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations using the Division and Multiplication Properties of Equality
- Solve an equation with variables and constants on both sides

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Solution of an equation

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

HOW TO: Determine whether a number is a solution to an equation

1. Substitute the number in for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side).
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.

EXAMPLE 1

Determine whether $x = -3$ is a solution of $4x + 5 = -7$.

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

Substitute $x = -3$ for x	$4(-3) + 5 = -7$
Multiply.	$-12 + 5 = -7$
Simplify.	$-7 = -7$

Since $x = -3$ results in a true statement (-7 is in fact equal to -7), -3 is a solution to the equation $4x + 5 = -7$.

TRY IT 1

Is $y = 2$ a solution of $9y - 2 = 6$?

Show answer
no

There are many types of equations. In this chapter we will focus on solving linear equations.

Linear Equation

A **linear equation** is a first degree equation in one variable that can be written as:

$ax + b = 0$, where *a* and *b* are real numbers and $a \neq 0$,

Solve Equations Using the Subtraction and Addition Properties of Equality

Let us review all the properties that will help us to solve equations algebraically. The first one is the **Subtraction Property of Equality**.

Subtraction Property of Equality

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & a-c & = & b-c \end{array}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Let’s see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

EXAMPLE 2

Solve: $y+37=-13$.

Solution

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

	$y+37=-13$
Subtract 37 from each side to ‘undo’ the addition.	$y+37-37=-13-37$
Simplify.	$y=-50$
Check:	$y+37=-13$
Substitute $y=-50$	$-50+37=-13$
	$-13\overset{?}{=}-13\ \checkmark$

Since $y=-50$ makes $y+37=-13$ a true statement, we have the solution to this equation.

TRY IT 2

Solve: $x+19=-27$.

Show answer

$x=-46$

What happens when an equation has a number subtracted from the variable, as in the equation

$x - 5 = 8$? We use another property to solve equations when a number is subtracted from the variable. We want to isolate the variable, so to ‘undo’ the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

Addition Property of Equality

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & a+c & = & b+c \end{array}$$

When you add the same quantity to both sides of an equation, you still have equality.

In Example 2, 37 was added to the *y* and so we subtracted 37 to ‘undo’ the addition. In Example 3, we will need to ‘undo’ subtraction by using the Addition Property of Equality.

EXAMPLE 3

Solve: $a - 28 = -37.$

Solution

	$a - 28 = -37$
Add 28 to each side to ‘undo’ the subtraction.	$a - 28 + 28 = -37 + 28$
Simplify.	$a = -9$
Check:	$a - 28 = -37$
Substitute $a = -9$	$-9 - 28 = -37$
	$-37 \overset{?}{=} -37$ ✓
The solution to $a - 28 = -37$ is $a = -9.$	

TRY IT 3

Solve: $n - 61 = -75.$

Show answer

$n = -14$

EXAMPLE 4

Solve: $x - \frac{5}{8} = \frac{3}{4}$.

Solution

	$x - \frac{5}{8} = \frac{3}{4}$
Use the Addition Property of Equality.	$x - \frac{5}{8} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$
Find the LCD to add the fractions on the right.	$x - \frac{5}{8} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$
Simplify.	$x = \frac{11}{8}$
Check:	$x - \frac{5}{8} = \frac{3}{4}$
Substitute $x = \frac{11}{8}$.	$\frac{11}{8} - \frac{5}{8} \overset{?}{=} \frac{3}{4}$
Subtract.	$\frac{6}{8} \overset{?}{=} \frac{3}{4}$
Simplify.	$\frac{3}{4} = \frac{3}{4}$ ✓
The solution to $x - \frac{5}{8} = \frac{3}{4}$ is $x = \frac{11}{8}$.	

TRY IT 4

Solve: $p - \frac{2}{3} = \frac{5}{6}$.

Show answer

$p = \frac{9}{6}\, p = \frac{3}{2}$

The next example will be an equation with decimals.

EXAMPLE 5

Solve: $n - 0.63 = -4.2$.

Solution

	$n - 0.63 = -4.2$
Use the Addition Property of Equality.	$n - 0.63 + 0.63 = -4.2 + 0.63$
Add.	$n = -3.57$
Check:	$n = -3.57$
Let $n = -3.57$.	$-3.57 - 0.63 \overset{?}{=} -4.2$
	$-4.2 = -4.2$ ✓

TRY IT 5

Solve: $b - 0.47 = -2.1$.

Show answer

$b = -1.63$

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.To solve those kind of equations we will use the **Division Property of Equality**.

The Division Property of Equality

For any numbers *a*, *b*, and *c*, and $c \neq 0$,

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c} \end{array}$$

When you divide both sides of an equation by any non-zero number, you still have equality.

The goal in solving an equation is to ‘undo’ the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to ‘undo’ the multiplication.

EXAMPLE 6

Solve: $5x = -27$.

Solution

To isolate x, “undo” the multiplication by 5.	$5x = -27$
Divide to ‘undo’ the multiplication.	$\frac{5x}{5} = -\frac{27}{5}$
Simplify.	$x = -\frac{27}{5}$
Check:	$5x = -27$
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right) \overset{?}{=} -27$
	$-27 = -27$ ✓
Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.	

TRY IT 6

Solve: $3y = -41$.

Show answer
$y = \frac{-41}{3}$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to “undo” the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

The Multiplication Property of Equality

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & ac & = & bc \end{array}$$

If you multiply both sides of an equation by the same number, you still have equality.

EXAMPLE 7

Solve: $\frac{y}{-7} = -14$.

Solution

Here y is divided by -7. We must multiply by -7 to isolate y.

	$\frac{y}{-7} = -14$
Multiply both sides by -7.	$-7\left(\frac{y}{-7}\right) = -7(-14)$
Multiply.	$\frac{-7y}{7} = 98$
Simplify.	$y = 98$
Check.	$\frac{y}{-7} = -14$
Substitute $y = 98$.	$\frac{98}{-7} \overset{?}{=} -14$
Divide	$-14 = -14$ ✓

TRY IT 7

Solve: $\frac{a}{-7} = -42$.

Show answer

$a = 294$

EXAMPLE 8

Solve: $\frac{3}{4}x = 12$.

Solution
Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

	$\frac{3}{4}x = 12$
Multiply by the reciprocal of $\frac{3}{4}$.	$\frac{4}{3}\cdot\frac{3}{4}x = \frac{4}{3}\cdot 12$
Reciprocals multiply to 1.	$1x = \frac{4}{3}\cdot\frac{12}{1}$
Multiply.	$x = 16$
Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.	
Check:	$\frac{3}{4}x = 12$
Substitute $x = 16$.	$\frac{3}{4}\cdot 16 \stackrel{?}{=} 12$ $12 = 12$ ✓

TRY IT 8

Solve: $\frac{2}{5}n = 14$.

Show answer
$n = 35$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We’ll list them all together here for easy reference.

Properties of Equality

Subtraction Property of Equality

For any real numbers $a, b,$ and $c,$

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a-c & = & b-c. \end{array}$$

Division Property of Equality

For any numbers $a, b,$ and $c,$ and $c \neq 0,$

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array}$$

Addition Property of Equality

For any real numbers $a, b,$ and $c,$

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a+c & = & b+c. \end{array}$$

Multiplication Property of Equality

For any numbers $a, b,$ and $c,$

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array}$$

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Now we will use those properties to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Solve Equations with Variables and Constants on Both Sides

Our strategy will involve choosing one side of the equation to be the “variable side”, and the other side of the equation to be the “constant side.” Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

EXAMPLE 9

Solve: $8y - 9 = 31.$

Solution

In this equation, the variable is found only on the left side. It makes sense to call the left side the “variable” side. Therefore, the right side will be the “constant” side. We will write the labels above the equation to help us remember what goes where. Since the left side is the “variable” side, the 9 is out of place. It is subtracted from the $8y$, so to “undo” subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

	$\overset{\text{variable}}{8y - 9} = \overset{\text{constant}}{31}$
Add 9 to both sides.	$8y - 9 + 9 = 31 + 9$
Simplify.	$8y = 40$ The variables are now on one side and the constants on the other. We continue from here as we did earlier.
Divide both sides by 8.	$\frac{8y}{8} = \frac{40}{8}$
Simplify.	$y = 5$
Check.	$8y - 9 = 31$
Let $y = 5$.	$8 \cdot 5 - 9 \overset{?}{=} 31$ $40 - 9 \overset{?}{=} 31$ $31 = 31$ ✓

TRY IT 9

Solve: $5y - 9 = 16$.

Show answer

$y = 5$

What if there are variables on both sides of the equation? For equations like this, begin as we did above—choose a “variable” side and a “constant” side, and then use the subtraction and addition properties of equality to collect all variables on one side and all constants on the other side.

EXAMPLE 10

Solve: $5y - 9 = 8y$.

Solution

The only constant is on the left and the y’s are on both sides. Let’s leave the constant on the left and get the variables to the right.

	$\overset{\text{constant}}{5y-9}=\overset{\text{variable}}{8y}$
Subtract $5y$ from both sides.	$5y-5y-9=8y-5y$
Simplify.	$-9=3y$
We have the *y*’s on the right and the constants on the left. Divide both sides by 3.	$\frac{-9}{3}=\frac{3y}{3}$
Simplify.	$-3=y$
Check:	$5y-9=8y$
Let $y=-3$.	$5(-3)-9\overset{?}{=}8(-3)$
	$-15-9\overset{?}{=}-24$
	$-24=-24\checkmark$

TRY IT 10

Solve: $3p-14=5p$.

Show answer

$p=-7$

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

EXAMPLE 11

How to Solve Equations with Variables and Constants on Both Sides

Solve: $7x+5=6x+2$.

Solution

Step 1. Choose which side will be the “variable” side—the other side will be the “constant” side.	The variable terms are 7*x* and 6*x*. Since 7 is greater than 6, we will make the left side the “*x*” side. The right side will be the “constant” side.	$\overset{\text{variable}}{7x+5} = \overset{\text{constant}}{6x+2}$
Step 2. Collect the variable terms to the “variable” side of the equation, using the addition or subtraction property of equality.	With the right side as the “constant” side, the 6*x* is out of place, so subtract 6*x* from both sides. Combine like terms. Now, the variable is only on the left side!	$7x - 6x + 5 = 6x - 6x + 2$ $x + 5 = 2$
Step 3. Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.	The right side is the “constant” side, so the 5 is out of place. Subtract 5 from both sides. Simplify.	$x + 5 - 5 = 2 - 5$ $x = -3$
Step 4. Make the coefficient of the variable equal 1, using the multiplication or division property of equality.	The coefficient of *x* is one. The equation is solved.	
Step 5. Check.	Let $x = -3$ Simplify. Add.	**Check:** $7x + 6 = 6x + 2$ $(-3) + 5 = 6(-3) + 2$ $-21 + 5 = -18 + 2$ $-16 = -16$ ✓

TRY IT 11

Solve: $12x + 8 = 6x + 2$.

Show answer

$x = -1$

We'll list the steps below so you can easily refer to them. But we'll call this the 'Beginning Strategy' because we'll be adding some steps later in this chapter.

HOW TO: Equations with variables and constants on both sides of the equation (beginning strategy)

1. Choose which side will be the "variable" side—the other side will be the "constant" side.
2. Collect the variable terms to the "variable" side of the equation, using the Addition or Subtraction Property of Equality.
3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the "variable" side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

EXAMPLE 12

Solve: $7a - 3 = 13a + 7.$

Solution
In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since 13 > 7, make the right side the "variable" side and the left side the "constant" side.

	$\overset{\text{constant}}{7a-3} = \overset{\text{variable}}{13a+7}$
Subtract $7a$ from both sides to remove the variable term from the left.	$7a - 7a - 3 = 13a - 7a + 7$
Combine like terms.	$-3 = 6a + 7$
Subtract 7 from both sides to remove the constant from the right.	$-3 - 7 = 6a + 7 - 7$
Simplify.	$-10 = 6a$
Divide both sides by 6 to make 1 the coefficient of a.	$\frac{-10}{6} = \frac{6a}{6}$
Simplify.	$-\frac{5}{3} = a$
Check:	$7a - 3 = 13a + 7$
Let $a = -\frac{5}{3}$.	$7\left(-\frac{5}{3}\right) - 3 \stackrel{?}{=} 13\left(-\frac{5}{3}\right) + 7$
	$-\frac{35}{3} - \frac{9}{3} \stackrel{?}{=} -\frac{65}{3} + \frac{21}{3}$
	$-\frac{44}{3} = -\frac{44}{3}$

TRY IT 12

Solve: $2a - 2 = 6a + 18$.

Show answer

$a = -5$

In the last example, we could have made the left side the “variable” side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

EXAMPLE 13

Solve: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$.

Solution
Since $\frac{5}{4} > \frac{1}{4}$, make the left side the "variable" side and the right side the "constant" side.

	$\overset{\text{variable}}{\frac{5}{4}x + 6} = \overset{\text{constant}}{\frac{1}{4}x - 2}$
Subtract $\frac{1}{4}x$ from both sides.	$\frac{5}{4}x - \frac{1}{4}x + 6 = \frac{1}{4}x - \frac{1}{4}x - 2$
Combine like terms.	$x + 6 = -2$
Subtract 6 from both sides.	$x + 6 - 6 = -2 - 6$
Simplify.	$x = -8$ Check: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$ Let $x = -8$. $\frac{5}{4}(-8) + 6 \overset{?}{=} \frac{1}{4}(-8) - 2$ $-10 + 6 \overset{?}{=} -2 - 2$ $-4 = -4$

TRY IT 13

Solve: $\frac{7}{8}x - 12 = -\frac{1}{8}x - 2$.

Show answer
$x = 10$

We will use the same strategy to find the solution for an equation with decimals.

EXAMPLE 14

Solve: $7.8x + 4 = 5.4x - 8$.

Solution
Since $7.8 > 5.4$, make the left side the "variable" side and the right side the "constant" side.

	variable side constant side $7.8x + 4 = 5.4x - 8$
Subtract $5.4x$ from both sides.	$7.8x - 5.4x + 4 = 5.4x - 5.4x - 8$
Combine like terms.	$2.4x + 4 = -8$
Subtract 4 from both sides.	$2.4x + 4 - 4 = -8 - 4$
Simplify.	$2.4x = -12$
Use the Division Propery of Equality.	$\frac{2.4x}{2.4} = \frac{-12}{2.4}$
Simplify.	$x = -5$
Check:	$7.8x + 4 = 5.4x - 5$
Let $x = -5$.	$7.8(-5) + 4 = 5.4(-5) - 8$
	$-39 + 4 \overset{?}{=} -27 - 8$
	$-35 = -35$ ✓

TRY IT 14

Solve: $2.8x + 12 = -1.4x - 9$.

Show answer
$x = -5$

Key Concepts

- **To determine whether a number is a solution to an equation**
    1. Substitute the number in for the variable in the equation.
    2. Simplify the expressions on both sides of the equation.
    3. Determine whether the resulting statement is true.
        - If it is true, the number is a solution.
        - If it is not true, the number is not a solution.
- **The addition property of equality**

  - For any numbers *a*, *b*, and *c*, if $a = b$, then $a + c = b + c$.
- **The subtraction property of equality**
  - For any numbers *a*, *b*, and *c*, if $a = b$, then $a - c = b - c$.
- **The division property of equality**
  - For any numbers *a*, *b*, and *c*, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
    When you divide both sides of an equation by any non-zero number, you still have equality.
- **The multiplication property of equality**
  - For any numbers *a*, *b*, and *c*, if *a* = *b* then *ac* = *bc*.

Glossary

solution of an equation

A value of a variable that makes a true statement when substituted into the equation.

2.1 Exercise Set

In the following exercises, determine whether the given value is a solution to the equation.

1. Is $y = -4$ a solution of $6y + 30 = 6$?
2. Is $u = 3$ a solution of $8u - 4 = 18$?

In the following exercises, solve each equation using the Subtraction and Addition Properties of Equality.

3. $x + 24 = 35$
4. $y + 45 = -66$
5. $b + \frac{1}{4} = \frac{3}{4}$
6. $p + 2.4 = -9.3$
7. $a - 45 = 76$
8. $m - 18 = -200$
9. $x - \frac{1}{3} = 2$
10. $y - 3.8 = 10$
11. $x - 165 = -420$
12. $z + 0.52 = -8.5$
13. $q + \frac{3}{4} = \frac{1}{2}$
14. $p - \frac{2}{5} = \frac{2}{3}$

In the following exercises, solve each equation using the Division and Multiplication Properties of Equality and check the solution.

15. $8x = 56$
16. $-5c = 55$
17. $-809 = 15y$
18. $-37p = -541$

19. $0.25z = 3.25$
20. $-13x = 0$
21. $\frac{x}{4} = 35$
22. $-20 = \frac{q}{-5}$
23. $\frac{y}{9} = -16$
24. $\frac{m}{-12} = 45$
25. $-y = 6$
26. $-v = -72$
27. $\frac{2}{3}y = 48$
28. $-\frac{5}{8}w = 40$
29. $-\frac{2}{5} = \frac{1}{10}a$
30. $-\frac{7}{10}x = -\frac{14}{3}$
31. $\frac{7}{12} = -\frac{3}{4}p$
32. $-\frac{5}{18} = -\frac{10}{9}u$

In the following exercises, solve the following equations with constants on both sides.

33. $21k = 20k - 11$
34. $8x + 27 = 11x$
35. $5z = 39 - 8z$
36. $4x + \frac{3}{4} = 3x$
37. $-11r - 8 = -7r$
38. $6x - 17 = 5x + 2$
39. $21 + 18f = 19f + 14$
40. $12q - 5 = 9q - 20$
41. $8c + 7 = -3c - 37$
42. $7x - 17 = -8x + 13$
43. $9p + 14 = 6 + 4p$
44. $3y - 4 = 12 - y$
45. $\frac{7}{4}m - 7 = \frac{3}{4}m - 13$
46. $11 - \frac{1}{5}a = \frac{4}{5}a + 4$
47. $\frac{5}{4}a + 15 = \frac{3}{4}a - 5$
48. $\frac{3}{5}p + 2 = \frac{4}{5}p - 1$
49. $13z + 6.45 = 8z + 23.75$
50. $6.6x - 18.9 = 3.4x + 54.7$

Answers

1. yes
2. no
3. $x = 11$
4. $y = -111$
5. $b = \frac{1}{2}$
6. $p = -11.7$
7. $a = 121$
8. $m = -182$
9. $x = \frac{7}{3}$
10. $y = 13.8$
11. $x = -255$
12. $z = -9.02$
13. $q = -1/4$
14. $p = \frac{16}{15}$
15. $x = 7$
16. $c = -11$
17. $y = -\frac{809}{15}$
18. $p = \frac{541}{37}$
19. $z = 13$
20. $x = 0$
21. $x = 140$
22. $q = 100$
23. $y = -144$
24. $m = -540$
25. $y = -6$
26. $v = 72$
27. $y = 72$
28. $w = -64$
29. $a = -4$
30. $x = \frac{20}{3}$
31. $p = -\frac{7}{9}$
32. $u = \frac{1}{4}$
33. $k = -11$
34. $x = 9$
35. $z = 3$
36. $x = -\frac{3}{4}$
37. $r = -2$
38. $x = 19$
39. $f = 7$
40. $q = -5$
41. $c = -4$
42. $x = 2$
43. $p = -\frac{8}{5}$
44. $y = 4$
45. $m = -6$

46. $a = 7$
47. $a = -40$
48. $p = 15$
49. $z = 3.46$
50. $x = 23$

Attributions

This chapter has been adapted from “Solve Equations Using the Subtraction and Addition Properties of Equality” in *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Adaptation Statement for more information.

2.2 Use a General Strategy to Solve Linear Equations

Learning Objectives

By the end of this section it is expected that you will be able to:

- Solve equations using a general strategy
- Classify equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

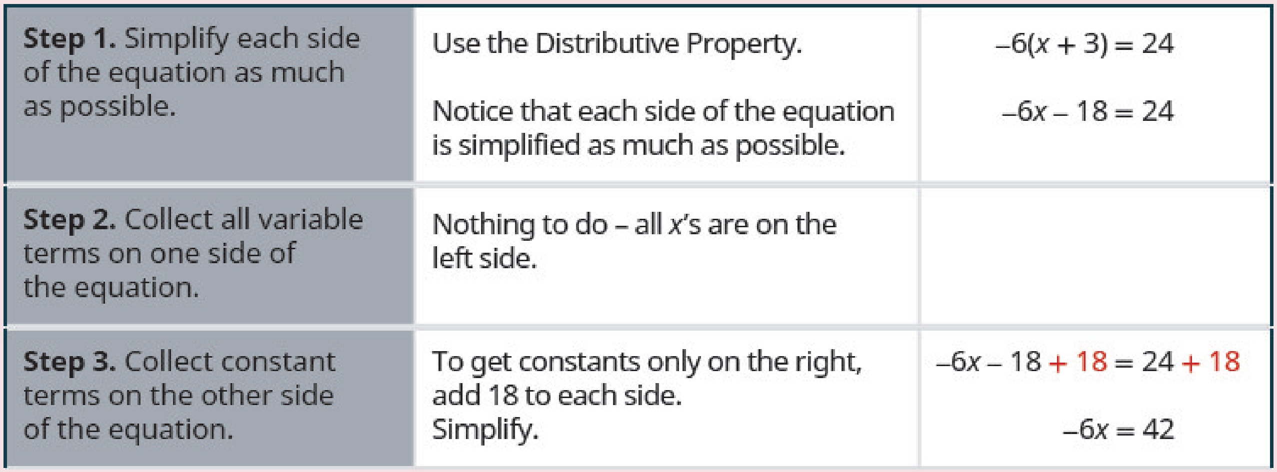
Beginning by simplifying each side of the equation makes the remaining steps easier.

EXAMPLE 1

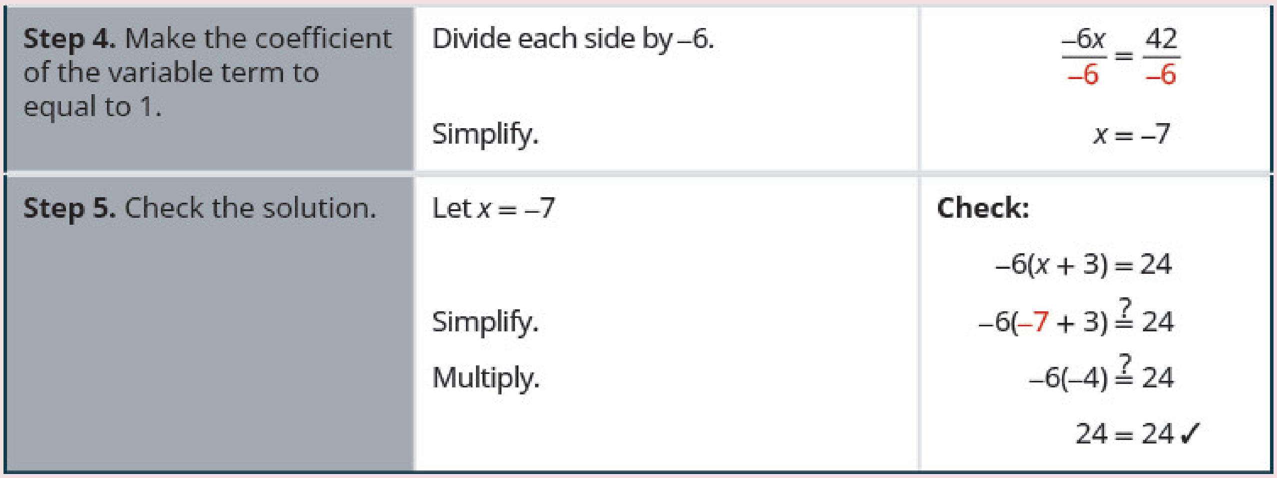
How to Solve Linear Equations Using the General Strategy

Solve: $-6\left(x+3\right)=24.$

Solution



Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all *x*'s are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$



Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by −6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\overset{?}{=}24$ $-6(-4)\overset{?}{=}24$ $24=24$ ✓

TRY IT 1

Solve: $5\left(x+3\right)=35.$

Show answer

$x=4$

General strategy for solving linear equations.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2

Solve: $(y+9)=8$.

Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$
Check:	$-(y+9)=8$
Let $y=-17$.	$-(-17+9)\overset{?}{=}8$
	$-(-8)\overset{?}{=}8$
	$8=8$ ✓

TRY IT 2

Solve: $(y+8)=-2$.

Show answer

$y=-6$

EXAMPLE 3

Solve: $5\left(a-3\right)+5=-10$.

Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$
Make the coefficient of the variable term to equal to 1 by dividing both sides by 5.	$\frac{5a}{5}=\frac{0}{5}$
Simplify.	$a=0$
Check:	$5\,(a-3)+5=-10$
Let $a=0$.	$5\,(0-3)+5\overset{?}{=}-10$
	$5(-3)+5\overset{?}{=}-10$
	$-15+5\overset{?}{=}-10$
	$-10=-10$ ✓

TRY IT 3

Solve: $2\left(m-4\right)+3=-1$.

Show answer

$m=2$

EXAMPLE 4

Solve: $\frac{2}{3}\left(6m-3\right)=8-m$.

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$
Let $m=2$.	$\frac{2}{3}(6\cdot 2-3)\overset{?}{=}8-2$
	$\frac{2}{3}(12-3)\overset{?}{=}6$
	$\frac{2}{3}(9)\overset{?}{=}6$
	$6=6$ ✓

TRY IT 4

Solve: $\frac{1}{3}\left(6u+3\right)=7-u$.

Show answer

$u=2$

EXAMPLE 5

Solve: $8-2\left(3y+5\right)=0$.

Solution

	$8-2(3y+5)=0$
Simplify—use the Distributive Property.	$8-6y-10=0$
Combine like terms.	$-6y-2=0$
Add 2 to both sides to collect constants on the right.	$-6y-2+2=0+2$
Simplify.	$-6y=2$
Divide both sides by -6.	$\frac{-6y}{-6}=\frac{2}{-6}$
Simplify.	$y=-\frac{1}{3}$
Check: Let $y=-\frac{1}{3}$.	$8-2(3y+5)=0$ $8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$ $8-2(-1+5)\overset{?}{=}0$ $8-2(4)\overset{?}{=}0$ $8-8\overset{?}{=}0$ $0=0$ ✓

TRY IT 5

Solve: $12-3\left(4j+3\right)=-17$.

Show answer

$j = \frac{5}{3}$

EXAMPLE 6

Solve: $4(x-1)-2=5(2x+3)+6$.

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\overset{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\overset{?}{=}5(-9+3)+6$
	$-22-2\overset{?}{=}5(-6)+6$
	$-24\overset{?}{=}-30+6$
	$-24=-24$ ✓

TRY IT 6

Solve: $6\left(p-3\right)-7=5\left(4p+3\right)-12$.

Show answer

$p=-2$

EXAMPLE 7

Solve: $10\left[3-8\left(2s-5\right)\right]=15\left(40-5s\right)$.

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to get the s’s to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$
Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute $s=-2$.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$
	$10[3-8(-9)]\stackrel{?}{=}15(50)$
	$10[3+72]\stackrel{?}{=}750$
	$10[75]\stackrel{?}{=}750$
	$750=750$ ✓

TRY IT 7

Solve: $6\left[4-2\left(7y-1\right)\right]=8\left(13-8y\right)$.

Show answer

$y=-\frac{17}{5}$

EXAMPLE 8

Solve: $0.36\left(100n+5\right)=0.6\left(30n+15\right)$.

Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract $18n$ to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$
Simplify.	$18n=7.2$
Divide.	$\frac{18n}{18}=\frac{7.2}{18}$
Simplify.	$n=0.4$
Check:	$0.36(100n+5)=0.6(30n+15)$
Let $n=0.4$.	$0.36(100(0.4)+5)\overset{?}{=}0.6(30(0.4)+15)$
	$0.36(40+5)\overset{?}{=}0.6(12+15)$
	$0.36(45)\overset{?}{=}0.6(27)$
	$16.2=16.2$ ✓

TRY IT 8

Solve: $0.55\left(100n+8\right)=0.6\left(85n+14\right)$.

Show answer

$n=1$

Classify Equations

When you solve the equation $7x+8=-13$, the solution is $x=-3$. This means the equation

$7x + 8 = -13$ is true when we replace the variable, *x*, with the value -3. We can show this by checking the solution $x = -3$ and evaluating $7x + 8 = -13$ for $x = -3$.

$$7(-3) + 8 \overset{?}{=} -13$$
$$-21 + 8 \overset{?}{=} -13$$
$$-13 = -13 \checkmark$$

If we evaluate $7x + 8$ for a different value of *x*, the left side will not be -13.

The equation $7x + 8 = -13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. Whether or not the equation $7x + 8 = -13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

Conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

Now let's consider the equation $2y + 6 = 2\left(y + 3\right)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

	$2y + 6 = 2(y + 3)$
Distribute.	$2y + 6 = 2y + 6$
Subtract $2y$ to get the y's to one side.	$2y - 2y + 6 = 2y - 2y + 6$
Simplify—the y's are gone!	$6 = 6$

But $6 = 6$ is true.

This means that the equation $2y + 6 = 2\left(y + 3\right)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an identity.

Identity

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is every real number.

What happens when we solve the equation $5z = 5z - 1$?

	$5z = 5z - 1$
Subtract $5z$ to get the constant alone on the right.	$5z - 5z = 5z - 5z - 1$
Simplify—the z's are gone!	$0 \neq -1$

But $0 \neq 1$.

Solving the equation $5z = 5z - 1$ led to the false statement $0 = -1$. The equation $5z = 5z - 1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

Contradiction

An equation that is false for all values of the variable is called a contradiction.

A contradiction has no solution.

EXAMPLE 9

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$6\left(2n-1\right)+3=2n-8+5\left(2n+1\right)$$

Solution

	$6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$
Distribute.	$12n - 6 + 3 = 2n - 8 + 10n + 5$
Combine like terms.	$12n - 3 = 12n - 3$
Subtract $12n$ to get the n's to one side.	$12n - 12n - 3 = 12n - 12n - 3$
Simplify.	$-3 = -3$
This is a true statement.	The equation is an identity. The solution is every real number.

TRY IT 9

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4+9\left(3x-7\right)=-42x-13+23\left(3x-2\right)$$

Show answer

identity; all real numbers

EXAMPLE 10

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$$10+4\left(p-5\right)=0$$

Solution

	$10+4(p-5)=0$
Distribute.	$10+4p-20=0$
Combine like terms.	$4p-10=0$
Add 10 to both sides.	$4p-10+10=0+10$
Simplify.	$4p=10$
Divide.	$\frac{4p}{4}=\frac{10}{4}$
Simplify.	$p=\frac{5}{2}$
The equation is true when $p=\frac{5}{2}$.	This is a conditional equation. The solution is $p=\frac{5}{2}$.

TRY IT 10

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$11\left(q+3\right)-5=19$$

Show answer

conditional equation; $q=\frac{9}{11}$

EXAMPLE 11

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$5m + 3\left(9 + 3m\right) = 2\left(7m - 11\right)$$

Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract $14m$ from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

TRY IT 11

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$12c + 5\left(5 + 3c\right) = 3\left(9c - 4\right)$$

Show answer

contradiction; no solution

Type of equation – Solution

Type of equation	**What happens when you solve it?**	**Solution**
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.
     Use the Multiplication or Division Property of Equality.
     State the solution to the equation.
  5. Check the solution.
     Substitute the solution into the original equation.

Glossary

conditional equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity
An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

2.2 Exercise Set

In the following exercises, solve each linear equation.

1. $21\left(y-5\right)=-42$
2. $-16\left(3n+4\right)=32$
3. $5\left(8+6p\right)=0$
4. $-\left(t-19\right)=28$
5. $21+2\left(m-4\right)=25$
6. $-6+6\left(5-k\right)=15$
7. $8\left(6t-5\right)-35=-27$
8. $-2\left(11-7x\right)+54=4$
9. $\frac{3}{5}\left(10x-5\right)=27$
10. $\frac{1}{4}\left(20d+12\right)=d+7$
11. $15-\left(3r+8\right)=28$
12. $-3-\left(m-1\right)=13$
13. $18-2\left(y-3\right)=32$
14. $35-5\left(2w+8\right)=-10$

15. $-2\left(a-6\right)=4\left(a-3\right)$
16. $5\left(8-r\right)=-2\left(2r-16\right)$
17. $9\left(2m-3\right)-8=4m+7$
18. $-15+4\left(2-5y\right)=-7\left(y-4\right)+4$
19. $5\left(x-4\right)-4x=14$
20. $-12+8\left(x-5\right)=-4+3\left(5x-2\right)$
21. $7\left(2n-5\right)=8\left(4n-1\right)-9$
22. $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$
23. $-\left(7m+4\right)-\left(2m-5\right)$ $=14-\left(5m-3\right)$
24. $5\left[9-2\left(6d-1\right)\right]=11\left(4-10d\right)-139$
25. $3\left[-14+2\left(15k-6\right)\right]=8\left(3-5k\right)-24$
26. $10\left[5\left(n+1\right)+4\left(n-1\right)\right]$ $=11\left[7\left(5+n\right)-\left(25-3n\right)\right]$
27. $4\left(2.5v-0.6\right)=7.6$
28. $0.2\left(p-6\right)=0.4\left(p+14\right)$
29. $0.5\left(16m+34\right)=-15$

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

30. $15y+32=2\left(10y-7\right)-5y+46$
31. $9\left(a-4\right)+3\left(2a+5\right)=7\left(3a-4\right)-6a+7$
32. $24\left(3d-4\right)+100=52$
33. $30\left(2n-1\right)=5\left(10n+8\right)$
34. $18u-51=9\left(4u+5\right)-6\left(3u-10\right)$
35. $5\left(p+4\right)+8\left(2p-1\right)=9\left(3p-5\right)-6\left(p-2\right)$
36. $9\left(4k-7\right)=11\left(3k+1\right)+4$
37. $60\left(2x-1\right)=15\left(8x+5\right)$
38. $36\left(4m+5\right)=12\left(12m+15\right)$
39. $11\left(8c+5\right)-8c=2\left(40c+25\right)+5$
40. **Coins.** Marta has $1.90 in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, *n*, by solving the equation $0.05n+0.10\left(2n-1\right)=1.90.$

Answers

1. $y=3$
2. $n=-2$
3. $p=-\frac{4}{3}$
4. $t=47$
5. $m=6$
6. $k=\frac{3}{2}$
7. $t=1$
8. $x=-2$
9. $x=5$
10. $d=1$
11. $r=-7$
12. $m=-15$
13. $y=-4$
14. $w=\frac{1}{2}$
15. $a=4$
16. $r=8$
17. $m=3$
18. $y=-3$
19. $x=34$
20. $x=-6$
21. $n=-1$
22. $a=-4$
23. $m=\frac{4}{5}$
24. $d=-3$
25. $k=\frac{3}{5}$
26. $n=-5$
27. $v=1$

28. $p = -34$
29. $m = -4$
30. identity; all real numbers
31. identity; all real numbers
32. conditional equation; $d = \frac{2}{3}$
33. conditional equation; $n = 7$
34. contradiction; no solution
35. contradiction; no solution
36. conditional equation; $k = 26$
37. contradiction; no solution
38. identity; all real numbers
39. identity; all real numbers
40. 8 nickels

Attributions

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2.3 Use a Problem Solving Strategy

Learning Objectives

By the end of this section it is expected that you will be able to:

- Translate to an equation and solve
- Translate and solve applications

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the equals sign. In the next few examples, we will translate sentences into equations and then solve the equations.

EXAMPLE 1

Translate and solve: Eleven more than *x* is equal to 54.

Solution

Translate.	$\underbrace{\text{Eleven more than } x}_{x+11} \quad \underbrace{\text{is equal to}}_{=} \quad \underbrace{54}_{54}$
Subtract 11 from both sides.	$x+11-11 \quad = \quad 54-11$
Simplify.	$x \quad = \quad 43$
Check: Is 54 eleven more than 43? $43+11 \overset{?}{=} 54$ $54 = 54$	

TRY IT 1

Translate and solve: Ten more than *x* is equal to 41.

Show answer

$x + 10 = 41; x = 31$

EXAMPLE 2

Translate and solve: The number 143 is the product of -11 and *y*.

Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The number 143}}_{143}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{the product of }-11\text{ and }y.}_{-11y}$
Divide by -11.	$\frac{143}{-11} = \frac{-11y}{-11}$
Simplify.	$-13 = y$
Check:	$143 = -11y$ $143 \overset{?}{=} -11(-13)$ $143 = 143$

TRY IT 2

Translate and solve: The number 132 is the product of −*12* and *y*.

Show answer

$132 = -12y; y = -11$

EXAMPLE 3

Translate and solve: The quotient of y and -4 is 68.

Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The quotient of } y \text{ and } -4}_{\frac{y}{-4}}\ \underbrace{\text{is}}_{=}\ \underbrace{68.}_{68}$
Multiply both sides by -4.	$-4\left(\frac{y}{-4}\right) = -4(68)$
Simplify.	$y = -272$
Check:	Is the quotient of y and -4 equal to 68?
Let $y = -272$.	Is the quotient of -272 and -4 equal to 68?
Translate.	$\frac{-272}{-4} \overset{?}{=} 68$
Simplify.	$68 = 68$

TRY IT 3

Translate and solve: The quotient of q and -8 is 72.

Show answer
$\frac{q}{-8} = 72; q = -576$

EXAMPLE 4

Translate and solve: Three-fourths of p is 18.

Solution
Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Translate.	$\underbrace{\text{Three-fourths of } p}_{\frac{3}{4}p}\ \underbrace{\text{is}}_{=}\ \underbrace{18.}_{18}$
Multiply both sides by $\frac{4}{3}$.	$\frac{4}{3}\cdot\frac{3}{4}p=\frac{4}{3}\cdot 18$
Simplify.	$p=24$
Check:	Is three-fourths of p equal to 18?
Let $p = 24.$	Is three-fourths of 24 equal to 18?
Translate.	$\frac{3}{4}\cdot\ 24\ \overset{?}{=}\ 18$
Simplify.	$18 = 18$

TRY IT 4

Translate and solve: Two-fifths of f is 16.

Show answer
$\frac{2}{5}f = 16; f = 40$

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real life situation. To begin, that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

EXAMPLE 5

How to Translate and Solve Applications

The Alec family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	The problem is about the weight of newspapers.	
Step 2. Identify what we are asked to find.	What are we asked to find?	"How much did the newspapers weigh the 2nd month?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Choose a variable.	Let w = weight of the newspapers the 1st month
Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.	Restate the problem. We know the weight of the newspapers the second month is 28 pounds. Translate into an equation, using the variable w.	Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds. Weight from 1st month plus 28 equals 57. $w + 28 = 57$
Step 5. Solve the equation using good algebra techniques.	Solve.	$w + 28 - 28 = 57 - 28$ $w = 29$
Step 6. Check the answer in the problem and make sure it makes sense.	Does 1st month's weight plus 2nd month's weight equal 57 pounds?	**Check:** Does 1st month's weight plus 2nd month's weight equal 57 pounds? $29 + 28 \stackrel{?}{=} 57$ $57 = 57$ ✓
Step 7. Answer the question with a complete sentence.	Write a sentence to answer "How much did the newspapers weigh the 2nd month?"	The 2nd month the newspapers weighed 29 pounds.

TRY IT 5

Translate into an algebraic equation and solve:

The Snider family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Show answer
7 pounds

HOW TO: Solve an application

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

EXAMPLE 6

Abdullah paid $28,675 for his new car. This was $875 less than the sticker price. What was the sticker price of the car?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	“What was the sticker price of the car?”
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let s = the sticker price of the car.
Step 4. Translate into an equation. Restate the problem in one sentence.	$28,675 is $875 less than the sticker price
Step 5. Solve the equation.	$28,675 is $875 less than s $\begin{aligned} 28,675 &= s-875 \\ 28,675+875 &= s-875+875 \\ 29,550 &= s \end{aligned}$
Step 6. Check the answer.	Is $875 less than $29,550 equal to $28,675? $\begin{aligned} 29,550-875 &\overset{?}{=} 28,675 \\ 28,675 &= 28,675 \end{aligned}$
Step 7. Answer the question with a complete sentence.	The sticker price of the car was $29,550.

TRY IT 6

Translate into an algebraic equation and solve:

Jaffrey paid $19,875 for her new car. This was $1,025 less than the sticker price. What was the sticker price of the car?

Show answer
$20,900

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems”.

EXAMPLE 7

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let n = the number.
Step 4. Translate.	
Restate the problem as one sentence.	$\underbrace{\text{The sum of twice a number and 7}}\quad\underbrace{\text{is}}\quad\underbrace{15}$
Translate into an equation.	$2n+7 \quad = \quad 15$
Step 5. Solve the equation.	$2n+7=15$
Subtract 7 from each side and simplify.	$2n=8$
Divide each side by 2 and simplify.	$n=4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	$2\cdot4+7 \overset{?}{=} 15$ $15 = 15$
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

TRY IT 7

The sum of four times a number and two is 14. Find the number.

Show answer
3

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 8

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.	
Choose a variable to represent the first number.	Let $n = 1^{\text{st}}$ number.
What do we know about the second number?	One number is five more than another.
	$n + 5 = 2^{\text{nd}}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.	The sum of the 1st number and the 2nd number is 21.
Translate into an equation.	$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}} \underbrace{=} \underbrace{21}$
Substitute the variable expressions.	$n \quad + \quad n+5 \quad = \quad 21$
Step 5. Solve the equation.	$n + n + 5 = 21$
Combine like terms.	$2n + 5 = 21$
Subtract 5 from both sides and simplify.	$2n = 16$
Divide by 2 and simplify.	$n = 8 \quad 1^{\text{st}}$ number
Find the second number, too.	$n + 5 \quad 2^{\text{nd}}$ number
	$8 + 5$
	13
Step 6. Check.	
Do these numbers check in the problem?	
Is one number 5 more than the other?	$13 \overset{?}{=} 8 + 5$
Is thirteen 5 more than 8? Yes.	$13 = 13$
Is the sum of the two numbers 21?	$8 + 13 \overset{?}{=} 21$
	$21 = 21$
Step 7. Answer the question.	The numbers are 8 and 13.

TRY IT 8

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

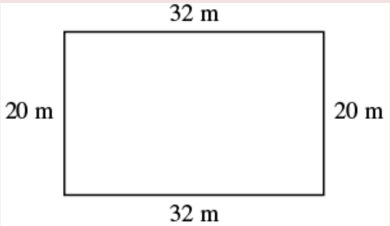
Show answer
9, 15

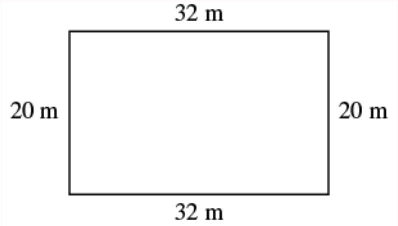
Now, we will use the problem solving strategy to solve some geometry problems.

EXAMPLE 9

The length of a rectangle is 32 metres and the width is 20 metres. Find a) the perimeter, and b) the area.

Solution

a)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the perimeter of a rectangle
Step 3. **Name.** Choose a variable to represent it.	Let *P* = the perimeter
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{P}_{P} \quad \underbrace{=}_{=} \quad \underbrace{2\ L}_{2(32)} \quad \underbrace{+}_{+} \quad \underbrace{2\ W}_{2(20)}$
Step 5. **Solve** the equation.	$P = 64 + 40$ $P = 104$
Step 6. **Check:**	$P \stackrel{?}{=} 104$ $20 + 32 + 20 + 32 \stackrel{?}{=} 104$ $104 = 104\ \checkmark$
Step 7. **Answer** the question.	The perimeter of the rectangle is 104 metres.

b)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of a rectangle
Step 3. **Name.** Choose a variable to represent it.	Let *A* = the area
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A} \underbrace{=}_{=} \underbrace{L}_{32\text{ m}} \underbrace{\cdot}_{\cdot} \underbrace{W}_{20\text{ m}}$
Step 5. **Solve** the equation.	$A = 640$
Step 6. **Check:**	$A \stackrel{?}{=} 640$ $32 \cdot 20 \stackrel{?}{=} 640$ $640 = 640$ ✓
Step 7. **Answer** the question.	The area of the rectangle is 60 square metres.

TRY IT 9

The length of a rectangle is 120 yards and the width is 50 yards. Find a) the perimeter and b) the area.

Show answer

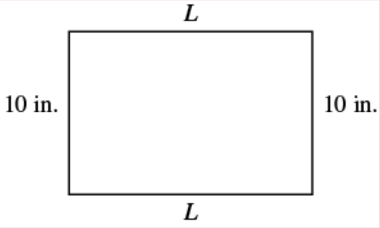
a) 340 yd

b) 6000 sq. yd

EXAMPLE 10

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length of the rectangle
Step 3. **Name.** Choose a variable to represent it.	Let L = the length
Step 4. **Translate.** Write the appropriate formula. Substitute.	$\underbrace{P}_{50} \quad \underbrace{=}_{=} \quad \underbrace{2L}_{2L} \quad \underbrace{+}_{+} \quad \underbrace{2W}_{2(10)}$
Step 5. **Solve** the equation.	$50 - 20 = 2L + 20 - 20$ $30 = 2L$ $\frac{30}{2} = \frac{2L}{2}$ $15 = L$
Step 6. **Check:**	$P = 50$ $15 + 10 + 15 + 10 \overset{?}{=} 50$ $50 = 50$ ✓
Step 7. **Answer** the question.	The length is 15 inches.

TRY IT 10

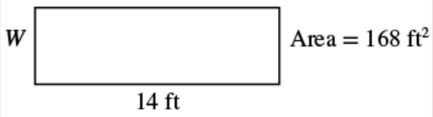
Find the length of a rectangle with a perimeter of 80 inches and width of 25 inches.

Show answer
15 in.

EXAMPLE 11

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the width of a rectangular room
Step 3. **Name.** Choose a variable to represent it.	Let *W* = width
Step 4.**Translate.** Write the appropriate formula and substitute in the given information.	$A = LW$ $168 = 14W$
Step 5. **Solve** the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. **Check:**	$A = LW$ $168 \overset{?}{=} 14 \cdot 12$ $168 = 168$ ✓
Step 7. **Answer** the question.	The width of the room is 12 feet.

TRY IT 11

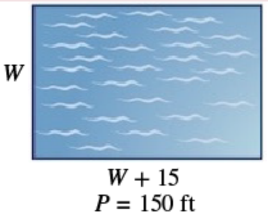
The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Show answer
26 ft

EXAMPLE 12

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length and width of the pool
Step 3. **Name.** Choose a variable to represent it. The length is 15 feet more than the width.	Let $W = \text{width}$ $W + 15 = \text{length}$
Step 4.**Translate.** Write the appropriate formula and substitute.	$\underbrace{P}_{150} \quad \underbrace{=}_{=} \quad \underbrace{2L}_{2(w+15)} \quad \underbrace{+}_{+} \quad \underbrace{2W}_{2w}$
Step 5. **Solve** the equation.	$150 = 2w + 30 + 2w$ $150 = 4w + 30$ $120 = 4w$ $30 = w$ the width of the pool $w + 15$ the length of the pool $30 + 15$ 45
Step 6. **Check:**	$p = 2L + 2W$ $150 \overset{?}{=} 2(45) + 2(30)$ $150 = 150$
Step 7. **Answer** the question.	The length of the pool is 45 feet and the width is 30 feet.

TRY IT 12

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Show answer
30 ft, 70 ft

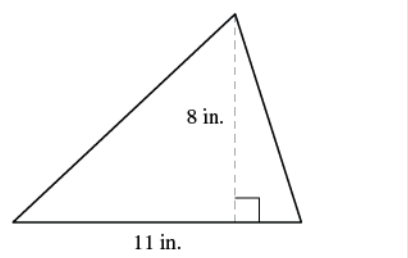
The formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height.

To find the area of the triangle, you need to know its base and height.

EXAMPLE 13

Find the area of a triangle whose base is 11 inches and whose height is 8 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the triangle
Step 3. **Name.** Choose a variable to represent it.	let *A* = area of the triangle
Step 4.**Translate.** Write the appropriate formula. Substitute.	$\underbrace{A}_{A}\ \underbrace{=}_{=}\ \underbrace{\frac{1}{2}}_{\frac{1}{2}}\ \underbrace{\cdot}_{\cdot}\ \underbrace{b}_{11}\ \underbrace{\cdot}_{\cdot}\ \underbrace{h}_{8}$
Step 5. **Solve** the equation.	$A = 44$ square inches
Step 6. **Check:**	$A = \frac{1}{2}bh$ $44 \overset{?}{=} \frac{1}{2}(11)8$ $44 = 44$ ✓
Step 7. **Answer** the question.	The area is 44 square inches.

TRY IT 13

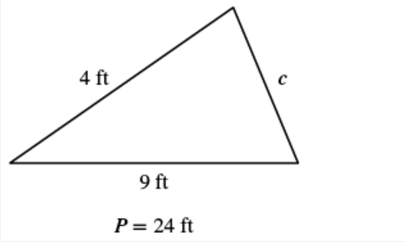
Find the area of a triangle with base 13 inches and height 2 inches.

Show answer
13 sq. in.

EXAMPLE 14

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 4 feet and 9 feet. How long is the third side?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	length of the third side of a triangle
Step 3. **Name.** Choose a variable to represent it.	Let *c* = the third side
Step 4.**Translate.** Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{24} = \underbrace{a}_{4} + \underbrace{b}_{9} + \underbrace{c}_{c}$
Step 5. **Solve** the equation.	$24 = 13 + c$ $11 = c$
Step 6. **Check:**	$P = a + b + c$ $24 \overset{?}{=} 4 + 9 + 11$ $24 = 24\checkmark$
Step 7. **Answer** the question.	The third side is 11 feet long.

TRY IT 14

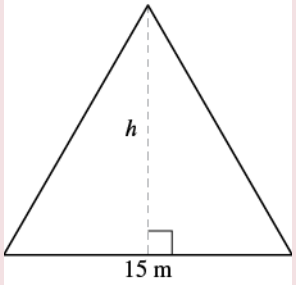
The perimeter of a triangular garden is 24 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Show answer
8 ft

EXAMPLE 15

The area of a triangular church window is 90 square metres. The base of the window is 15 metres. What is the window's height?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	height of a triangle
Step 3. **Name.** Choose a variable to represent it.	Let *h* = the height
Step 4.**Translate.** Write the appropriate formula. Substitute in the given information.	$\underbrace{A}_{90} \underbrace{=}_{=} \underbrace{\frac{1}{2}}_{\frac{1}{2}} \underbrace{\cdot}_{\cdot} \underbrace{b}_{15} \underbrace{\cdot}_{\cdot} \underbrace{h}_{h}$
Step 5. **Solve** the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. **Check:**	$A = \frac{1}{2}bh$ $90 \overset{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90$ ✓
Step 7. **Answer** the question.	The height of the triangle is 12 metres.

TRY IT 15

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Show answer
14 in.

Key Concepts

- **To translate a sentence to an equation**
    1. Locate the "equals" word(s). Translate to an equal sign (=).
    2. Translate the words to the left of the "equals" word(s) into an algebraic expression.
    3. Translate the words to the right of the "equals" word(s) into an algebraic expression.
- **To solve an application**
    1. Read the problem. Make sure all the words and ideas are understood.
    2. Identify what we are looking for.
    3. Name what we are looking for. Choose a variable to represent that quantity.
    4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
    5. Solve the equation using good algebra techniques.
    6. Check the answer in the problem and make sure it makes sense.
    7. Answer the question with a complete sentence.

2.3 Exercise Set

In the following exercises, translate to an equation and then solve it.

1. Nine more than x is equal to 52.
2. Ten less than m is -14.
3. The sum of y and -30 is 40.
4. The difference of n and $\frac{1}{6}$ is $\frac{1}{2}$.
5. The sum of $-4n$ and $5n$ is -82.
6. 133 is the product of -19 and *n*.
7. The quotient of b and -6 is 18.
8. Three-tenths of *x* is 15.
9. The sum of two-fifths and *f* is one-half.
10. The difference of *q* and one-eighth is three-fourths

In the following exercises, translate into an equation and solve.

11. Avril rode her bike a total of 18 miles, from home to the library and then to the beach. The distance from Avril's house to the library is 7 miles. What is the distance from the library to the beach?
12. Eva's daughter is 15 years younger than her son. Eva's son is 22 years old. How old is her daughter?
13. For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?

14. Arjun's temperature was 0.7 degrees higher this morning than it had been last night. His temperature this morning was 101.2 degrees. What was his temperature last night?
15. Ron's paycheck this week was $17.43 less than his paycheck last week. His paycheck this week was $103.76. How much was Ron's paycheck last week?

In the following exercises, solve each number word problem

16. The sum of a number and eight is 12. Find the number.
17. The difference of twice a number and seven is 17. Find the number.
18. Three times the sum of a number and nine is 12. Find the number.
19. One number is six more than the other. Their sum is 42. Find the numbers.
20. The sum of two numbers is -45. One number is nine more than the other. Find the numbers.
21. One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.
22. One number is one more than twice another. Their sum is -5. Find the numbers.

In the following exercises, find the a) perimeter and b) area of each rectangle.

23. The length of a rectangle is 85 feet and the width is 45 feet.
24. A rectangular room is 15 feet wide by 14 feet long.

In the following exercises, solve.

25. Find the length of a rectangle with perimeter 124 inches and width 38 inches.
26. Find the width of a rectangle with perimeter 92 metres and length 19 metres.
27. The area of a rectangle is 414 square metres. The length is 18 metres. What is the width?
28. The length of a rectangle is 9 inches more than the width. The perimeter is 46 inches. Find the length and the width.
29. The perimeter of a rectangle is 58 metres. The width of the rectangle is 5 metres less than the length. Find the length and the width of the rectangle.
30. The width of the rectangle is 0.7 metres less than the length. The perimeter of a rectangle is 52.6 metres. Find the dimensions of the rectangle.
31. The perimeter of a rectangle is 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.
32. The length of a rectangle is 3 metres less than twice the width. The perimeter is 36 metres. Find the length and width.
33. The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?
34. The area of a rectangular roof is 2310 square metres. The length is 42 metres. What is the width?

35. The perimeter of a rectangular courtyard is 160 feet. The length is 10 feet more than the width. Find the length and the width.
36. The width of a rectangular window is 40 inches less than the height. The perimeter of the doorway is 224 inches. Find the length and the width.

In the following exercises, solve using the properties of triangles.

37. A triangular flag has base of 8 foot and height of 1.5 feet. What is its area?
38. What is the base of a triangle with an area of 207 square inches and height of 18 inches?
39. The perimeter of a triangular reflecting pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?
40. The perimeter of a triangle is 39 feet. One side of the triangle is 1 foot longer than the second side. The third side is 2 feet longer than the second side. Find the length of each side.

Answers

1. $x+9=52; x=43$
2. $m-10=-14; m=-4$
3. $y+(-30)=40; y=70$
4. $n-\frac{1}{6}=\frac{1}{2}; \frac{2}{3}$
5. $-4n+5n=-82; -82$
6. 133=-19n; n=-7
7. $\frac{b}{-6}=18; b=-108$
8. 3/10x=15; x=50
9. $\frac{2}{5}+f=\frac{1}{2}; f=\frac{1}{10}$
10. $q-\frac{1}{8}=\frac{3}{4}; q=\frac{7}{8}$
11. 11 miles
12. 7 years old
13. 21 pounds
14. 100.5 degrees
15. $\$121.19$
16. 4
17. 12
18. -5
19. 18,24
20. $-18, -27$
21. 32,46
22. $-2, -3$
23.
    a. 260 ft
    b. 3825 sq. ft
24.
    a. 58 ft
    b. 210 sq. ft
25. 24 inches
26. 27 metres
27. 23 m
28. 7 in., 16 in.
29. 17 m, 12 m
30. 13.5 m, 12.8 m
31. 25 ft, 50 ft
32. 7 m, 11 m
33. 26 in.
34. 55 m
35. 35 ft, 45 ft
36. 76 in., 36 in.
37. 6 sq. ft
38. 23 in.
39. 11 ft
40. 12 ft, 13 ft, 14 ft

2.4 Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section it is expected that you will be able to:

- Solve a formula for a specific variable
- Use formulas to solve applications

Solve a Formula for Specific Variable

In the last section we have worked with some geometry formulas when solving problems. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

EXAMPLE 1

Solve the formula $d = rt$ for t:

a. when $d = 520$ and $r = 65$
b. in general

Solution
We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

a) when $d = 520$ and $r = 65$		b) in general	
Write the formula.	$d = rt$	Write the formula.	$d = rt$
Substitute.	$520 = 65t$		
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$	Divide, to isolate t.	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$	Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

TRY IT 1

Solve the formula $d = rt$ for r:

a) when $d = 180$ and $t = 4$ b) in general

Show answer

a) $r = 45$ b) $r = \frac{d}{t}$

EXAMPLE 2

Solve the formula $A = \frac{1}{2}bh$ for h:

a) when $A = 90$ and $b = 15$ b) in general

Solution

a) when $A = 90$ and $b = 15$		b) in general	
Write the formula.	$A = \frac{1}{2}bh$	Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$		
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2} 15h$	Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$
Simplify.	$180 = 15h$	Simplify.	$2A = bh$
Solve for h.	$12 = h$	Solve for h.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$.

TRY IT 2

Use the formula $A = \frac{1}{2}bh$ to solve for h:

a) when $A = 170$ and $b = 17$ b) in general

Show answer
a) $h = 20$ b) $h = \frac{2A}{b}$

The formula $I = Prt$ is used to calculate simple interest, *I*, for a principal, *P*, invested at rate, *r*, for *t* years.

EXAMPLE 3

Solve the formula $I = Prt$ to find the principal, P:

a) when $I = \$5,600, r = 4\%, t = 7\ years$ b) in general

Solution

a) $I = \$5,600, r = 4\%, t = 7 years$		b) in general	
Write the formula.	$I = Prt$	Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$		
Simplify.	$5600 = P(0.28)$	Simplify.	$I = P(rt)$
Divide, to isolate *P*.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	Divide, to isolate *P*.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20,000 = P$	Simplify.	$\frac{I}{rt} = P$
The principal is	$20,000		$P = \frac{I}{rt}$

TRY IT 3

Use the formula $I = Prt$ to find the principal, P:

a) when $I = \$2,160, r = 6\%, t = 3\ years$ b) in general

Show answer

a) $12,000 b) $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

EXAMPLE 4

Solve the formula $3x + 2y = 18$ for *y*:

a) when $x = 4$ b) in general

Solution

a) when $x = 4$		b) in general	
	$3x + 2y = 18$		$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$		
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$	Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$	Simplify.	$y = -\frac{3x}{2} + 9$

TRY IT 4

Solve the formula $3x + 4y = 10$ for *y*:

a) when $x = \frac{14}{3}$ b) in general

Show answer

a) $y = 1$ b) $y = \frac{10-3x}{4}$

Now we will solve a formula in general without using numbers as a guide.

EXAMPLE 5

Solve the formula $P = a + b + c$ for a.

Solution

We will isolate a on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

TRY IT 5

Solve the formula $P = a + b + c$ for *b*.

Show answer
$b = P - a - c$

EXAMPLE 6

Solve the formula $6x + 5y = 13$ for *y*.

Solution

	$6x + 5y = 13$
Subtract $6x$ from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$
Simplify.	$y = \frac{13 - 6x}{5}$

The fraction is simplified. We cannot divide $13 - 6x$ by 5

TRY IT 6

Solve the formula $4x + 7y = 9$ for *y*.

Show answer
$y = \frac{9-4x}{7}$

Geometric formulas often need to be solved for another variable, too. The formula $V = \frac{1}{3}\pi r^2 h$ is used to find the volume of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

EXAMPLE 7

Solve the formula $V = \frac{1}{3}\pi r^2 h$ for *h*.

Write the formula.	$V = \frac{1}{3}\pi r^2 h$
Remove the fraction on the right.	$3 \cdot V = 3 \cdot \frac{1}{3}\pi r^2 h$
Simplify.	$3V = \pi r^2 h$
Divide both sides by πr^2.	$\frac{3V}{\pi r^2} = h$

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h = \frac{3V}{\pi r^2}$.

TRY IT 7

Use the formula $A = \frac{1}{2}bh$ to solve for *b*.

Show answer

$b = \frac{2A}{h}$

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

EXAMPLE 8

Solve the formula $C = \frac{5}{9}\left(F - 32\right)$ for *F*.

Write the formula.	$C=\frac{5}{9}(F-32)$
Remove the fraction on the right.	$\frac{9}{5}C=\frac{9}{5}\cdot\frac{5}{9}(F-32)$
Simplify.	$\frac{9}{5}C=(F-32)$
Add 32 to both sides.	$\frac{9}{5}C+32=F$

We can now use the formula $F=\frac{9}{5}C+32$ to find the Fahrenheit temperature when we know the Celsius temperature.

TRY IT 8

Solve the formula $F=\frac{9}{5}C+32$ for *C*.

Show answer

$C=\frac{5}{9}(F-32)$

Use Formulas to Solve Applications

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for “speed.” The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car’s cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

Distance, Rate, and Time

For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:

$$d=rt \qquad \text{where} \qquad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

HOW TO: Solve an application (with a formula).

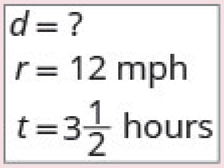
1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

EXAMPLE 9

Adam rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let *d* = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	 $d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
	 12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 48 miles in 4 hours. 42 miles in $3\frac{1}{2}$ hours is reasonable
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

TRY IT 9

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

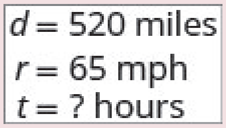
Show answer

330 miles

EXAMPLE 10

Rey is planning to drive from his house in Saskatoon to visit his grandmother in Winnipeg, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let *t* = time.
	 $d = 520$ miles $r = 65$ mph $t = ?$ hours *d = 600 km r = 75 km/h t = ? hours*
Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	$d = rt$ $520 \overset{?}{=} 65 \cdot 8$ $520 = 520$
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

TRY IT 10

Lee wants to drive from Kamloops to his brother's apartment in Banff, a distance of 495 km. If he drives at a steady rate of 90 km/h, how many hours will the trip take?

Show answer
5 1/2 hours

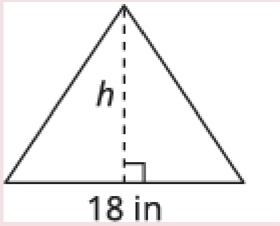
When we solve geometry applications, we adapt our problem solving strategy, use some common geometry formulas, and draw a figure and label it with given information.

The next example involves the area of a triangle. The area of a triangle is one-half the base times the height. We can write this as $A = \frac{1}{2}bh$, where *b* = length of the base and *h* = height.

EXAMPLE 11

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area = 126 sq. in.
	
Step 4. Translate.	
Write the appropriate formula.	$A = \frac{1}{2}bh$
Substitute in the given information.	$126 = \frac{1}{2} \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$
Divide both sides by 9.	$14 = h$
Step 6. Check. $A = \frac{1}{2}bh$ $126 \stackrel{?}{=} \frac{1}{2} \cdot 18 \cdot 14$ $126 = 126$	
Step 7. Answer the question.	The height of the triangle is 14 inches.

TRY IT 11

The area of a triangular church window is 90 square metres. The base of the window is 15 metres. What is the window's height?

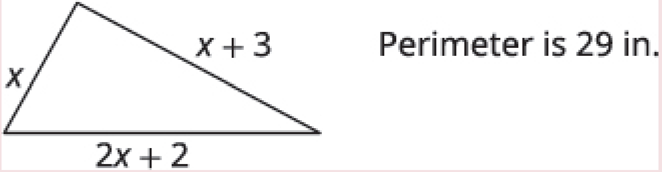
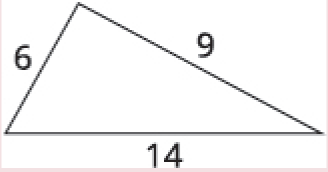
Show answer
The window's height is 12 metres.

The next example is about the perimeter of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where *a*, *b*, and *c* are the lengths of the sides.

EXAMPLE 12

One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the lengths of the three sides of a triangle
Step 3. Name. Choose a variable to represent the length of the first side.	$\text{Let } x = \text{length of } 1^{\text{st}} \text{ side.}$ $x + 3 = \text{length of } 2^{\text{nd}} \text{ side}$ $2x + 2 = \text{length of } 3^{\text{rd}} \text{ side}$ 
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$P = a + b + c$ $29 = x + (x + 3) + (2x + 2)$
Step 5. Solve the equation.	$29 = 4x + 5$ $24 = 4x$ $6 = x$ length of first side $x + 3$ length of second side $6 + 3$ 9 $2x + 2$ length of second side $2 \cdot 6 + 2$ 14
Step 6. Check.  $29 \overset{?}{=} 6 + 9 + 14$ $29 = 29$	
Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

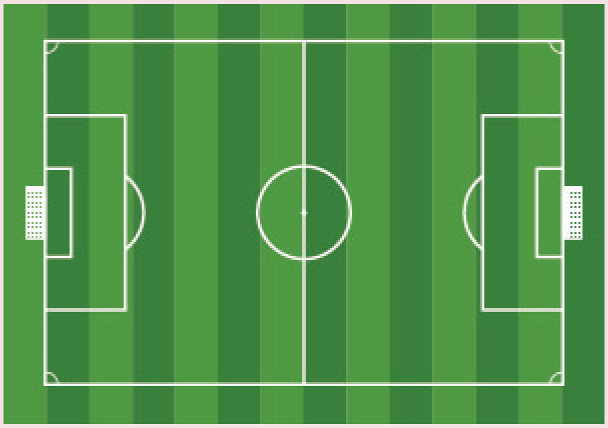
TRY IT 12

One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

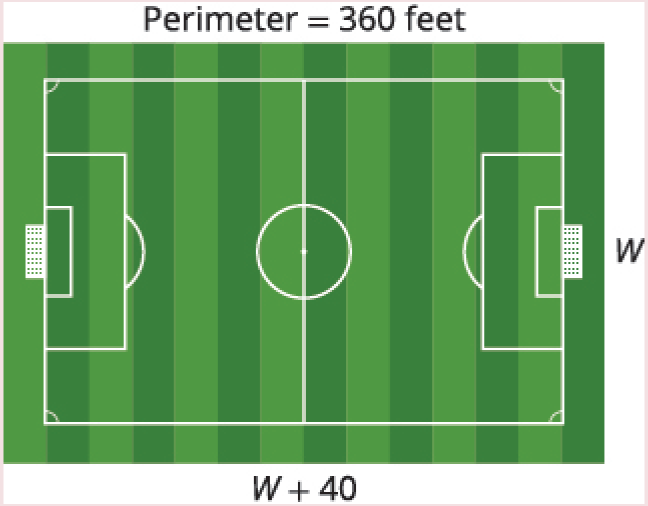
Show answer
The lengths of the sides of the triangle are 5, 11 and 12 inches.

EXAMPLE 13

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.



Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and width of the soccer field
Step 3. Name. Choose a variable to represent it. The length is 40 feet more than the width. Draw the figure and label it with the given information.	Let w = width. $w + 40 =$ length 
Step 4. Translate. Write the appropriate formula and substitute.	$P = 2L + 2W$ $360 = 2(w + 40) + 2w$
Step 5. Solve the equation.	$360 = 2w + 80 + 2w$ $360 = 4w + 80$ $280 = 4w$ $70 = w$ the width of the field $w + 40$ the length of the field $70 + 40$ 110
Step 6. Check. $P = 2L + 2W$ $360 \stackrel{?}{=} 2(110) + 2(70)$ $360 = 360$	
Step 7. Answer the question.	The length of the soccer field is 110 feet and the width is 70 feet.

TRY IT 13

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Show answer
The length of the swimming pool is 70 feet and the width is 30 feet.

Key Concepts

- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.
- **To Solve an Application (with a formula)**
    1. **Read** the problem. Make sure all the words and ideas are understood.
    2. **Identify** what we are looking for.
    3. **Name** what we are looking for. Choose a variable to represent that quantity.
    4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
    5. **Solve** the equation using good algebra techniques.
    6. **Check** the answer in the problem and make sure it makes sense.
    7. **Answer** the question with a complete sentence.
- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: $d = rt$ where *d* = distance, *r* = rate, *t* = time.

2.4 Exercise Set

In the following exercises, use the formula $d = rt$.

1. Solve for t
    a. when $d = 240$ and $r = 60$
    b. in general
2. Solve for r
    a. when $d = 420$ and $t = 6$
    b. in general

In the following exercises, solve the formula .

3. Solve the formula $A = \frac{1}{2}bh$ for h
    a. when $A = 176$ and $b = 22$
    b. in general
4. Solve the formula I=Prt for the principal, P for

   a. $I = \$5,480$,$r = 4\%$,$t = 7$ years
   b. in general
5. Solve for the time, t for
   a. $I = \$2,376,$ $P = \$9,000,$ $r = 4.4\%$
   b. in general
6. Solve the formula $2x + 3y = 12$ for y
   a. when $x = 3$
   b. in general
7. Solve the formula $3x - y = 7$ for y
   a. when $x = -2$
   b. in general
8. Solve $180 = a + b + c$ for a.
9. Solve the formula $-4x + y = -6$ for y.
10. Solve the formula $x - y = -4$ for y.
11. Solve the formula $P = 2L + 2W$ for L.
12. Solve the formula $C = \pi d$ for d.
13. Solve the formula $V = LWH$ for L.
14. Solve the formula $A = \frac{1}{2}d_1d_2$ for d_1.
15. Solve the formula $A = \frac{1}{2}d_1d_2$ for d_2.
16. Solve the formula $A = \frac{1}{2}h\left(b_1 + b_2\right)$ for b_1.
17. Solve the formula $A = \frac{1}{2}h\left(b_1 + b_2\right)$ for b_2.
18. Solve the formula $h = 54t + \frac{1}{2}at^2$ for *a*.
19. Solve the formula $h = 48t + \frac{1}{2}at^2$ for *a*.

In the following exercises, solve using a geometry formula.

20. A triangular flag has area 0.75 square feet and height 1.5 foot. What is its base?
21. What is the base of triangular window with area 207 square inches and height 18 inches?
22. The width of a rectangle is seven metres less than the length. The perimeter is 58 metres. Find the length and width.
23. The width of the rectangle is 0.7 metres less than the length. The perimeter of a rectangle is 52.6 metres. Find the dimensions of the rectangle.
24. The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.
25. The length of the rectangle is three metres less than twice the width. The perimeter of a rectangle is 36 metres. Find the dimensions of the rectangle.
26. The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.
27. One side of a triangle is twice the smallest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.
28. The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.
29. A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

In the following exercises, solve.

30. Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?
31. Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?

32. Marta is taking the bus from Abbotsford to Cranbrook. The distance is 774 km and the bus travels at a steady rate of 86 miles per hour. How long will the bus ride be?
33. Halle wants to ride his bike from Golden, BC to Banff, AB. The distance is 140 km. If he rides at a steady rate of 20 km/h, how many hours will the trip take?
34. Alejandra is driving to Prince George, 450 km away. If she wants to be there in 6 hours, at what rate does she need to drive?
35. Philip got a ride with a friend from Calgary to Kelowna, a distance of 890 km. If the trip took 10 hours, how fast was the friend driving?
36. **Converting temperature.** Yon was visiting the United States and he saw that the temperature in Seattle one day was 50^{o} Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

Answers

1. a. $t = 4$
   b. $t = \frac{d}{r}$
2. a. $r = 70$
   b. $r = \frac{d}{t}$
3. a. $h = 8$
   b. $h = \frac{2A}{b}$
4. a. $P = \$19571.43$
   b. $P = \frac{I}{rt}$
5. a. $t = 6$
   b. $t = \frac{I}{Pr}$
6. a. $y = 2$
   b. $y = \frac{12-2x}{3}$
7. a. $y = -13$
   b. $y = 3x - 7$
8. $a = 180 - b - c$
9. $y = 4x - 6$
10. $y = x + 4$
11. $L = \frac{P-2W}{2}$
12. $d = \frac{C}{pi}$
13. $L = \frac{V}{WH}$
14. $d_1 = \frac{2A}{d_2}$
15. $d_2 = \frac{2A}{d_1}$
16. $b_1 = \frac{2A}{h} - b_2$
17. $b_2 = \frac{2A}{h} - b_1$
18. $a = \frac{2h-108t}{t^2}$
19. $a = \frac{2h-96t}{t^2}$
20. 1 foot
21. 23 inches
22. 18 metres, 11 metres
23. 13.5 m, 12.8 m
24. 25 ft, 50 ft
25. 7 m, 11 m
26. 12 ft, 13 ft, 14 ft
27. 3 ft, 6 ft, 8 ft
28. 120 yd, 160 yd
29. 40 ft, 85 ft
30. 290 miles
31. 30 miles
32. 9 hours.
33. 7 hours
34. 75 km/h
35. 89 km/h
36. 10°C

2.5 Solve Inequalities

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section, you will be able to:

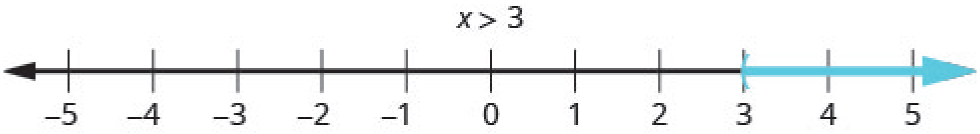
- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Graph Inequalities on the Number Line

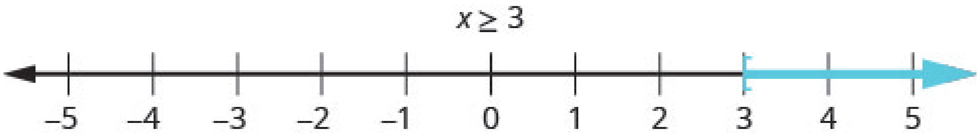
Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, '*x* could be 4'? That's correct, but *x* could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in (Figure). Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.
The inequality $x > 3$ is graphed on this number line.



The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in (Figure).
The inequality $x \geq 3$ is graphed on this number line.



Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

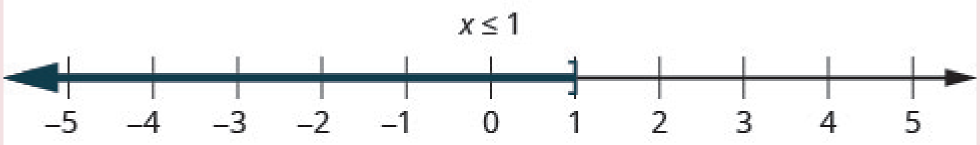
EXAMPLE 1

Graph on the number line:

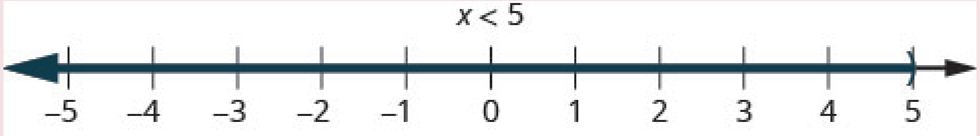
a) $x \leq 1$ b) $x < 5$ c) $x > -1$

Solution

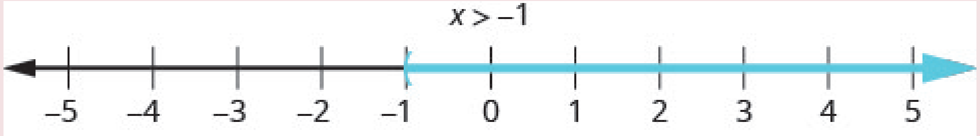
a) $x \leq 1$
This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at $x = 1$ to show that it is included.



b) $x < 5$
This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at $x = 5$ to show it is not included.



c) $x > -1$
This means all numbers greater than -1, but not including -1. We shade in all the numbers on the number line to the right of -1, then put a parenthesis at $x = -1$ to show it is not included.

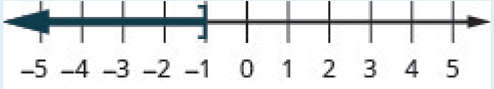


TRY IT 1

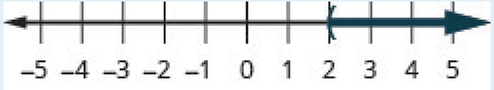
Graph on the number line: a) $x \leq -1$ b) $x > 2$ c) $x < 3$

Show answer

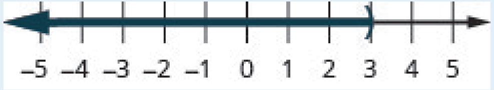
a)



b)

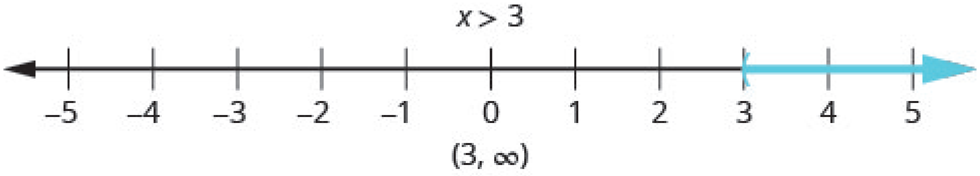


c)



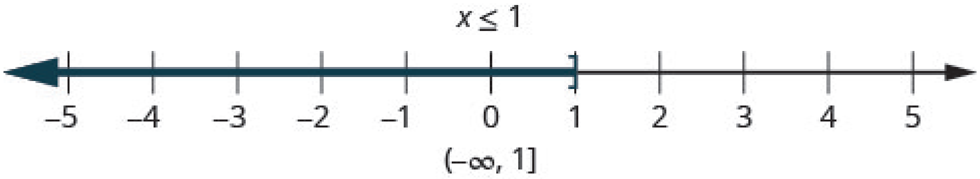
We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as 'infinity'. It is not an actual number. (Figure) shows both the number line and the interval notation.

The inequality $x > 3$ is graphed on this number line and written in interval notation.



The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as 'negative infinity'. (Figure) shows both the number line and interval notation.

The inequality $x \leq 1$ is graphed on this number line and written in interval notation.



Inequalities, Number Lines, and Interval Notation

Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in (Figure).

The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

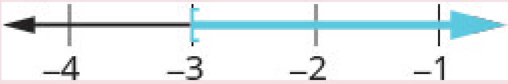
EXAMPLE 2

Graph on the number line and write in interval notation.

a) $x \geq -3$ b) $x < 2.5$ c) $x \leq -\frac{3}{5}$

Solution

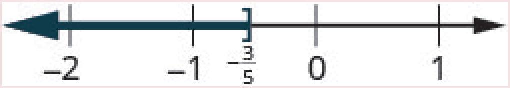
a)

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

b)

	$x < 2.5$
Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

c)

	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

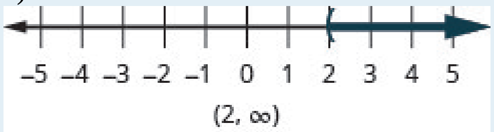
TRY IT 2

Graph on the number line and write in interval notation:

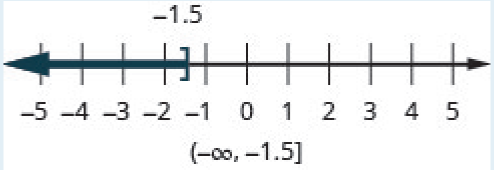
a) $x > 2$ b) $x \leq -1.5$c) $x \geq \frac{3}{4}$

Show answer

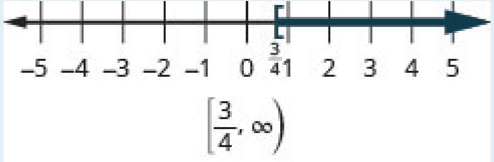
a)



b)



c)



Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

Properties of Equality

$$\begin{array}{llll} \text{Subtraction Property of Equality} & & & \\ \text{For any numbers } a, b, \text{and } c, & & & \\ \text{if} & a & = & b, \\ \text{then} & a - c & = & b - c. \end{array} \qquad \begin{array}{llll} \text{Addition Property of Equality} & & & \\ \text{For any numbers } a, b, \text{and } c, & & & \\ \text{if} & a & = & b, \\ \text{then} & a + c & = & b + c. \end{array}$$

Similar properties hold true for inequalities.

For example, we know that −4 is less than 2.	$-4 < 2$
If we subtract 5 from both quantities, is the left side still less than the right side?	$-4-5 \; ? \; 2-5$
We get −9 on the left and −3 on the right.	$-9 \; ? \; -3$
And we know −9 is less than −3.	$-9 < -3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

Properties of Inequality

Subtraction Property of Inequality

For any numbers $a, b,$ and $c,$

$$\begin{array}{llcl} \text{if} & a & < & b \\ \text{then} & a-c & < & b-c. \end{array}$$

$$\begin{array}{llcl} \text{if} & a & > & b \\ \text{then} & a-c & > & b-c. \end{array}$$

Addition Property of Inequality

For any numbers $a, b,$ and $c,$

$$\begin{array}{llcl} \text{if} & a & < & b \\ \text{then} & a+c & < & b+c. \end{array}$$

$$\begin{array}{llcl} \text{if} & a & > & b \\ \text{then} & a+c & > & b+c. \end{array}$$

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x + 5 > 9$, the steps would look like this:

	$x+5>9$
Subtract 5 from both sides to isolate x.	$x+5-5>9-5$
Simplify.	$x>4$

Any number greater than 4 is a solution to this inequality.

EXAMPLE 3

Solve the inequality $n - \frac{1}{2} \leq \frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution

	$n - \frac{1}{2} \leq \frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	$n - \frac{1}{2} + \frac{1}{2} \leq \frac{5}{8} + \frac{1}{2}$
Simplify.	$n \leq \frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	.

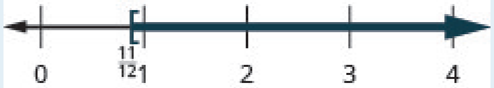
TRY IT 3

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p - \frac{3}{4} \geq \frac{1}{6}$

Show answer

$p \geq \frac{11}{12}$



$\left[\frac{11}{12}, \infty\right)$

Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don’t divide by 0).

Properties of Equality

Division Property of Equality
For any numbers a, b, c, and $c \neq 0$,

$$\begin{array}{llcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array}$$

Multiplication Property of Equality
For any real numbers a, b, c,

$$\begin{array}{llcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array}$$

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} \, ? \, \frac{15}{5}$	Multiply both sides by 5.	$10(5) \, ? \, 15(5)$
Simplify.	$2 \, ? \, 3$		$50 \, ? \, 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

	$10 < 15$		$10 < 15$
Divide both sides by −5.	$\frac{10}{-5} \, ? \, \frac{15}{-5}$	Multiply both sides by −5.	$10(-5) \, ? \, 15(-5)$
Simplify.	$-2 \, ? \, -3$		$-50 \, ? \, -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

The inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Here are the Division and Multiplication Properties of Inequality for easy reference.

Division and Multiplication Properties of Inequality

For any numbers *a*, *b*, and *c*,

$$\text{multiply or divide by a positive}$$

$$\text{if } a < b \text{ and } c > 0, \text{then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$
$$\text{if } a > b \text{ and } c > 0, \text{then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

$$\text{multiply or divide by a negative}$$

$$\text{if } a < b \text{ and } c < 0, \text{then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

$$\text{if } a > b \text{ and } c < 0, \text{then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

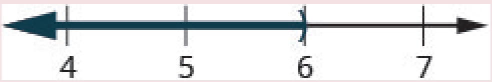
When we **divide or multiply** an inequality by a:

- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

EXAMPLE 4

Solve the inequality $7y$ < 42, graph the solution on the number line, and write the solution in interval notation.

Solution

	$7y < 42$
Divide both sides of the inequality by 7. Since $7 > 0$, the inequality stays the same.	$\frac{7y}{7} < \frac{42}{7}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

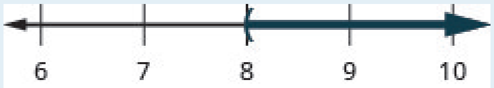
TRY IT 4

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$(8, \infty)$

Show answer

$c > 8$



EXAMPLE 5

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Solution

	$-10a \geq 50$
Divide both sides of the inequality by −10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

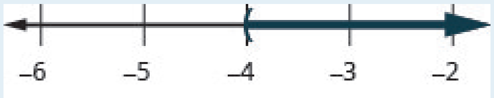
TRY IT 5

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Show answer

$q > -4$



$(-4, \infty)$

Solving Inequalities

Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.

$$x > a \text{ has the same meaning as } a < x$$

Think about it as "If Xavier is taller than Alex, then Alex is shorter than Xavier."

EXAMPLE 6

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Solution

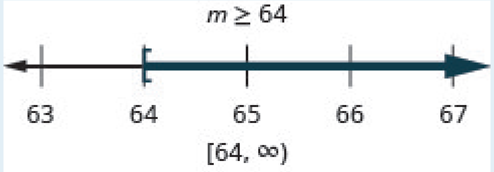
	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

TRY IT 6

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \leq \frac{3}{8}m$

Show answer



EXAMPLE 7

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Solution

	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

TRY IT 7

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Show answer

$k \geq -180$

$[-180, \infty)$

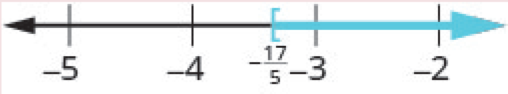
Solve Inequalities That Require Simplification

Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

EXAMPLE 8

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

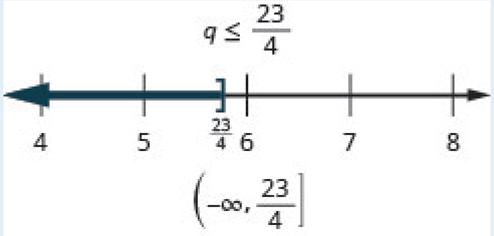
Solution

	$4m \le 9m + 17$
Subtract $9m$ from both sides to collect the variables on the left.	$4m - 9m \le 9m - 9m + 17$
Simplify.	$-5m \le 17$
Divide both sides of the inequality by −5, and reverse the inequality.	$\frac{-5m}{-5} \ge \frac{17}{-5}$
Simplify.	$m \ge -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

TRY IT 8

Solve the inequality $3q \ge 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

Show answer



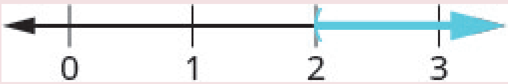
$q \le \frac{23}{4}$

$\left(-\infty, \frac{23}{4}\right]$

EXAMPLE 9

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

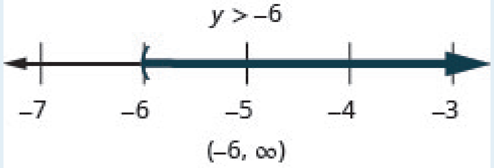
Solution

Simplify each side as much as possible.	$8p+3\left(p-12\right)>7p-28$
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract $7p$ from both sides to collect the variables on the left.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to collect the constants on the right.	$4p-36+36>-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interal notation.	$\left(2,\infty\right)$

TRY IT 9

Solve the inequality $9y+2\left(y+6\right)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

Show answer

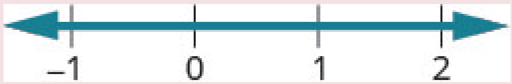


Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

EXAMPLE 10

Solve the inequality $8x - 2(5 - x) < 4(x + 9) + 6x$, graph the solution on the number line, and write the solution in interval notation.

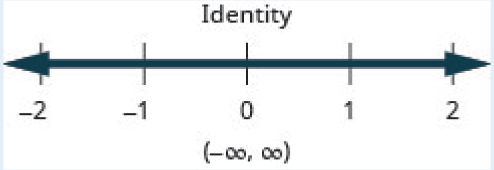
Solution

Simplify each side as much as possible.	$8x - 2(5 - x) < 4(x + 9) + 6x$
Distribute.	$8x - 10 + 2x < 4x + 36 + 6x$
Combine like terms.	$10x - 10 < 10x + 36$
Subtract $10x$ from both sides to collect the variables on the left.	$10x - 10 - 10x < 10x + 36 - 10x$
Simplify.	$-10 < 36$
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \infty)$

TRY IT 10

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Show answer



EXAMPLE 11

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

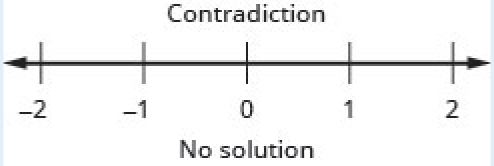
Solution

	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

TRY IT 11

Solve the inequality $\frac{1}{4}x-\frac{1}{12}x>\frac{1}{6}x+\frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

Show answer



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

(Figure) shows some common phrases that indicate inequalities.

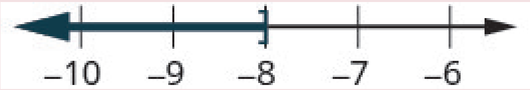
$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

EXAMPLE 12

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times *c* is no more than 96.

Solution

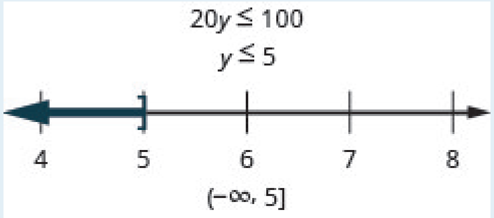
Translate.	Twelve times c $\underbrace{\text{is no more than}}$ 96 $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$
Graph on the number line.	

TRY IT 12

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times *y* is at most 100

Show answer



$20y \leq 100$

$y \leq 5$

$(-\infty, 5]$

EXAMPLE 13

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than *x* is at least 45.

Solution

Translate.	Thirty less than x $\underbrace{\text{is at least}}$ 45. $x - 30 \geq 45$
Solve—add 30 to both sides.	$x - 30 + 30 \geq 45 + 30$
Simplify.	$x \geq 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

TRY IT 13

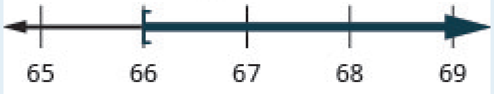
Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than *p* is no less than 47

Show answer

$p - 19 \geq 47$

$p \geq 66$



$[66, \infty)$

Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a - c < b - c$ and
  if $a > b$ then $a - c > b - c$.
- **Addition Property of Inequality**
  For any numbers a, b, and c,

if $a < b$ then $a + c < b + c$ and
if $a > b$ then $a + c > b + c$.

- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac < bc$.
- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

2.5 Exercise Set

In the following exercises, graph each inequality on the number line.

1.
   a. $x > 1$
   b. $x < -2$
   c. $x \geq -3$

2.
   a. $x \leq 0$
   b. $x > -4$
   c. $x \geq -1$

In the following exercises, graph each inequality on the number line and write in interval notation.

3.
   a. $x > 3$
   b. $x \leq -0.5$
   c. $x \geq \frac{1}{3}$

4.
   a. $x \leq 5$
   b. $x \geq -1.5$
   c. $x < -\frac{7}{3}$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

5. $m - 45 \leq 62$
6. $v + 12 > 3$
7. $b + \frac{7}{8} \geq \frac{1}{6}$
8. $g - \frac{11}{12} < -\frac{5}{18}$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

9. $6y < 48$
10. $9s \geq 81$
11. $-8v \leq 96$
12. $-7d > 105$
13. $40 < \frac{5}{8}k$
14. $\frac{9}{4}g \leq 36$
15. $\frac{b}{-10} \geq 30$
16. $-18 > \frac{q}{-6}$
17. $7s < -28$
18. $\frac{3}{5}x \leq -45$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

19. $5u \le 8u - 21$
20. $9p > 14p + 18$
21. $9y + 5(y + 3) < 4y - 35$
22. $4k - (k - 2) \ge 7k - 26$
23. $6n - 12(3 - n) \le 9(n - 4) + 9n$
24. $9u + 5(2u - 5) \ge 12(u - 1) + 7u$
25. $\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$
26. $12v + 3(4v - 1) \le 19(v - 2) + 5v$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

27. $35k \ge -77$
28. $18q - 4(10 - 3q) < 5(6q - 8)$
29. $-\frac{21}{8}y \le -\frac{15}{28}$
30. $d + 29 > -61$
31. $\frac{n}{13} \le -6$

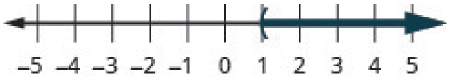
In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

32. Ninety times *c* is less than 450.
33. Ten times *y* is at most -110.
34. Six more than *k* exceeds 25.
35. Twelve less than *x* is no less than 21.
36. Negative two times *s* is lower than 56.
37. Fifteen less than *a* is at least -7.
38. The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

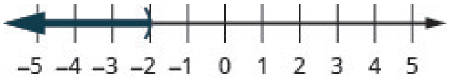
Answers

1.

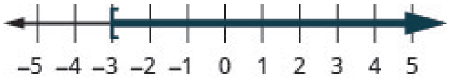
a.



b.

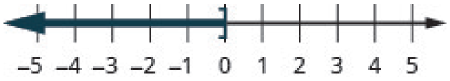


c.

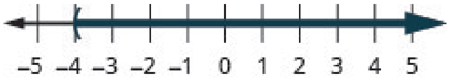


2.

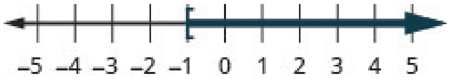
a.



b.

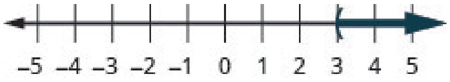


c.



3.

a.

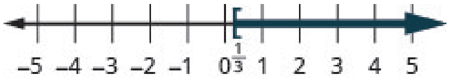


$(3, \infty)$

b.

$(-\infty, -0.5]$

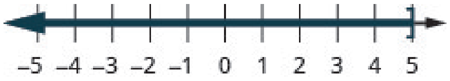
c.



$\left[\frac{1}{3}, \infty\right)$

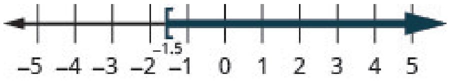
4.

a.



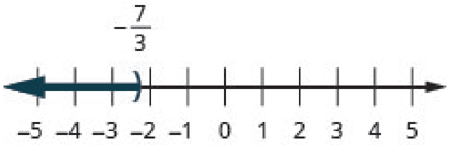
$(-\infty, 5]$

b.



$[-1.5, \infty)$

c.



$\left(-\infty, -\frac{7}{3}\right)$

5.

$m \leq 107$

$(-\infty, 107]$

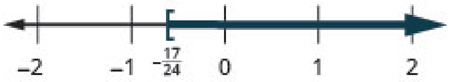
6.

$v > -9$

$(-9, \infty)$

7.

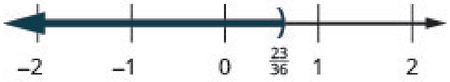
$b \geq -\frac{17}{24}$



$\left[-\frac{17}{24}, \infty\right)$

8.

$g < \frac{23}{36}$



$\left(-\infty, \frac{23}{36}\right)$

9.

$y < 8$

$(-\infty, 8)$

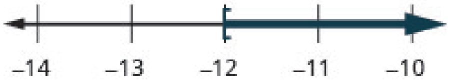
10.

$s \geq 9$

$[9, \infty)$

11.

$v \geq -12$



$[-12, \infty)$

12.

$d < -15$

$(-\infty, -15)$

13.

$k > 64$

$(64, \infty)$

14.

$g \leq 16$

$(-\infty, 16]$

15.

$b \leq -300$

$(-\infty, -300]$

16.

$q > 108$

$(108, \infty)$

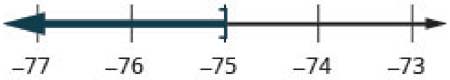
17.

$s < -4$

$(-\infty, -4)$

18.

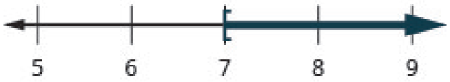
$x \leq -75$



$(-\infty, -75]$

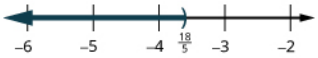
19.

$u \geq 7$



$[7, \infty)$

$p < -\frac{18}{5}$

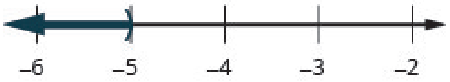


$\left(-\infty, \frac{18}{5}\right)$

20.

21.

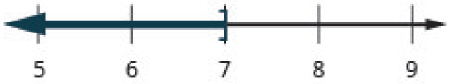
$y < -5$



$(-\infty, -5)$

22.

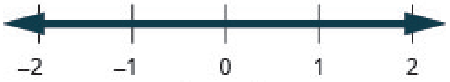
$x \leq 7$



$(-\infty, 7]$

23.

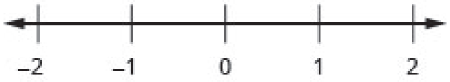
identity



$(-\infty, \infty)$

24.

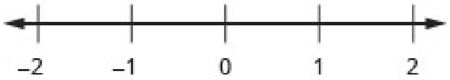
Contradiction



No solution

25.

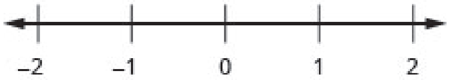
Contradiction



No solution

26.

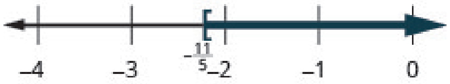
Contradiction



No solution

27.

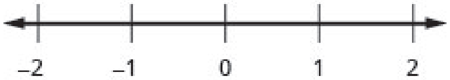
$k \geq -\frac{11}{5}$



$\left[-\frac{11}{5}, \infty\right)$

28.

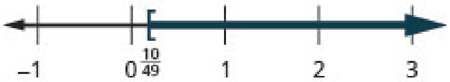
Contradiction



No solution

29.

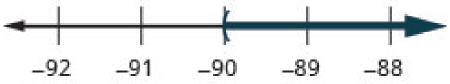
$y \geq \frac{10}{49}$



$\left[\frac{10}{49}, \infty\right)$

30.

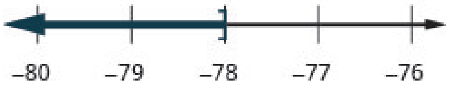
$d > -90$



$(-90, \infty)$

31.

$n \leq -78$



$(-\infty, -78]$

32.

$90c < 450$

$c < 5$

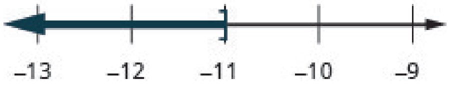
$(-\infty, 5)$

33.

$-10y \leq -110$

$y \leq -11$

$(-\infty, -11]$



34.

$k + 6 > 25$

$k > 19$

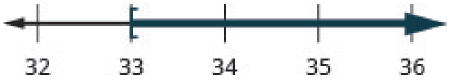
$(19, \infty)$

35.

$x - 12 \geq 21$

$x \geq 33$

$[33, \infty)$



36.

$-2s < 56$

$s > -28$

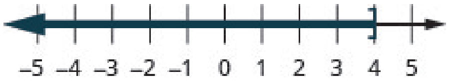
$(-28, \infty)$

37.

$a - 15 \geq -7$

$a \geq 8$

$[8, \infty)$



38. $h \leq 77$

3. Equations and their Graphs

Graphs are found in all areas of our lives. In this chapter, we will study the rectangular coordinate system, which is the basis for most graphs. We will look at linear graphs, slopes of lines, equations of lines, and linear inequalities.

3.1 Use the Rectangular Coordinate System

Learning Objectives

By the end of this section it is expected that you will be able to:

- Plot points in a rectangular coordinate system
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to a linear equation in two variables

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a **rectangular coordinate system**. The rectangular coordinate system is also called the *xy*-plane or the ‘coordinate plane.’

The horizontal number line is called the *x-axis*. The vertical number line is called the *y-axis.* The *x*-axis and the *y*-axis together form the rectangular coordinate system. These axes divide a plane into four regions, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See (Figure 1).

‘Quadrant’ has the root ‘quad,’ which means ‘four.’

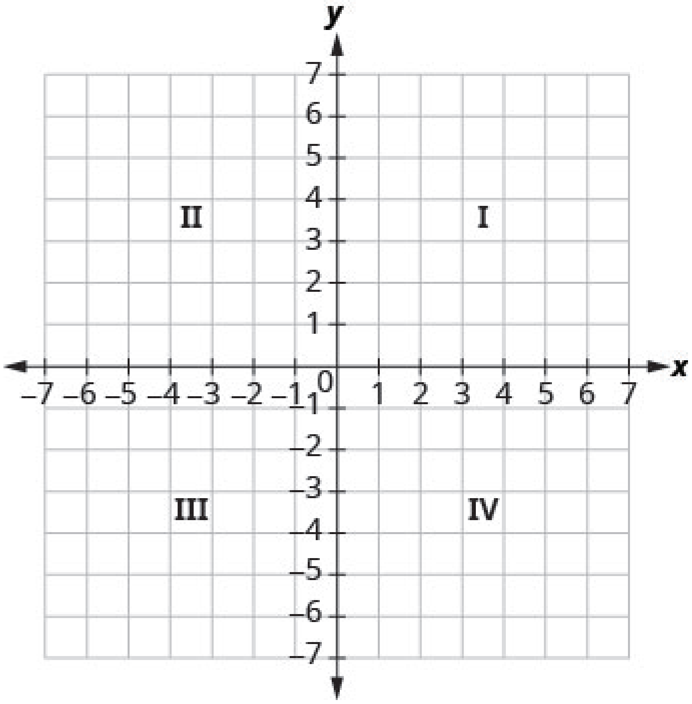
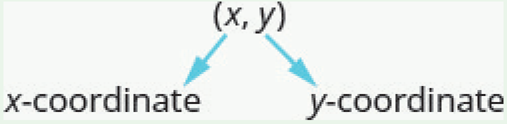


Figure .1

In the rectangular coordinate system, every point is represented by an *ordered pair*. The first number in the ordered pair is the ***x*-coordinate** of the point, and the second number is the ***y*-coordinate** of the point.

Ordered pair

An ordered pair, (x, y), gives the coordinates of a point in a rectangular coordinate system.



The first number is the x-coordinate.

The second number is the y-coordinate.

The phrase 'ordered pair' means the order is important. What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

The origin

The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.

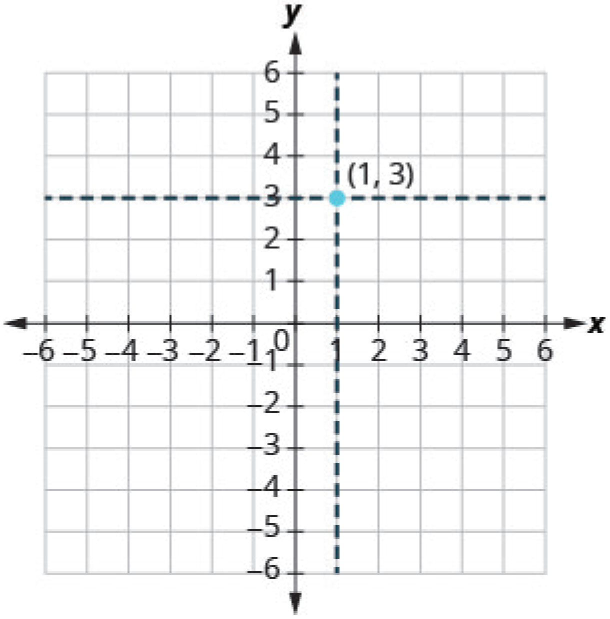


Figure .2

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

EXAMPLE 1

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

A $(-5, 4)$ B $(-3, -4)$ C $(2, -3)$ D $(-2, 3)$ E $\left(3, \frac{5}{2}\right)$.

Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate.

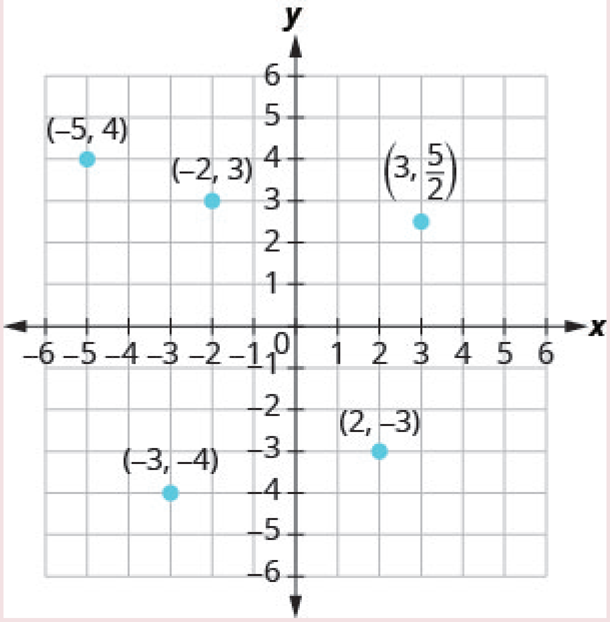
A. Since $x = -5$, the point is to the left of the *y*-axis. Also, since $y = 4$, the point is above the *x*-axis. The point $(-5, 4)$ is in Quadrant II.

B. Since $x = -3$, the point is to the left of the *y*-axis. Also, since $y = -4$, the point is below the *x*-axis. The point $(-3, -4)$ is in Quadrant III.

C. Since $x = 2$, the point is to the right of the *y*-axis. Since $y = -3$, the point is below the *x*-axis. The point $(2, -3)$ is in Quadrant lV.

D. Since $x = -2$, the point is to the left of the *y*-axis. Since $y = 3$, the point is above the *x*-axis. The point $(-2, 3)$ is in Quadrant II.

E. Since $x = 3$, the point is to the right of the *y*-axis. Since $y = \frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.

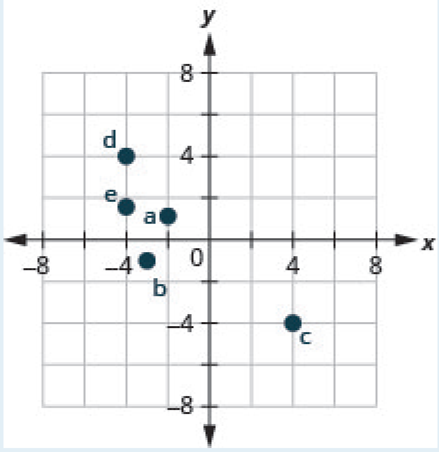


TRY IT 1

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

A $(-2, 1)$ B $(-3, -1)$ C $(4, -4)$ D $(-4, 4)$ E $\left(-4, \frac{3}{2}\right)$.

Show answer



How do the signs affect the location of the points? You may have noticed some patterns as you graphed the points in the previous example.

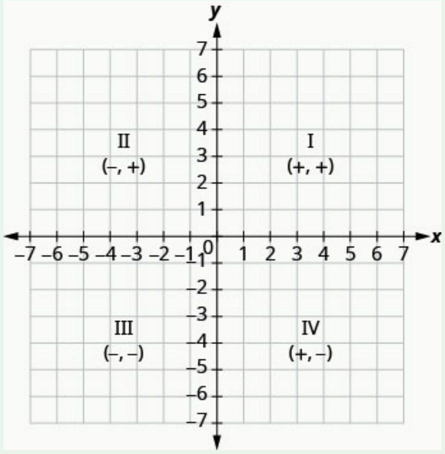
For the point in (Figure 2) in Quadrant IV, what do you notice about the signs of the coordinates? What about the signs of the coordinates of points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point $(-2, 5)$ is located? In which quadrant is $(2, -5)$ located?

Quadrants

We can summarize sign patterns of the quadrants in this way.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(, +)$	$(,)$	$(+,)$



What if one coordinate is zero as shown in (Figure 3)? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located?

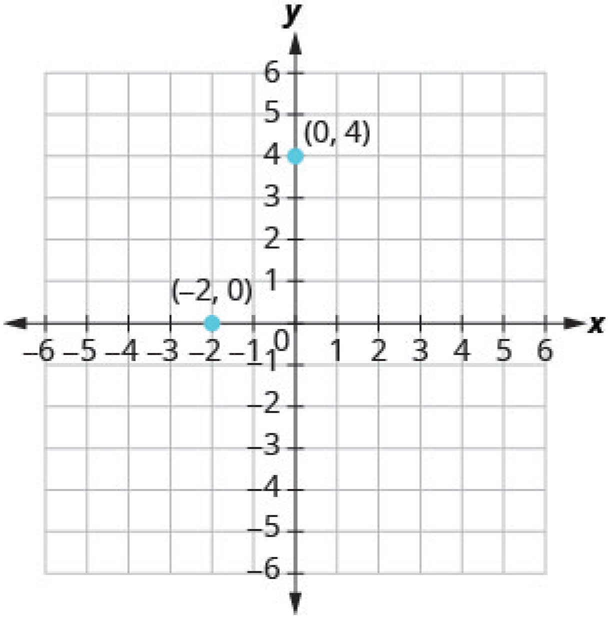


Figure .3

The point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

Points on the axes

Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.

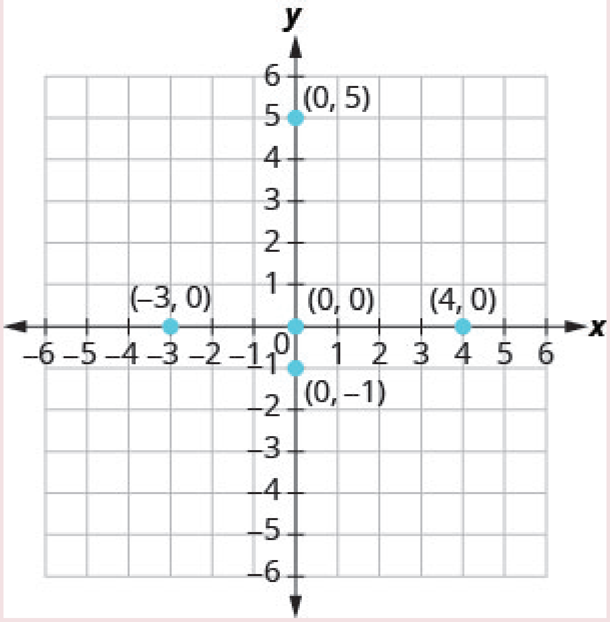
Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

EXAMPLE 2

Plot each point:A $(0, 5)$ B $(4, 0)$ C $(-3, 0)$ D $(0, 0)$ E $(0, -1)$.

Solution

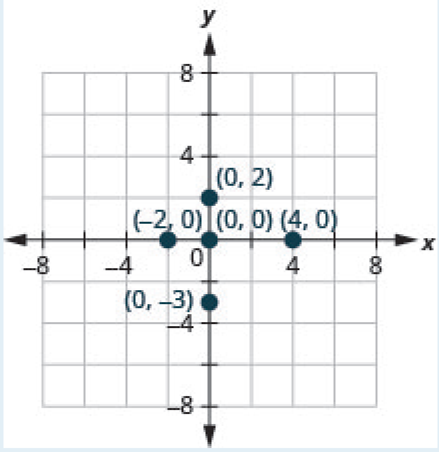
A. Since $x = 0$, the point whose coordinates are $(0, 5)$ is on the *y*-axis.
B. Since $y = 0$, the point whose coordinates are $(4, 0)$ is on the *x*-axis.
C. Since $y = 0$, the point whose coordinates are $(-3, 0)$ is on the *x*-axis.
D. Since $x = 0$ and $y = 0$, the point whose coordinates are $(0, 0)$ is the origin.
E. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the *y*-axis.



TRY IT 2

Plot each point:A $(4, 0)$ B $(-2, 0)$ C $(0, 0)$ D $(0, 2)$ E $(0, -3)$.

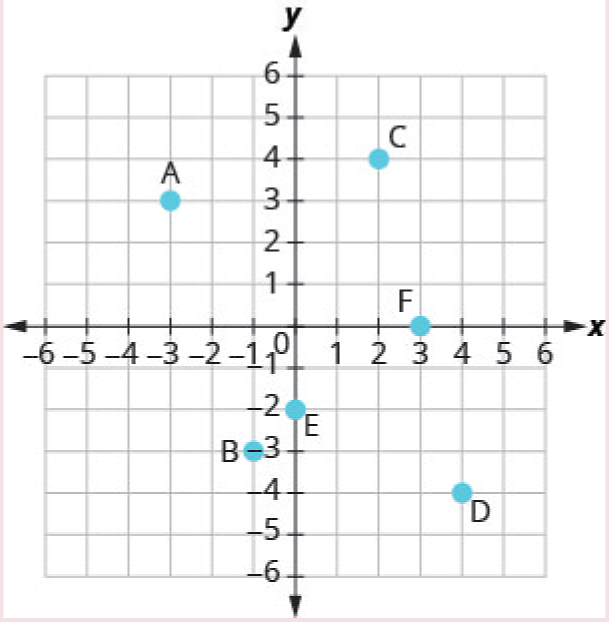
Show answer



In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, when you write the ordered pair use the correct order, (x, y).

EXAMPLE 3

Name the ordered pair of each point shown in the rectangular coordinate system.



Solution

Point A is above -3 on the *x*-axis, so the *x*-coordinate of the point is -3.

- The point is to the left of 3 on the *y*-axis, so the *y*-coordinate of the point is 3.
- The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the *x*-axis, so the *x*-coordinate of the point is -1.

- The point is to the left of -3 on the *y*-axis, so the *y*-coordinate of the point is -3.
- The coordinates of the point are $(-1, -3)$.

Point C is above 2 on the *x*-axis, so the *x*-coordinate of the point is 2.

- The point is to the right of 4 on the *y*-axis, so the *y*-coordinate of the point is 4.
- The coordinates of the point are $(2, 4)$.

Point D is below 4 on the *x*-axis, so the *x*-coordinate of the point is 4.

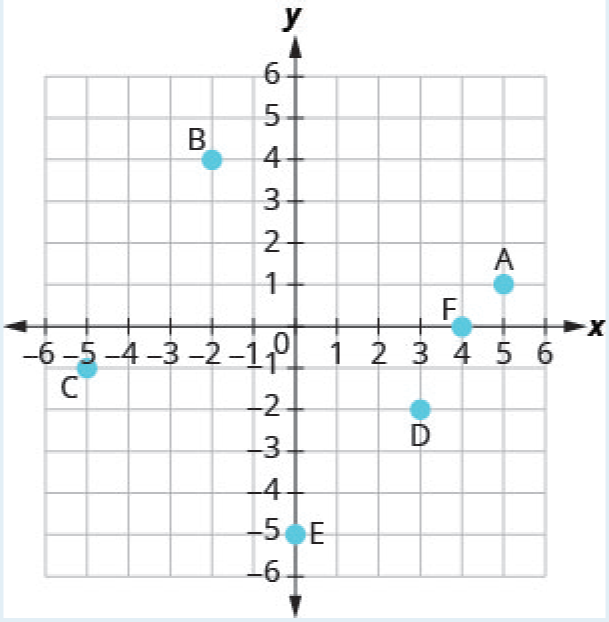
- The point is to the right of -4 on the *y*-axis, so the *y*-coordinate of the point is -4.
- The coordinates of the point are $(4, -4)$.

Point E is on the *y*-axis at $y = -2$. The coordinates of point E are $(0, -2)$.

Point F is on the *x*-axis at $x = 3$. The coordinates of point F are $(3, 0)$.

TRY IT 3

Name the ordered pair of each point shown in the rectangular coordinate system.



Show answer

A: $(5,1)$ B: $(-2,4)$ C: $(-5,-1)$ D: $(3,-2)$ E: $(0,-5)$ F: $(4,0)$

Verify Solutions to an Equation in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. The process of solving an equation ended with a statement like $x = 4$. (Then, you checked the solution by substituting back into the equation.)

Here’s an example of an equation in one variable, and its one solution.

$$\begin{aligned} 3x+5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. Equations of this form are called **linear equations in two variables**.

Linear equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation **in two variables**.

Notice the word *line* in **linear**. Here is an example of a linear equation in two variables, x and y.

$$Ax + By = C$$

$$x + 4y = 8$$

$$A = 1,\ B = 4,\ C = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	$y = -3x + 5$
Add to both sides.	$y + 3x = -3x + 5 + 3x$
Simplify.	$y + 3x = 5$
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in *standard form*.

Standard Form of Linear Equation

A linear equation is in standard form when it is written $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a **solution** of the linear equation $Ax + By = C$, if the equation is a true statement when the *x*– and *y*-values of the ordered pair are substituted into the equation.

EXAMPLE 4

Determine which ordered pairs are solutions to the equation $x + 4y = 8$.

A (\left(0,2\right)\) B $(2, -4)$ C $(-4, 3)$

Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if the result is a true statement.

(a)	(b)	(c)
$(0, 2)$	$(2, -4)$	$(-4, 3)$
$x = 0, y = 2$	$x = 2, y = -4$	$x = -4, y = 3$
$x + 4y = 8$	$x + 4y = 8$	$x + 4y = 8$
$0 + 4 \cdot 2 \overset{?}{=} 8$	$2 + 4(-4) \overset{?}{=} 8$	$-4 + 4 \cdot 3 \overset{?}{=} 8$
$0 + 8 \overset{?}{=} 8$	$2 + (-16) \overset{?}{=} 8$	$-4 + 12 \overset{?}{=} 8$
$8 = 8$ ✓	$-14 \neq 8$	$8 = 8$ ✓
(0, 2) is a solution.	(2, –4) is not a solution.	(–4, 3) is a solution.

TRY IT 4

Which of the following ordered pairs are solutions to $2x + 3y = 6$?
A $(3, 0)$ B $(2, 0)$ C $(6, -2)$

Show answer
A, C

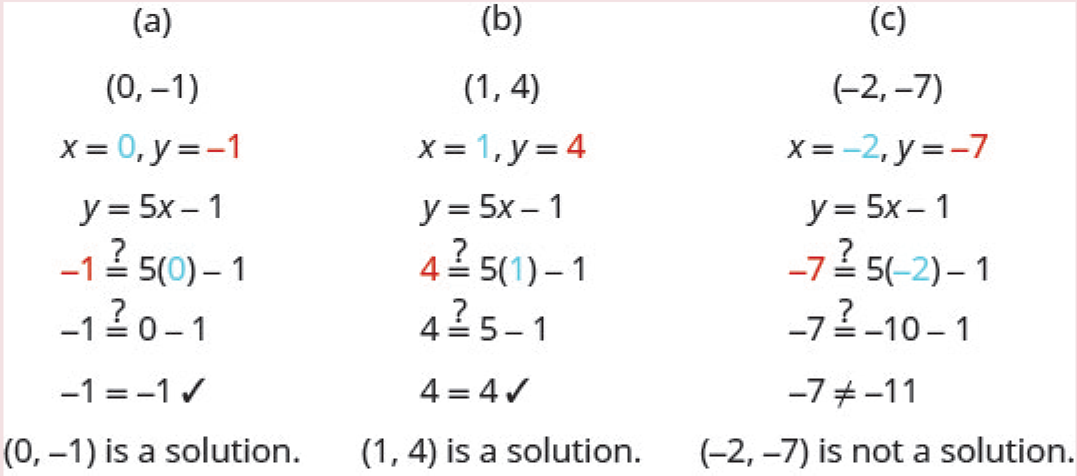
EXAMPLE 5

Which of the following ordered pairs are solutions to the equation $y = 5x - 1$?

A $(0, -1)$ B $(1, 4)$ C $(-2, -7)$

Solution

Substitute the *x*– and *y*-values from each ordered pair into the equation and determine if it results in a true statement.



(a)	(b)	(c)
$(0, -1)$	$(1, 4)$	$(-2, -7)$
$x = 0, y = -1$	$x = 1, y = 4$	$x = -2, y = -7$
$y = 5x - 1$	$y = 5x - 1$	$y = 5x - 1$
$-1 \overset{?}{=} 5(0) - 1$	$4 \overset{?}{=} 5(1) - 1$	$-7 \overset{?}{=} 5(-2) - 1$
$-1 \overset{?}{=} 0 - 1$	$4 \overset{?}{=} 5 - 1$	$-7 \overset{?}{=} -10 - 1$
$-1 = -1$ ✓	$4 = 4$ ✓	$-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

TRY IT 5

Which of the following ordered pairs are solutions to the equation $y = 4x - 3$? A $(0, 3)$ B $(1, 1)$ C $(-1, -1)$

Show answer

B

Complete a Table of Solutions to a Linear Equation in Two Variables

In the examples above, we substituted the *x*– and *y*-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do you find the ordered pairs if they are not given? It's easier than you might think—you can just pick a value for x and then solve the equation for y. Or, pick a value for y and then solve for x.

We'll start by looking at the solutions to the equation $y = 5x - 1$ that we found in (Example 5). We can summarize this information in a table of solutions, as shown in (Table 1).

$$y = 5x - 1$$

x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$

To find a third solution, we'll let $x = 2$ and solve for y.

	$y = 5x - 1$
Substitute $x = 2$.	$y = 5(2) - 1$
Multiply.	$y = 10 - 1$
Simplify.	$y = 9$

The ordered pair $(2, 9)$ is a solution to $y = 5x - 1$. We will add it to (Table 2).

$$y = 5x - 1$$

x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$
2	9	$(2, 9)$

We can find more solutions to the equation by substituting in any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are infinitely many solutions of this equation.

EXAMPLE 6

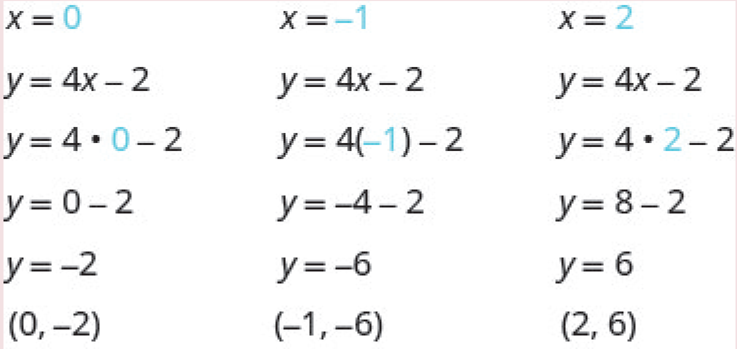
Complete the table to find three solutions to the equation $y = 4x - 2$.

$$y = 4x - 2$$

x	y	(x, y)
0		
-1		
2		

Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.



$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in the table below.

$$y = 4x - 2$$

x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

TRY IT 6

Complete the table to find three solutions to this equation: $y = 3x - 1$.

$$y = 3x - 1$$

x	y	(x, y)
0		
-1		
2		

Show answer

$$y = 3x - 1$$

x	y	(x, y)
0	-1	$(0, -1)$
-1	-4	$(-1, -4)$
2	5	$(2, 5)$

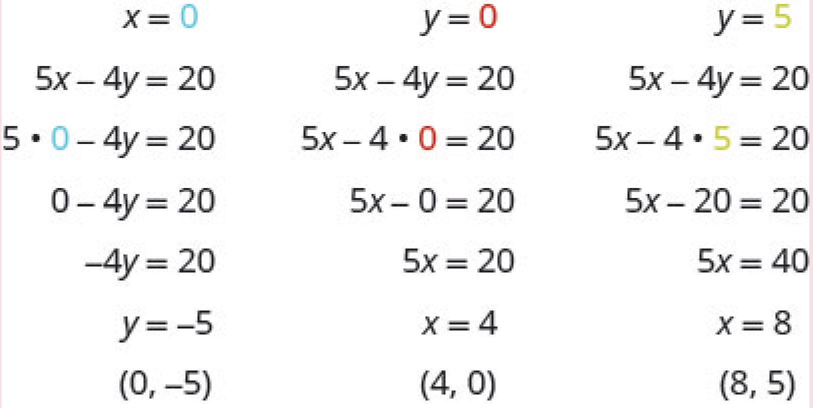
EXAMPLE 7

Complete the table to find three solutions to the equation $5x - 4y = 20$.

$$5x - 4y = 20$$

x	y	(x, y)
0		
	0	
	5	

Solution
Substitute the given value into the equation $5x - 4y = 20$ and solve for the other variable. Then, fill in the values in the table.



$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in the table below.

$$5x - 4y = 20$$

x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

TRY IT 7

Complete the table to find three solutions to this equation: $2x - 5y = 20$.

$$2x - 5y = 20$$

x	y	(x, y)
0		
	0	
-5		

Show answer

$$2x - 5y = 20$$

x	y	(x, y)
0	-4	$(0, -4)$
10	0	$(10, 0)$
-5	-6	$(-5, -6)$

Find Solutions to a Linear Equation

To find a solution to a linear equation, you really can pick *any* number you want to substitute into the equation for x or y. But since you'll need to use that number to solve for the other variable it's a good idea to choose a number that's easy to work with.

When the equation is in *y*-form, with the *y* by itself on one side of the equation, it is usually easier to choose values of x and then solve for y.

EXAMPLE 8

Find three solutions to the equation $y = -3x + 2$.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in *y*-form, it will be easier to substitute in values of x. Let's pick $x = 0$, $x = 1$, and $x = -1$.

	$x = 0$	$x = 1$	$x = -1$
Substitute the value into the equation.	$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$
Simplify.	$y = -3 \cdot 0 + 2$	$y = -3 \cdot 1 + 2$	$y = -3(-1) + 2$
Simplify.	$y = 0 + 2$	$y = -3 + 2$	$y = 3 + 2$
Write the ordered pair.	$y = 2$	$y = -1$	$y = 5$
Check.	(0, 2)	(1, −1)	(−1, 5)
	$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$
	$2 \overset{?}{=} -3 \cdot 0 + 2$	$-1 \overset{?}{=} -3 \cdot 1 + 2$	$5 \overset{?}{=} -3\left(-1\right) + 2$
	$2 \overset{?}{=} 0 + 2$	$-1 \overset{?}{=} -3 + 2$	$5 \overset{?}{=} 3 + 2$
	$2 = 2\checkmark$	$-1 = -1\checkmark$	$5 = 5\checkmark$

So, $(0, 2)$, $(1, -1)$ and $(-1, 5)$ are all solutions to $y = -3x + 2$. We show them in table below.

$$y = -3x + 2$$

x	y	(x, y)
0	2	$(0, 2)$
1	-1	$(1, -1)$
-1	5	$(-1, 5)$

TRY IT 8

Find three solutions to this equation: $y = -2x + 3$.

Show answer

Answers will vary.

We have seen how using zero as one value of x makes finding the value of y easy. When an equation is in standard form, with both the x and y on the same side of the equation, it is usually easier to first find one solution when $x = 0$ find a second solution when $y = 0$, and then find a third solution.

EXAMPLE 9

Find three solutions to the equation $3x + 2y = 6$.

Solution
We can substitute any value we want for x or any value for y. Since the equation is in standard form, let's pick first $x = 0$, then $y = 0$, and then find a third point.

	$x = 0$	$y = 0$	$x = 1$
	$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
Substitute the value into the equation.	$3(0) + 2y = 6$	$3x + 2(0) = 6$	$3(1) + 2y = 6$
Simplify.	$0 + 2y = 6$	$3x + 0 = 6$	$3 + 2y = 6$
Solve.	$2y = 6$	$3x = 6$	$2y = 3$
	$y = 3$	$x = 2$	$y = \frac{3}{2}$
Write the ordered pair.	(0, 3)	(2, 0)	$\left(1, \frac{3}{2}\right)$
Check.	$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
	$3 \cdot 0 + 2 \cdot 3 \stackrel{?}{=} 6$	$3 \cdot 2 + 2 \cdot 0 \stackrel{?}{=} 6$	$3 \cdot 1 + 2 \cdot \frac{3}{2} \stackrel{?}{=} 6$
	$0 + 6 \stackrel{?}{=} 6$	$6 + 0 \stackrel{?}{=} 6$	$3 + 3 \stackrel{?}{=} 6$
	$6 = 6\checkmark$	$6 = 6$	$6 = 6$

So $(0, 3)$, $(2, 0)$, and $\left(1, \frac{3}{2}\right)$ are all solutions to the equation $3x + 2y = 6$. We can list these three solutions in the table below.

$$3x + 2y = 6$$

x	y	(x, y)
0	3	$(0, 3)$
2	0	$(2, 0)$
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

EXAMPLE 9

Find three solutions to the equation $2x + 3y = 6$.

Show answer
Answers will vary.

Glossary

linear equation
A linear equation is of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair
An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system.

origin
The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

quadrant
The *x*-axis and the *y*-axis divide a plane into four regions, called quadrants.

rectangular coordinate system
A grid system is used in algebra to show a relationship between two variables; also called the *xy*-plane or the ‘coordinate plane.’

***x*-coordinate**
The first number in an ordered pair (x, y).

***y*-coordinate**
The second number in an ordered pair (x, y).

3.1 Exercise Set.

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

1.
   - A. $(-4, 2)$
   - B. $(-1, -2)$
   - C. $(3, -5)$
   - D. $(-3, 5)$
   - E. $\left(\frac{5}{3}, 2\right)$

2.
   - A. $(3, -1)$
   - B. $(-3, 1)$
   - C. $(-2, 2)$
   - D. $(0, 4)$
   - E. $\left(1, \frac{14}{5}\right)$

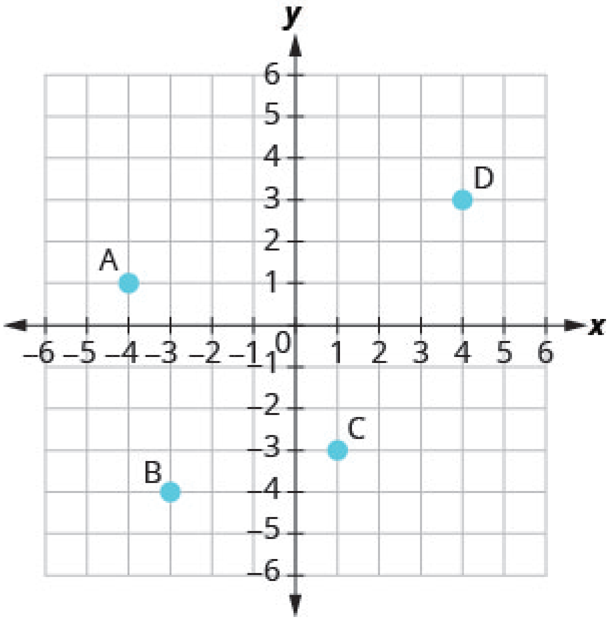
In the following exercises, plot each point in a rectangular coordinate system.

3.
   - A. $(-2, 0)$
   - B. $(-3, 0)$
   - C. $(0, 0)$
   - D. $(-3, 5)$
   - E. $(0, 2)$

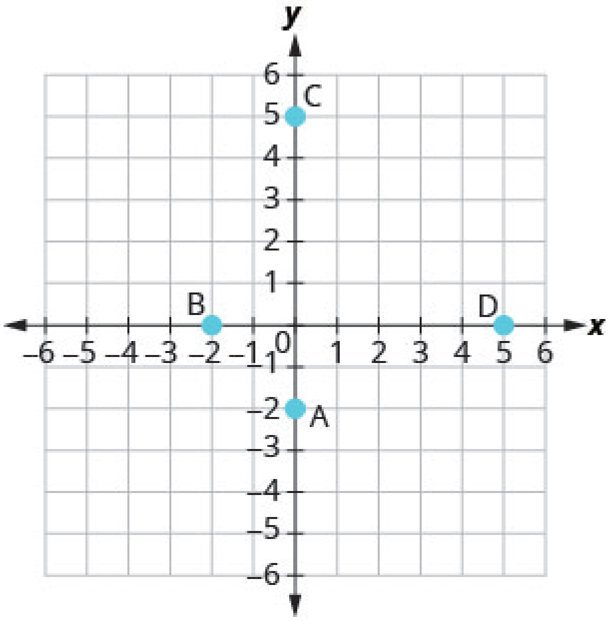
4.
   - A. $(0, 0)$
   - B. $(0, -3)$
   - C. $(-4, 0)$
   - D. $(1, 0)$
   - E. $(0, -2)$

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

5.



6.



In the following exercises, which ordered pairs are solutions to the given equations?

7. $2x + y = 6$
   - A. $(1, 4)$
   - B. $(3, 0)$
   - C. $(2, 3)$

8. $4x - 2y = 8$
   - A. $(3, 2)$
   - B. $(1, 4)$
   - C. $(0, -4)$

9. $y = 4x + 3$
   - A. $(4, 3)$
   - B. $(-1, -1)$
   - C. $\left(\frac{1}{2}, 5\right)$

10. $y = \frac{1}{2}x - 1$
   - A. $(2, 0)$
   - B. $(-6, -4)$
   - C. $(-4, -1)$

In the following exercises, complete the table to find solutions to each linear equation.

11. $y = 2x - 4$

x	y	(x, y)
0		
2		
-1		

12. $y = x + 5$

x	y	(x, y)
0		
3		
-2		

13. $y = \frac{1}{3}x + 1$

x	y	(x, y)
0		
3		
6		

14. $y = -\frac{3}{2}x - 2$

x	y	(x, y)
0		
2		
-2		

15. $x + 3y = 6$

x	y	(x, y)
0		
3		
	0	

16. $2x - 5y = 10$

x	y	(x, y)
0		
10		
	0	

In the following exercises, find three solutions to each linear equation.

17. $y = 5x - 8$

18. $y = -4x + 5$

19. $x + y = 8$

20. $x + y = -2$

21. $3x + y = 5$

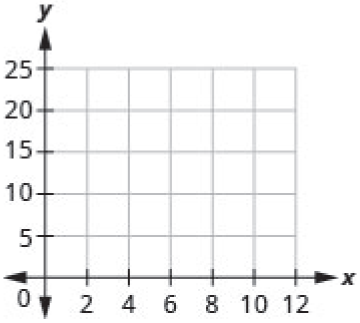
22. $4x - y = 8$

23. $2x + 4y = 8$

24. $5x - 2y = 10$

25. Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

a) Plot the points on a coordinate plane.

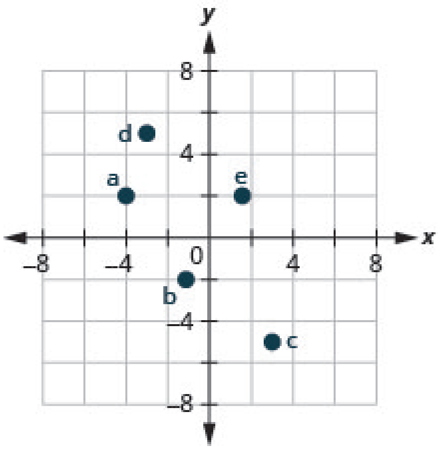


b) Why is only Quadrant I needed?

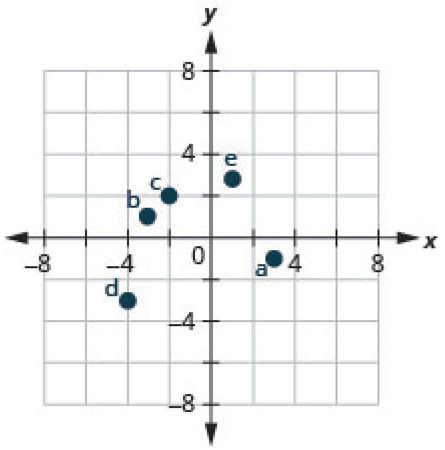
Age x	**Weight** y	(x, y)
0	7	(0, 7)
2	11	(2, 11)
4	15	(4, 15)
6	16	(6, 16)
8	19	(8, 19)
10	20	(10, 20)
12	21	(12, 21)

Answers

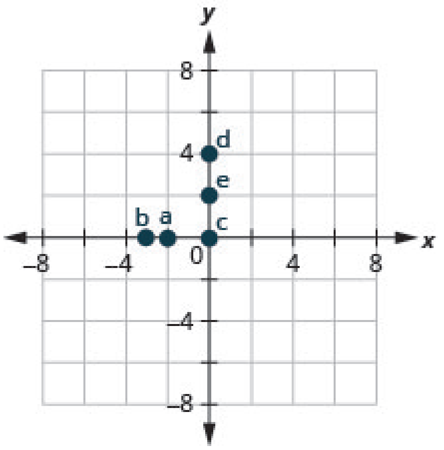
1.



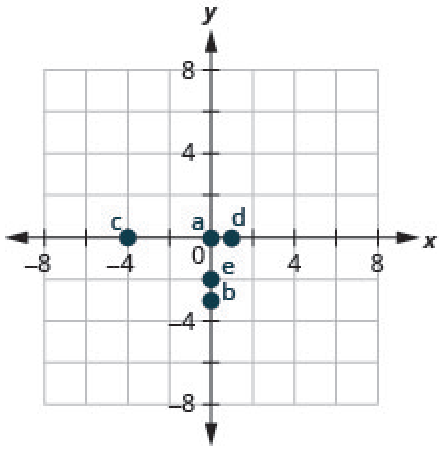
2.



3.



4.



5. A: $(-4, 1)$ B: $(-3, -4)$ C: $(1, -3)$ D: $(4, 3)$
6. A: $(0, -2)$ B: $(-2, 0)$ C: $(0, 5)$ D: $(5, 0)$
7. A, B
8. A, C
9. B, C
10. A, B

11.

x	y	(x, y)
0	-4	$(0, -4)$
2	0	$(2, 0)$
-1	-6	$(-1, -6)$

12.

x	y	(x, y)
0	5	$(0, 5)$
3	2	$(3, 2)$
-2	7	$(-2, 7)$

13.

x	y	(x, y)
0	1	$(0, 1)$
3	2	$(3, 2)$
6	3	$(6, 3)$

14.

x	y	(x, y)
0	-2	$(0, -2)$
2	-5	$(2, -5)$
-2	1	$(-2, 1)$

15.

x	y	(x, y)
0	2	$(0, 2)$
3	4	$(3, 1)$
6	0	$(6, 0)$

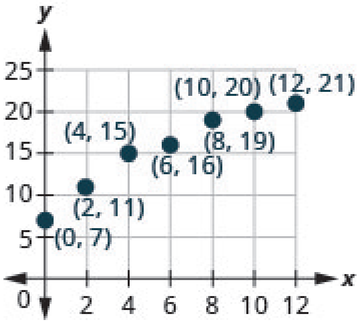
16.

x	y	(x, y)
0	-2	$(0, -2)$
10	2	$(10, 2)$
5	0	$(5, 0)$

17. Answers will vary.
18. Answers will vary.
19. Answers will vary.
20. Answers will vary.
21. Answers will vary.
22. Answers will vary.
23. Answers will vary.
24. Answers will vary.

25.

a)



b) Age and weight are only positive.

Attributions

This chapter has been adapted from “Use the Rectangular Coordinate System” in *Elementary Algebra*

3.2 Graph Linear Equations in Two Variables

Learning Objectives

By the end of this section it is expected that you will be able to:

- Recognize the relationship between the solutions of an equation and its graph.
- Graph a linear equation by plotting points.
- Graph vertical and horizontal lines.

Recognize the Relationship Between the Solutions of an Equation and its Graph

In the previous section, we found several solutions to the equation $3x+2y=6$. They are listed in the table below. So, the ordered pairs $(0,3)$, $(2,0)$, and $\left(1,\frac{3}{2}\right)$ are some solutions to the equation $3x+2y=6$. We can plot these solutions in the rectangular coordinate system as shown in (Figure 1).

$3x+2y=6$

x	y	(x,y)
0	3	$(0,3)$
2	0	$(2,0)$
1	$\frac{3}{2}$	$\left(1,\frac{3}{2}\right)$

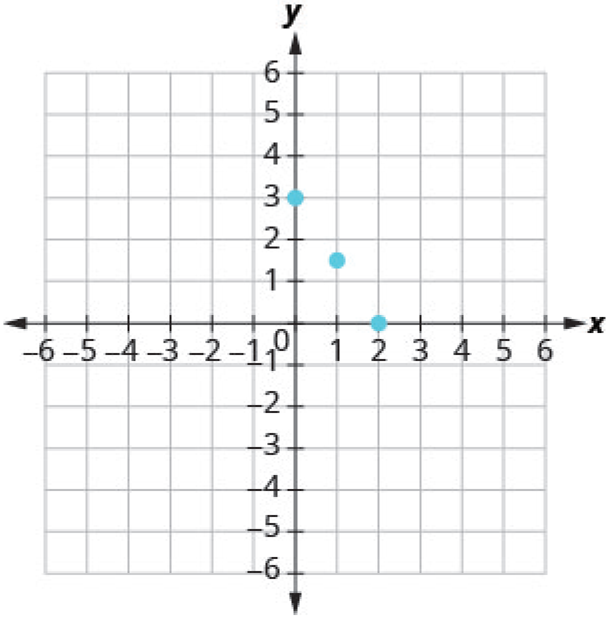


Figure .1

Notice how the points line up perfectly? We connect the points with a line to get the graph of the equation $3x + 2y = 6$. See (Figure 2). Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.

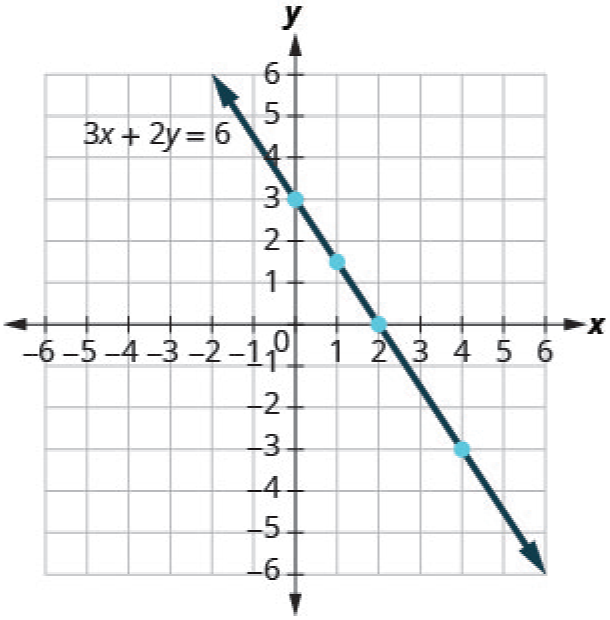


Figure .2

Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points *not* on the line are not solutions.

Notice that the point whose coordinates are $(-2, 6)$ is on the line shown in (Figure 3). If you substitute $x = -2$ and $y = 6$ into the equation, you find that it is a solution to the equation.

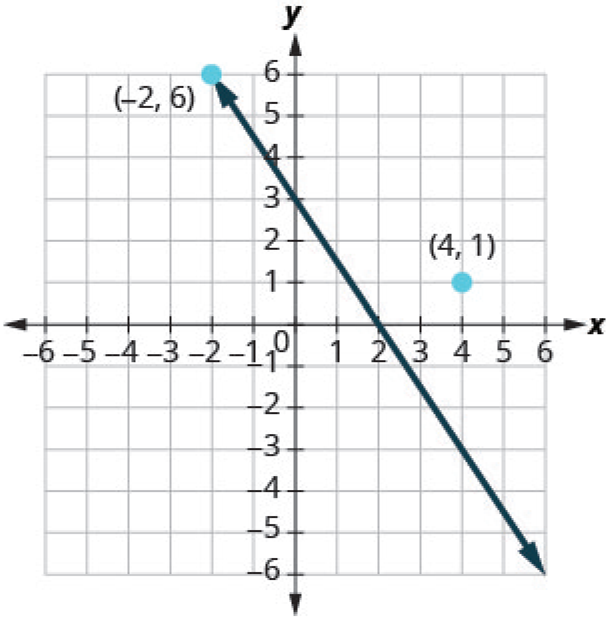


Figure .3

Test $(-2, 6)$

$$3x + 2y = 6$$

$$3(-2) + 2(6) = 6$$

$$-6 + 12 = 6$$

$$6 = 6 \checkmark$$

So the point $(-2, 6)$ is a solution to the equation $3x + 2y = 6$. (The phrase “the point whose coordinates are $(-2, 6)$” is often shortened to “the point $(-2, 6)$.”)

What about $(4, 1)$?

$$3x + 2y = 6$$

$$3 \cdot 4 + 2 \cdot 1 = 6$$

$$12 + 2 \overset{?}{=} 6$$

$$14 \neq 6$$

So $(4, 1)$ is not a solution to the equation $3x + 2y = 6$. Therefore, the point $(4, 1)$ is not on the line. See (Figure 2). This is an example of the saying, “A picture is worth a thousand words.” The line shows you *all* the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation $3x + 2y = 6$.

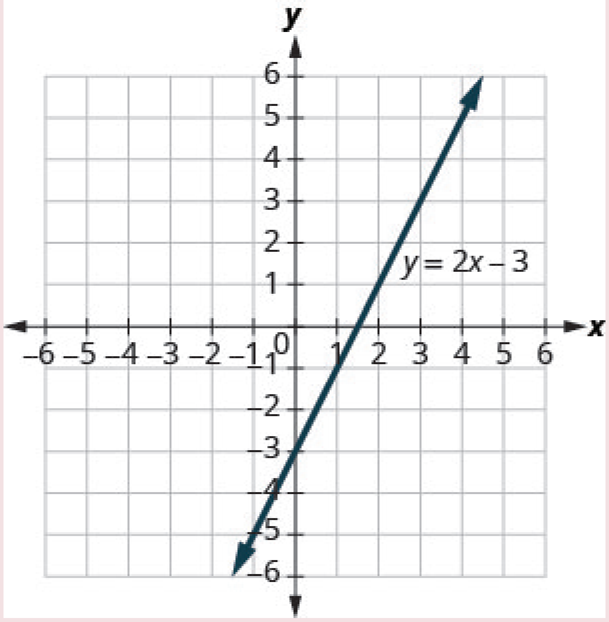
Graph of a linear equation

The graph of a linear equation $Ax + By = C$ is a line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

EXAMPLE 1

The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

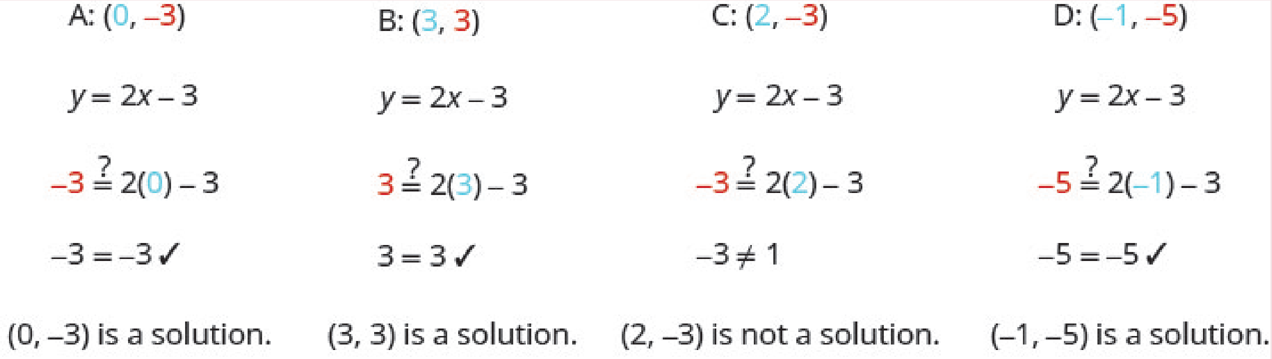
a) Is the ordered pair a solution to the equation?
b) Is the point on the line?

A $(0, -3)$ B $(3, 3)$ C $(2, -3)$ D $(-1, -5)$

Solution

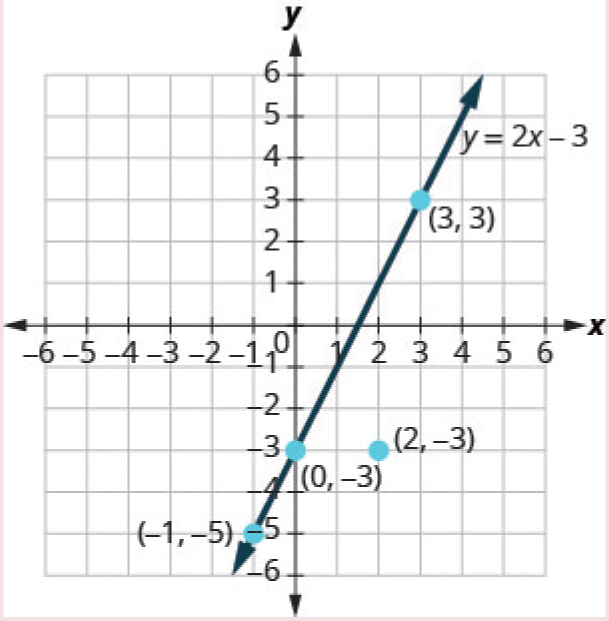
Substitute the *x*– and *y*– values into the equation to check if the ordered pair is a solution to the equation.

a)



A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \overset{?}{=} 2(0) - 3$	$3 \overset{?}{=} 2(3) - 3$	$-3 \overset{?}{=} 2(2) - 3$	$-5 \overset{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0, -3)$ is a solution.	$(3, 3)$ is a solution.	$(2, -3)$ is not a solution.	$(-1, -5)$ is a solution.

b) Plot the points A $(0, 3)$, B $(3, 3)$, C $(2, -3)$, and D $(-1, -5)$.



The points $(0, 3)$, $(3, 3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

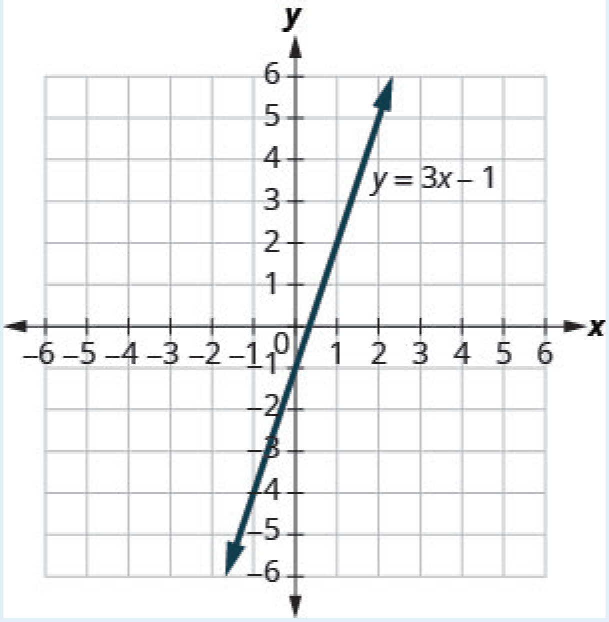
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

TRY IT 1

Use the graph of $y = 3x - 1$ to decide whether each ordered pair is:

- a solution to the equation.
- on the line.

a) $(0, -1)$ b) $(2, 5)$



Show answer
a) yes, yes b) yes, yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The method we used to graph $3x + 2y = 6$ is called plotting points, or the Point–Plotting Method.

EXAMPLE 2

How To Graph an Equation By Plotting Points

Graph the equation $y = 2x + 1$ by plotting points.

Solution

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for *x* or *y*.

In this case, since *y* is isolated on the left side of the equation, it is easier to choose values for *x*.

$$y = 2x + 1$$

$$x = 0$$
$$y = 2x + 1$$
$$y = 2 \cdot 0 + 1$$
$$y = 0 + 1$$
$$y = 1$$

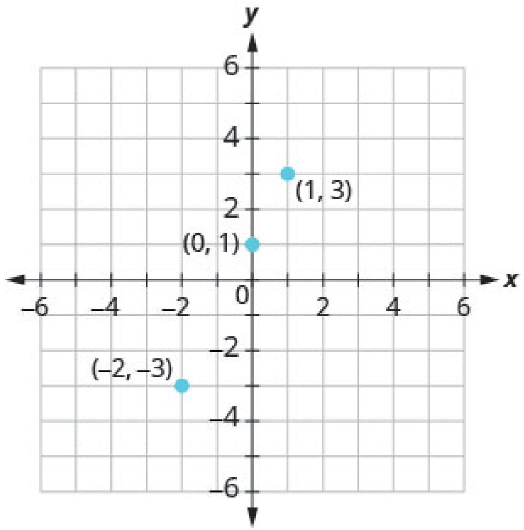
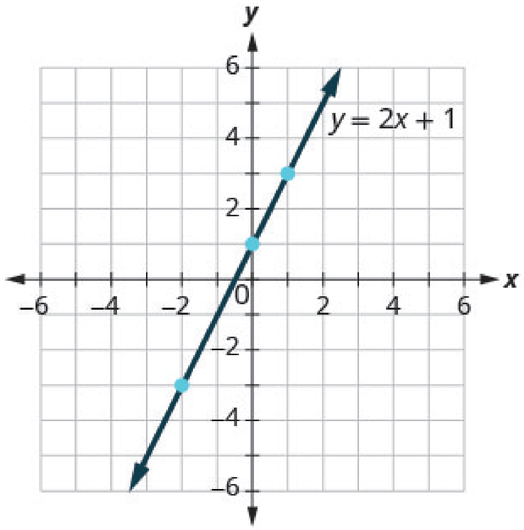
$$x = 1$$
$$y = 2x + 1$$
$$y = 2 \cdot 1 + 1$$
$$y = 2 + 1$$
$$y = 3$$

$$x = -2$$
$$y = 2x + 1$$
$$y = 2(-2) + 1$$
$$y = -4 + 1$$
$$y = -3$$

Organize the solutions in a table.

Put the three solutions in a table.

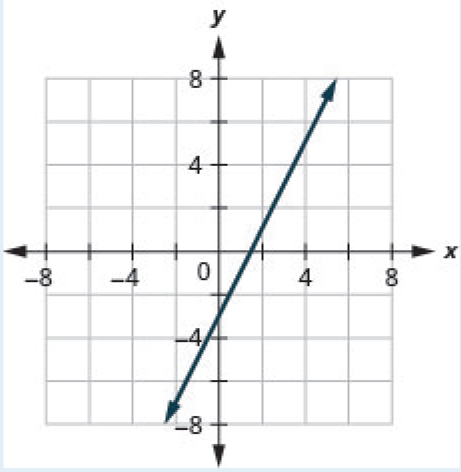
$y = 2x + 1$		
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
–2	–3	(–2, –3)

Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!	Plot: (0, 1), (1, 3), (–2, –3). Do the points line up? Yes, the points line up.	
Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of *y* = 2*x* + 1.	

TRY IT 2

Graph the equation by plotting points: $y = 2x - 3$.

Show answer



HOW TO: Graph a linear equation by plotting points.

The steps to take when graphing a linear equation by plotting points are summarized below.

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between part (a) and part (b) in (Figure 4).

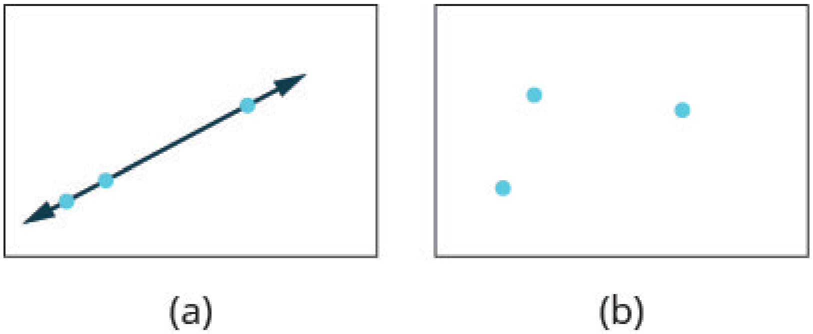


Figure .4

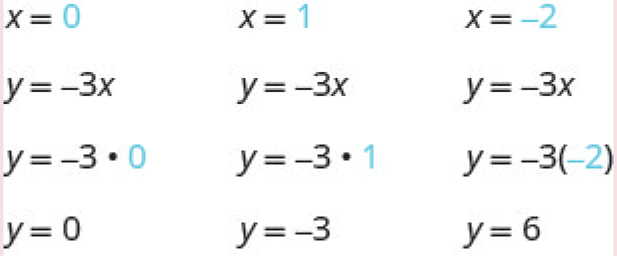
Let’s do another example. This time, we’ll show the last two steps all on one grid.

EXAMPLE 3

Graph the equation $y = -3x$.

Solution

Find three points that are solutions to the equation. Here, again, it’s easier to choose values for x. Do you see why?



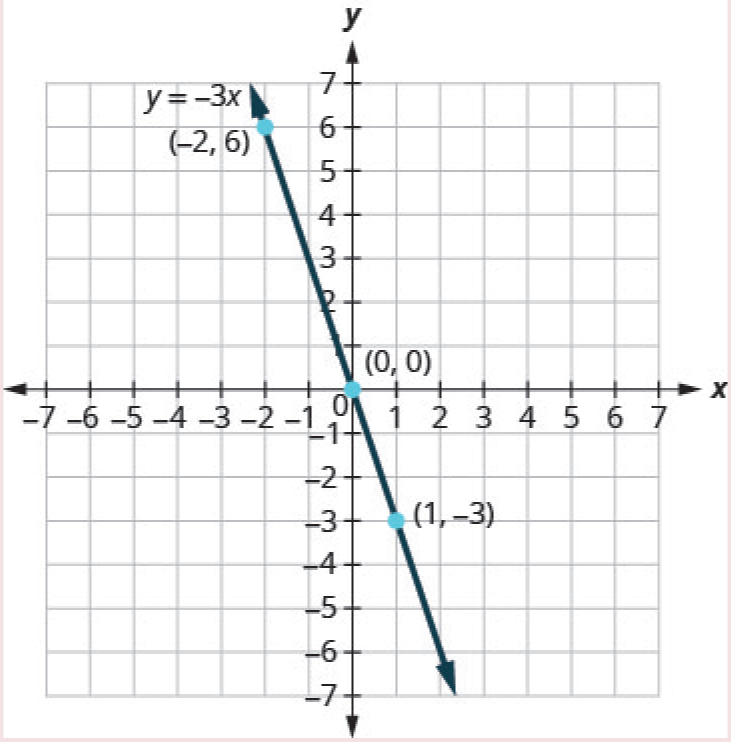
$x = 0$	$x = 1$	$x = -2$
$y = -3x$	$y = -3x$	$y = -3x$
$y = -3 \cdot 0$	$y = -3 \cdot 1$	$y = -3(-2)$
$y = 0$	$y = -3$	$y = 6$

We list the points in the table below.

$$y = -3x$$

x	y	(x, y)
0	0	$(0, 0)$
1	-3	$(1, -3)$
-2	6	$(-2, 6)$

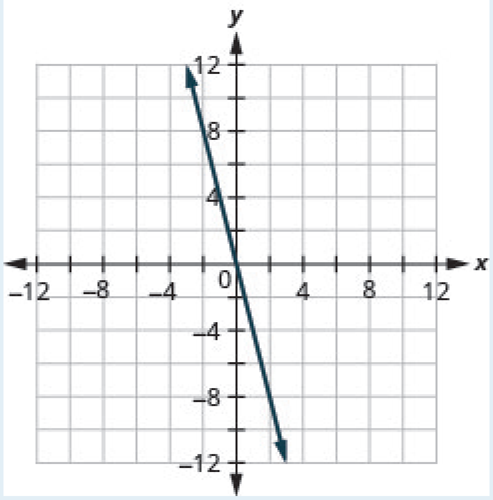
Plot the points, check that they line up, and draw the line.



TRY IT 3

Graph the equation by plotting points: $y = -4x$.

Show answer



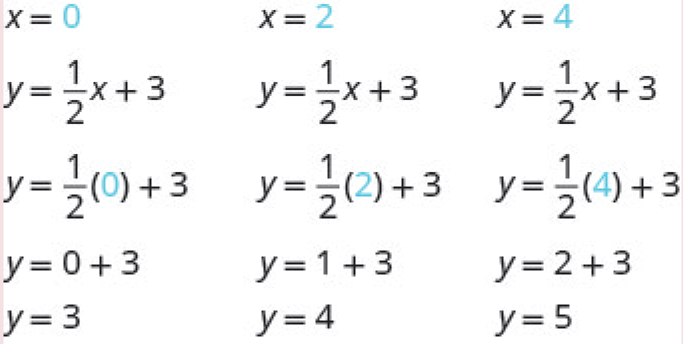
When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the math is easier if we make ‘good’ choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

EXAMPLE 4

Graph the equation $y = \frac{1}{2}x + 3$.

Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of 2 a good choice for values of x?



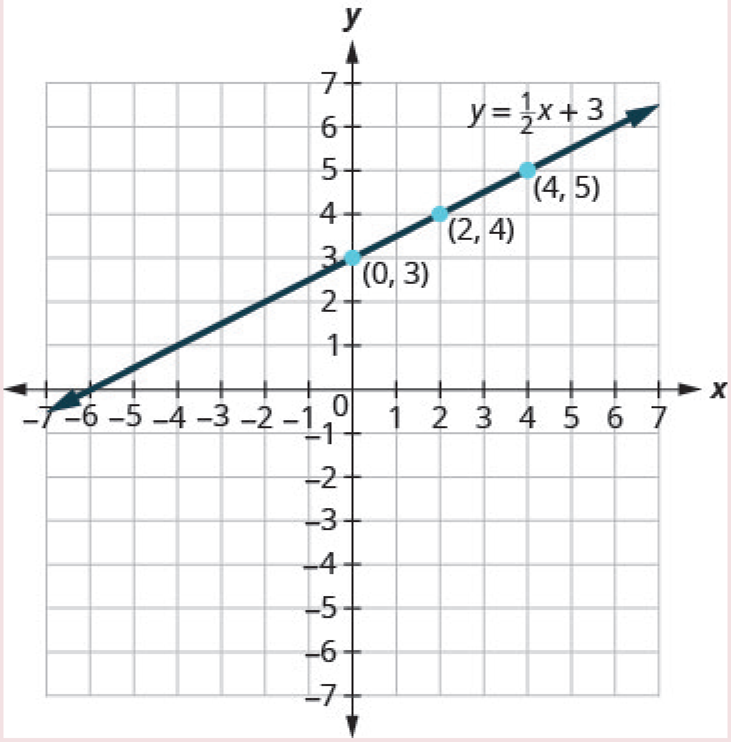
$$\begin{array}{lll}
x = 0 & x = 2 & x = 4 \\
y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 \\
y = \frac{1}{2}(0) + 3 & y = \frac{1}{2}(2) + 3 & y = \frac{1}{2}(4) + 3 \\
y = 0 + 3 & y = 1 + 3 & y = 2 + 3 \\
y = 3 & y = 4 & y = 5
\end{array}$$

The points are shown in the table below.

$$y = \frac{1}{2}x + 3$$

x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

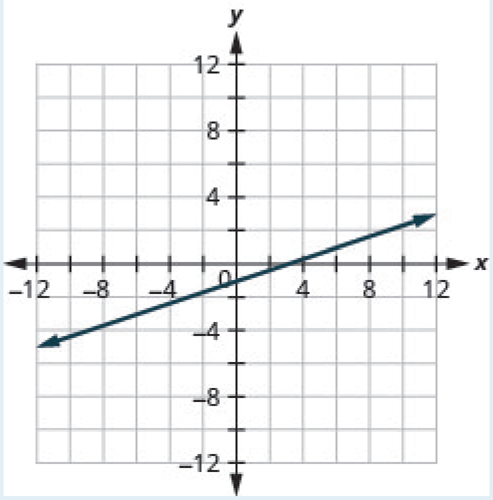
Plot the points, check that they line up, and draw the line.



TRY IT 4

Graph the equation $y = \dfrac{1}{3}x - 1$.

Show answer



So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side. Let's see what happens in the equation $2x + y = 3$. If $y = 0$ what is the value of x?

$$
\begin{aligned}
y &= 0 \\
2x + y &= 3 \\
2x + 0 &= 3 \\
2x &= 3 \\
x &= \frac{3}{2} \\
&\left(\frac{3}{2}, 0\right)
\end{aligned}
$$

This point has a fraction for the *x*– coordinate and, while we could graph this point, it is hard to be precise graphing fractions. Remember in the example $y = \frac{1}{2}x + 3$, we carefully chose values for x so as not to graph fractions at all. If we solve the equation $2x + y = 3$ for y, it will be easier to find three solutions to the equation.

$$
\begin{aligned}
2x + y &= 3 \\
y &= -2x + 3
\end{aligned}
$$

The solutions for $x = 0$, $x = 1$, and $x = -1$ are shown in the table below. The graph is shown in (Figure 5).

$2x + y = 3$

x	y	(x, y)
0	3	$(0, 3)$
1	1	$(1, 1)$
-1	5	$(-1, 5)$

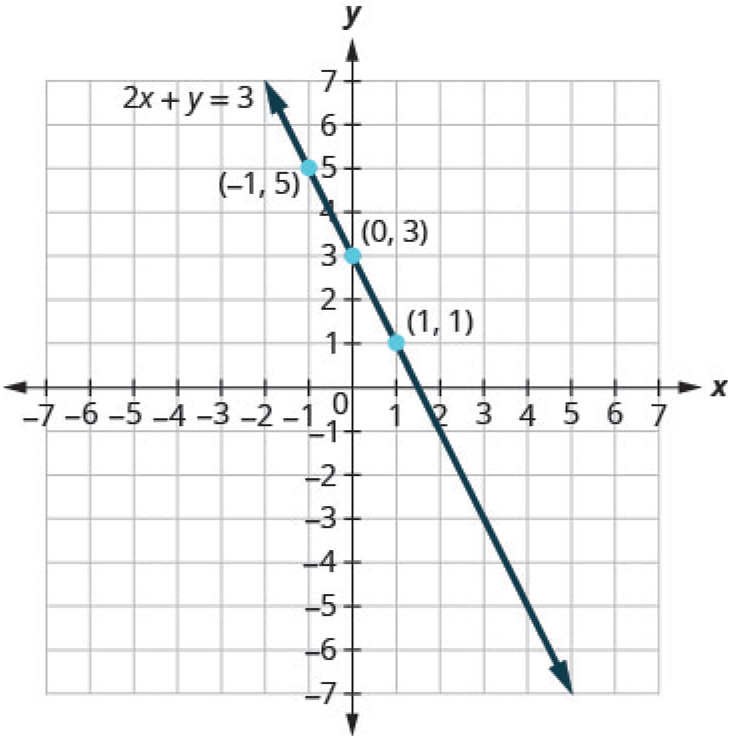


Figure .5

Can you locate the point $\left(\frac{3}{2}, 0\right)$, which we found by letting $y = 0$, on the line?

EXAMPLE 5

Graph the equation $3x + y = -1$.

Solution

Find three points that are solutions to the equation.	$3x + y \quad = \quad -1$
First, solve the equation for y.	$y \quad = \quad -3x - 1$

We'll let x be 0, 1, and -1 to find 3 points. The ordered pairs are shown in the table below. Plot the points, check that they line up, and draw the line. See (Figure 6).

$$3x + y = -1$$

x	y	(x, y)
0	-1	$(0, -1)$
1	-4	$(1, -4)$
-1	2	$(-1, 2)$

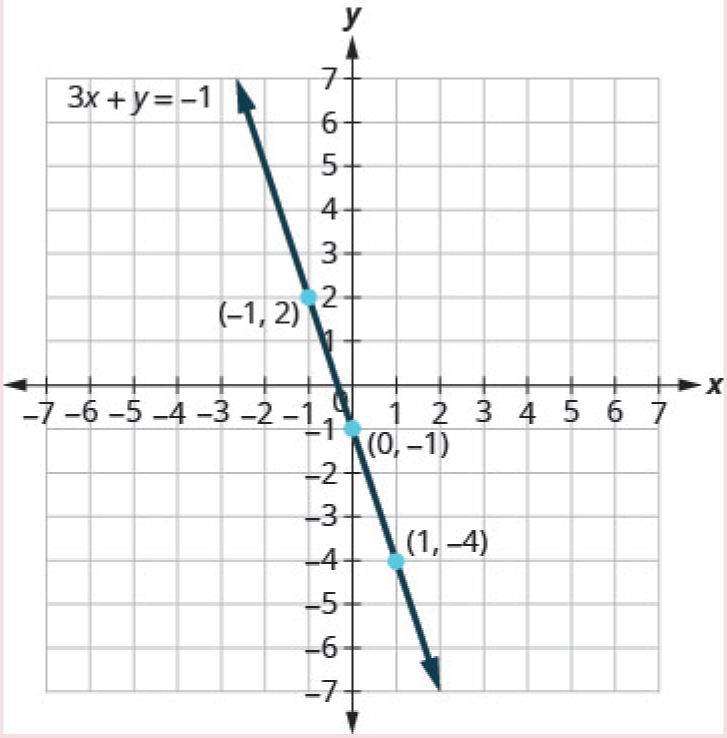
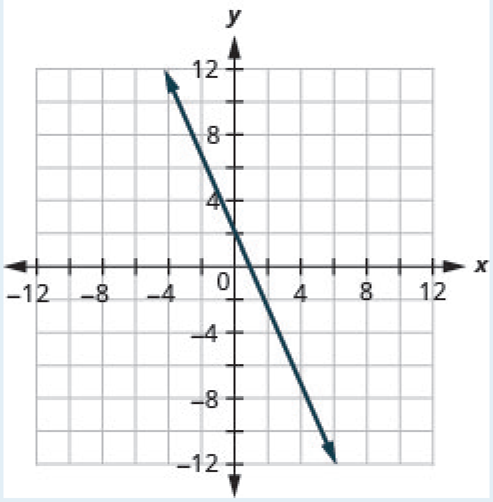


Figure .6

EXAMPLE 5

Graph the equation $2x + y = 2$.

Show answer



If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the *x*– and *y*-axis are the same, the graphs match!

The equation in (Example 5) was written in standard form, with both x and y on the same side. We solved that equation for y in just one step. But for other equations in standard form it is not that easy to solve for y, so we will leave them in standard form. We can still find a first point to plot by letting $x = 0$ and solving for y. We can plot a second point by letting $y = 0$ and then solving for x. Then we will plot a third point by using some other value for x or y.

EXAMPLE 6

Graph the equation $2x - 3y = 6$.

Solution

Find three points that are solutions to the equation.	$2x-3y\ =\ 6$
First, let $x=0$.	$2(0)-3y\ =\ 6$
Solve for y.	$-3y\ =\ 6$ $y\ =\ -2$
Now let $y=0$.	$2x-3(0)\ =\ 6$
Solve for x.	$2x\ =\ 6$ $x\ =\ 3$
We need a third point. Remember, we can choose any value for x or y. We'll let $x=6$.	$2(6)-3y\ =\ 6$
Solve for y.	$12-3y\ =\ 6$ $-3y\ =\ -6$ $y\ =\ 2$

We list the ordered pairs in the table below. Plot the points, check that they line up, and draw the line. See (Figure 7).

$$2x-3y=6$$

x	y	(x,y)
0	-2	$(0,-2)$
3	0	$(3,0)$
6	2	$(6,2)$

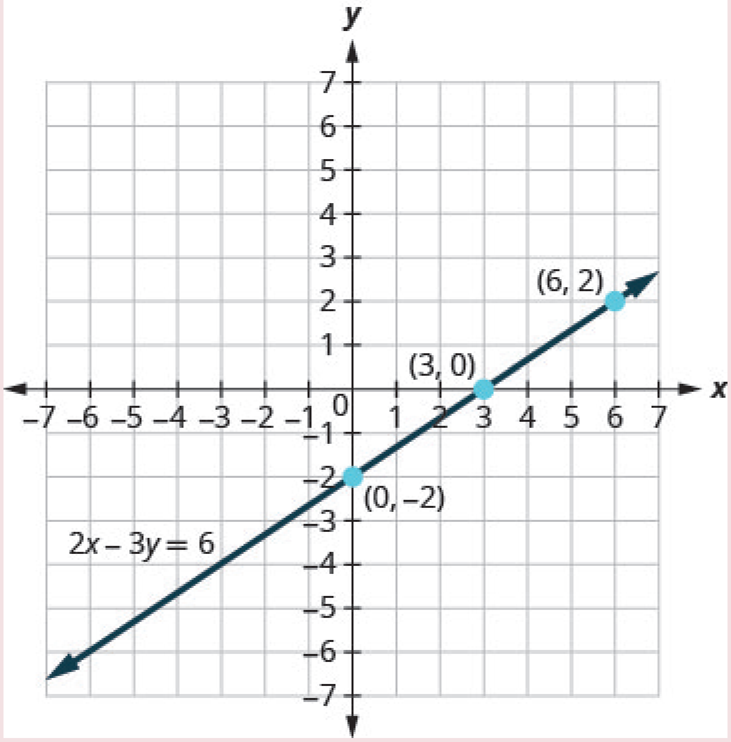
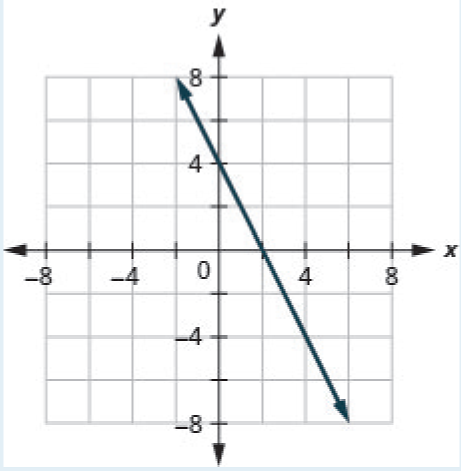


Figure .7

TRY IT 6

Graph the equation $4x + 2y = 8$.

Show answer



Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x? How will we make a table of values to get the points to plot?

Let's consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is *always* equal to -3, so its value does not depend on y. No matter what y is, the value of x is always -3.

So to make a table of values, write -3 in for all the x values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the *y*-coordinates. See the table below.

$x = -3$

x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from the table and connect them with a straight line. Notice in (Figure 8) that we have graphed a *vertical line*.

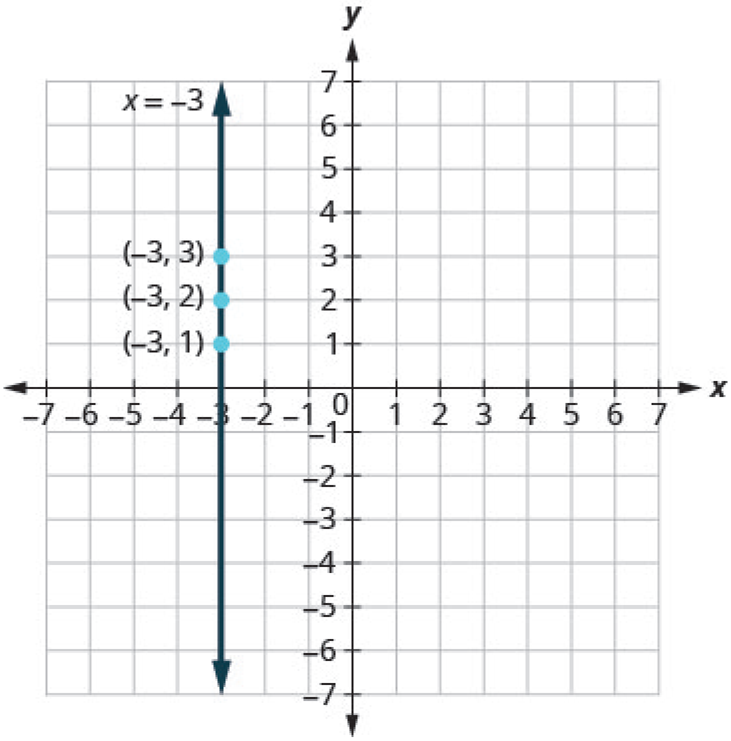


Figure .8

Vertical line

A vertical line is the graph of an equation of the form $x = a$.

The line passes through the *x*-axis at $(a, 0)$.

EXAMPLE 7

Graph the equation $x = 2$.

Solution
The equation has only one variable, x, and x is always equal to 2. We create the table below where x is always 2 and then put in any values for y. The graph is a vertical line passing through the *x*-axis at 2. See (Figure 9).

$$x = 2$$

x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$

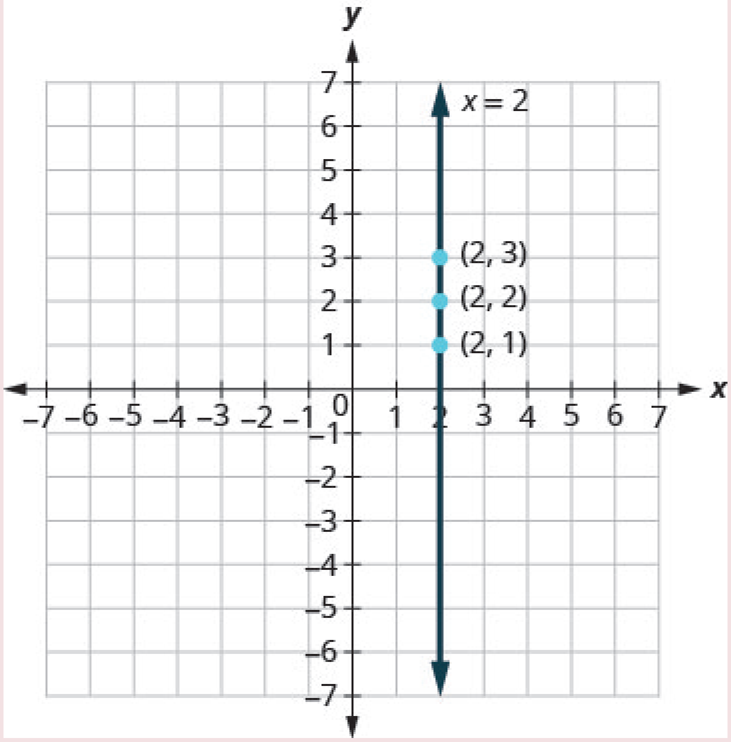
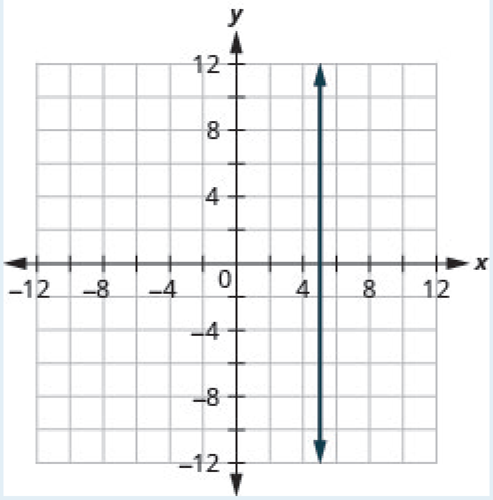


Figure .9

TRY IT 7

Graph the equation $x = 5$.

Show answer



What if the equation has y but no x? Let's graph the equation $y = 4$. This time the *y*– value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in the table below and then choose any values for x. We'll use 0, 2, and 4 for the *x*-coordinates.

$$y = 4$$

x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

The graph is a horizontal line passing through the *y*-axis at 4. See (Figure 10).

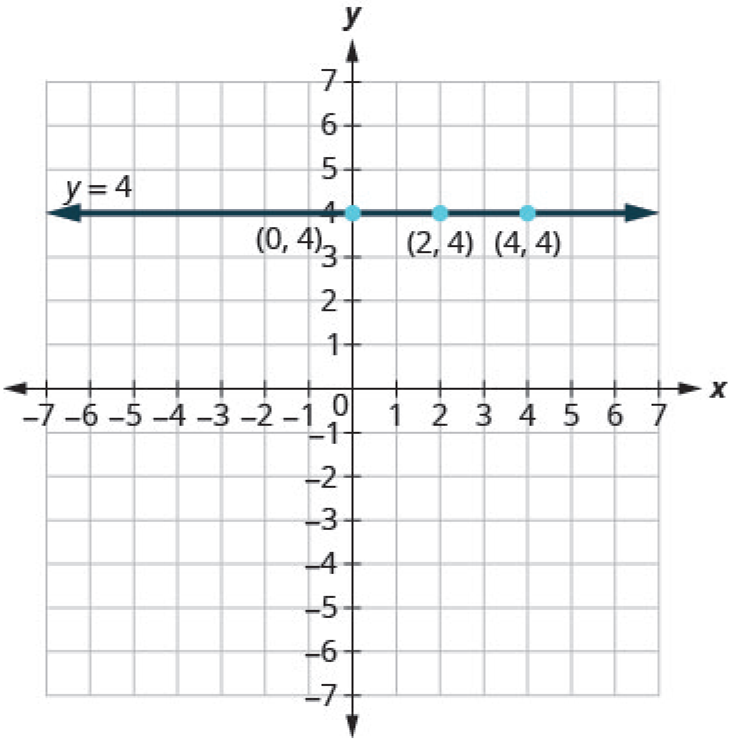


Figure .10

Horizontal line

A horizontal line is the graph of an equation of the form $y = b$.
The line passes through the *y*-axis at $(0, b)$.

EXAMPLE 8

Graph the equation $y = -1$.

Solution

The equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the table below have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at -1, as shown in (Figure 11).

$y = -1$

x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

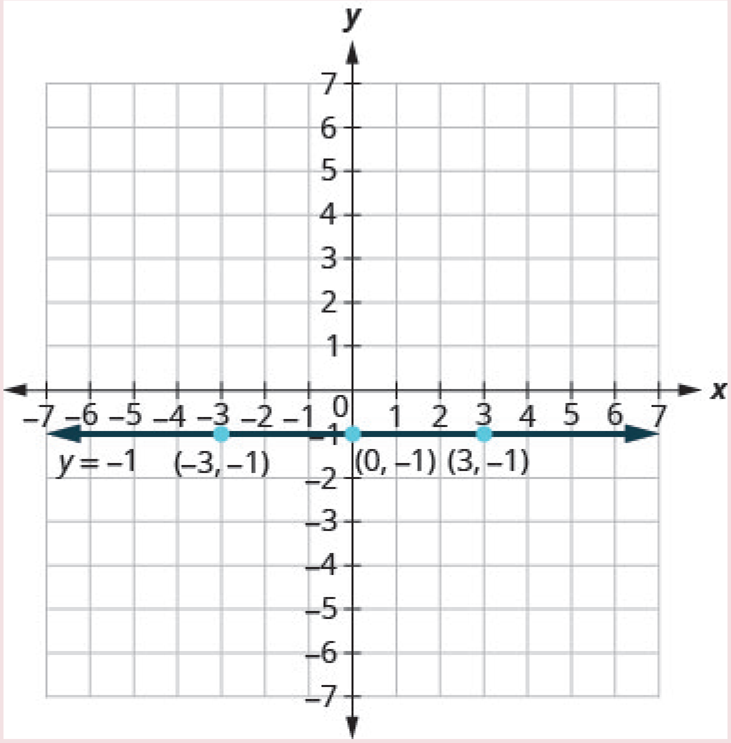
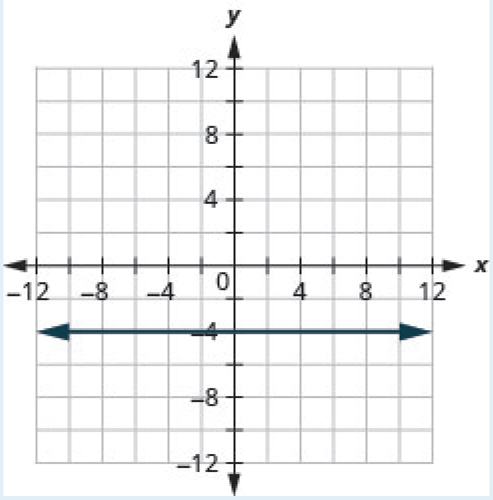


Figure .11

TRY IT 8

Graph the equation $y = -4$.

Show answer



The equations for vertical and horizontal lines look very similar to equations like $y = 4x$. What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x. The *y*-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant. The *y*-coordinate is always 4. It does not depend on the value of x. See the tables below.

$y = 4x$

x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

$y = 4$

x	y	(x, y)
0	0	$(0, 0)$
1	4	$(1, 4)$
2	8	$(2, 8)$

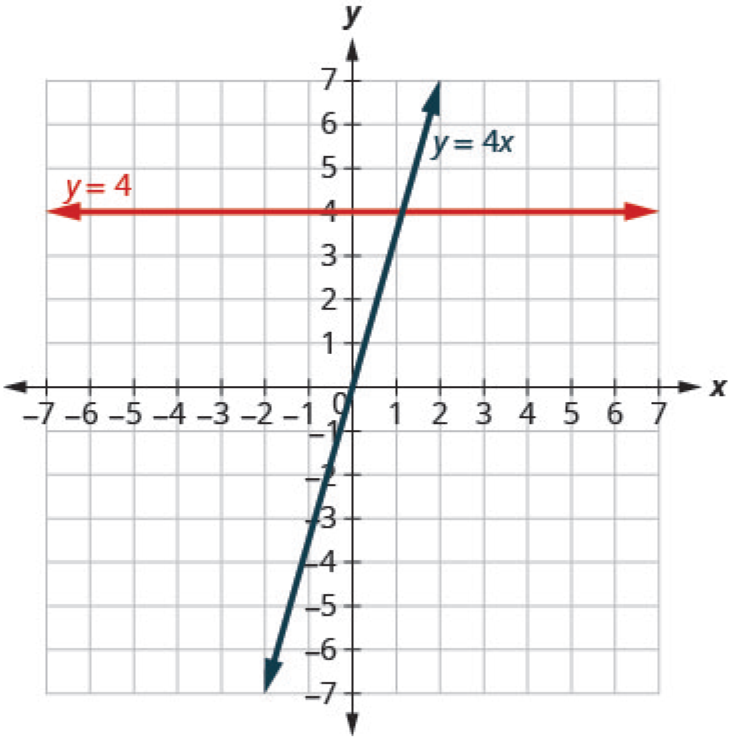


Figure .12

Notice, in (Figure 12), the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

EXAMPLE 9

Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution
Notice that the first equation has the variable x, while the second does not. See the tables below. The two graphs are shown in (Figure 13).

$$y = -3x$$

x	y	(x, y)
0	0	$(0, 0)$
1	-3	$(1, -3)$
2	-6	$(2, -6)$

$$y = -3$$

x	y	(x, y)
0	-3	$(0, -3)$
1	-3	$(1, -3)$
2	-3	$(2, -3)$

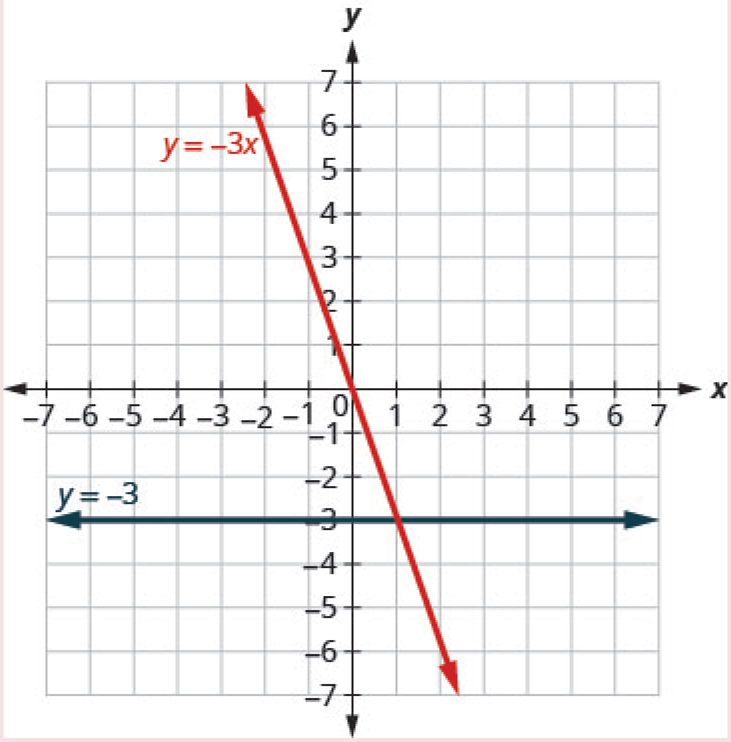
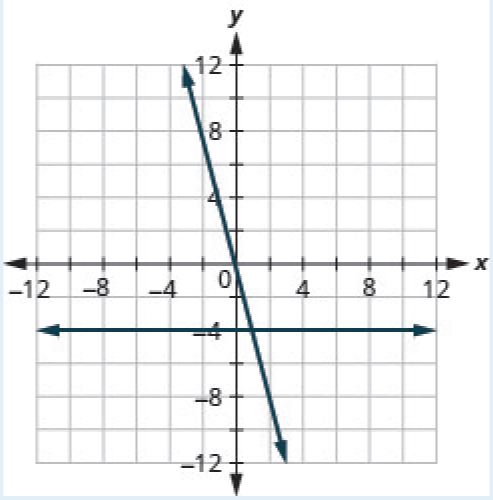


Figure .13

TRY IT 9

Graph $y = -4x$ and $y = -4$ in the same rectangular coordinate system.

Show answer



Key Concepts

- **Graph a Linear Equation by Plotting Points**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

Glossary

graph of a linear equation
The graph of a linear equation $Ax + By = C$ is a straight line. Every point on the line is a solution of the equation. Every solution of this equation is a point on this line.

horizontal line
A horizontal line is the graph of an equation of the form $y = b$. The line passes through the *y*-axis at $(0, b)$.

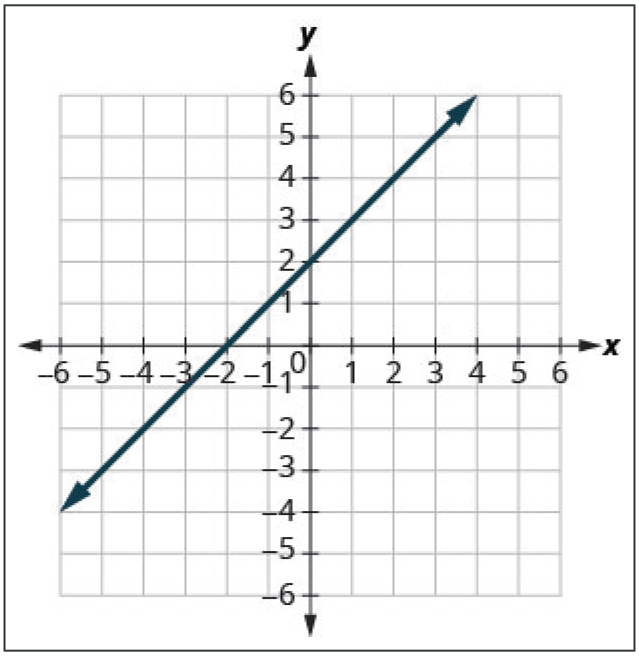
vertical line
A vertical line is the graph of an equation of the form $x = a$. The line passes through the *x*-axis at $(a, 0)$.

3.2 Exercise Set

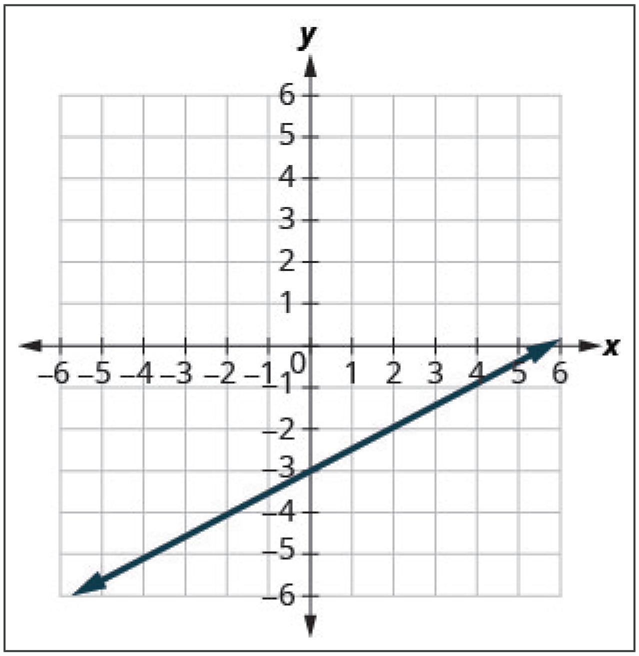
In the following exercises, for each ordered pair, decide:

a) Is the ordered pair a solution to the equation? b) Is the point on the line?

1. $y = x + 2$
   - A. $(0, 2)$
   - B. $(1, 2)$
   - C. $(-1, 1)$
   - D. $(-3, -1)$



2. $y = \frac{1}{2}x - 3$
   - A. $(0, -3)$
   - B. $(2, -2)$
   - C. $(-2, -4)$
   - D. $(4, 1)$



In the following exercises, graph by plotting points.

3. $y = 3x - 1$
4. $y = -3x + 3$
5. $y = x + 2$
6. $y = -x - 3$
7. $y = 2x$
8. $y = 3x$
9. $y = \frac{1}{2}x + 2$
10. $y = \frac{4}{3}x - 5$
11. $y = -\frac{2}{5}x + 1$
12. $y = -\frac{3}{2}x + 2$
13. $x + y = 6$
14. $x + y = -3$
15. $x - y = 2$
16. $x - y = -1$
17. $3x + y = 7$
18. $2x + y = -3$
19. $\frac{1}{3}x + y = 2$
20. $-\frac{1}{2}x - y = -3$
21. $2x + 3y = 12$
22. $3x - 4y = 12$
23. $x - 6y = 3$
24. $3x + y = 2$

In the following exercises, graph each equation.

25. $x = 4$
26. $x = -2$
27. $y = 3$
28. $y = -5$
29. $x = \frac{7}{3}$
30. $y = -\frac{15}{4}$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

31. $y = 2x$ and $y = 2$

32. $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$

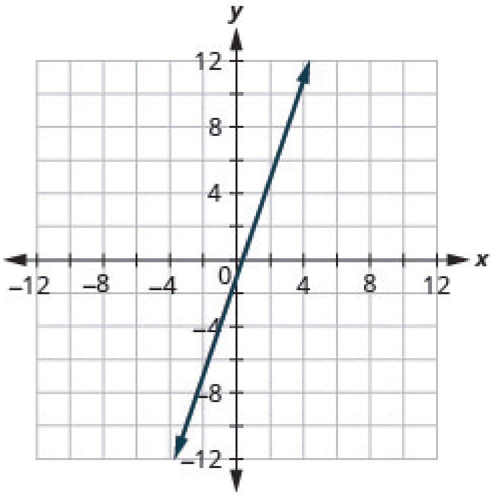
33. The Stonechilds rented a motor home for one week to go on vacation. It cost them $594 plus $0.32 per mile to rent the motor home, so the linear equation $y = 594 + 0.32x$ gives the cost, y, for driving x miles. Calculate the rental cost for driving 400, 800, and 1200 miles, and then graph the line.

Answers

1.
   - A. yes; no
   - B. no; no
   - C. yes; yes
   - D. yes; yes

2.
   - A. yes; yes
   - B. yes; yes
   - C. yes; yes
   - D. no; no

3.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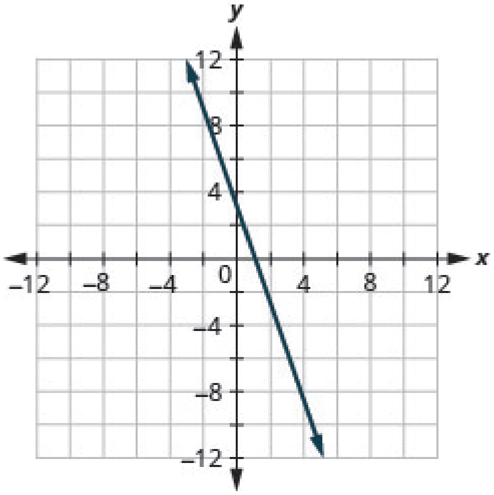
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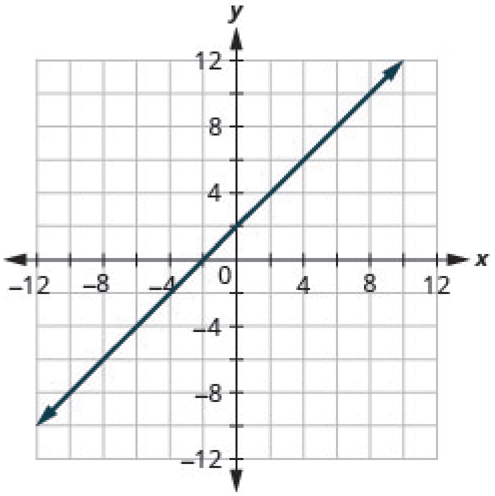
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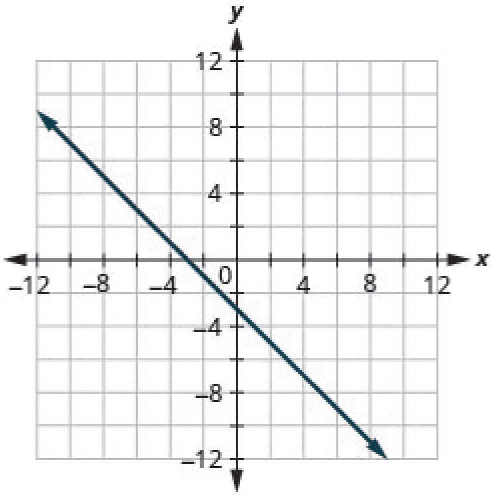
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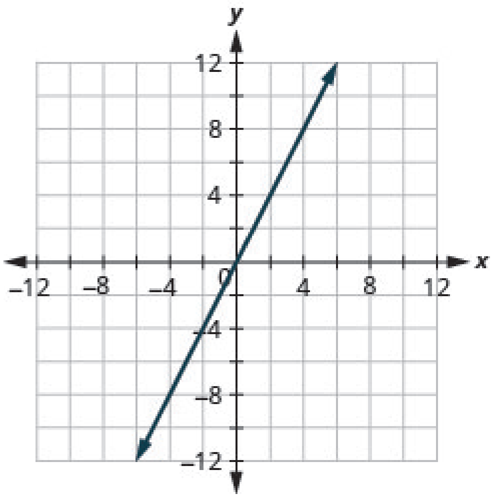
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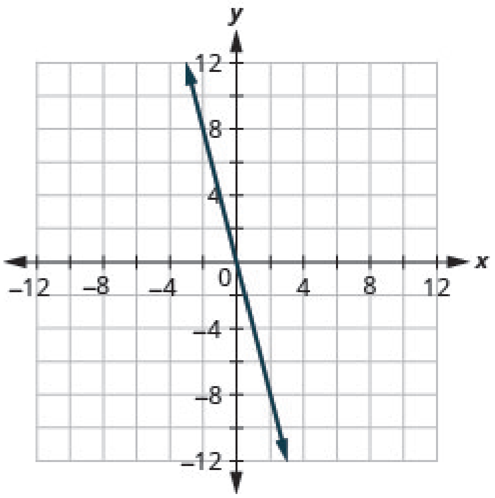
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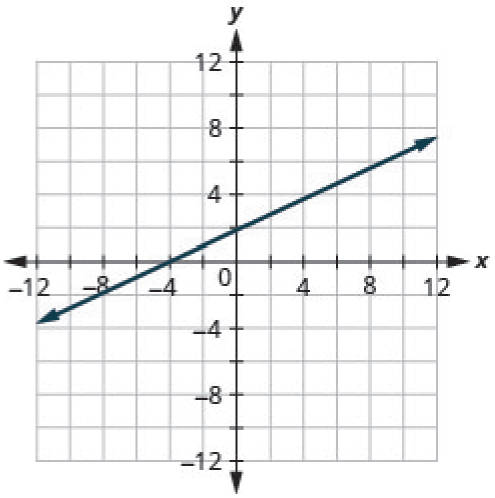
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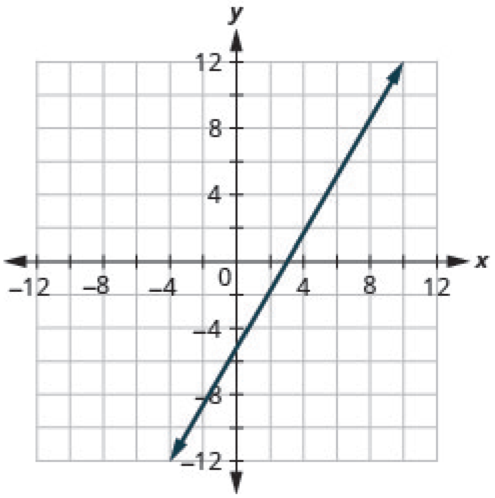
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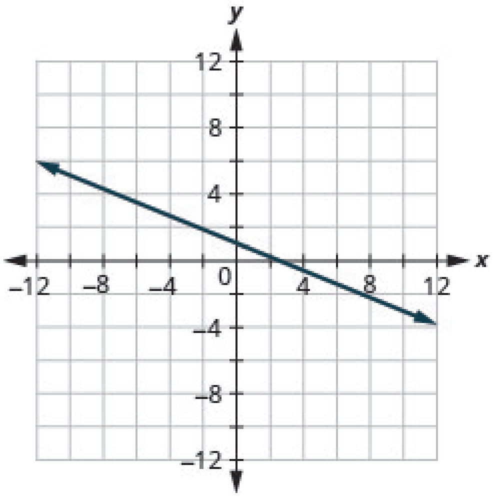
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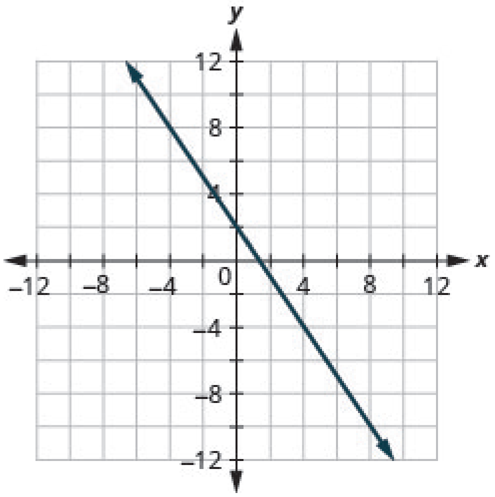
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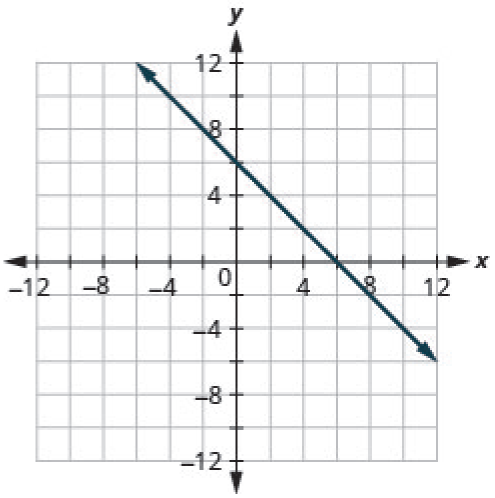

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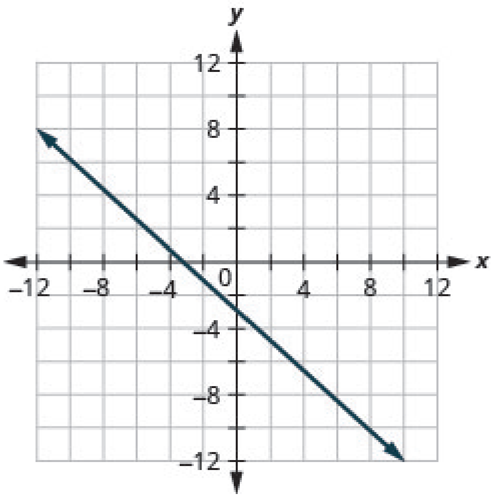

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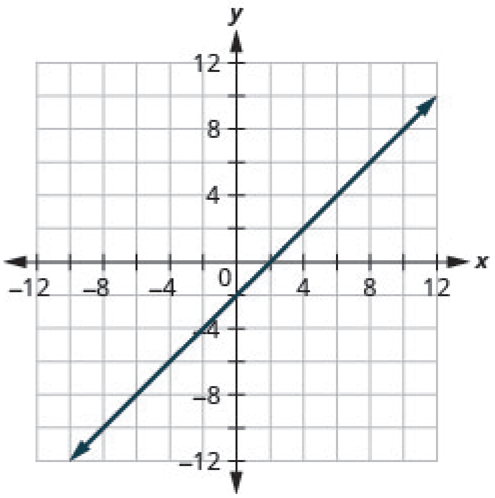

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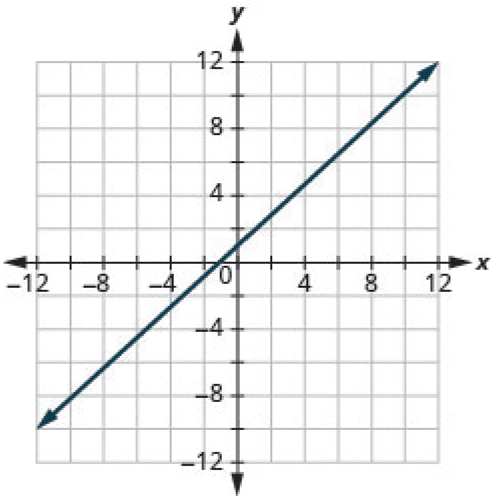

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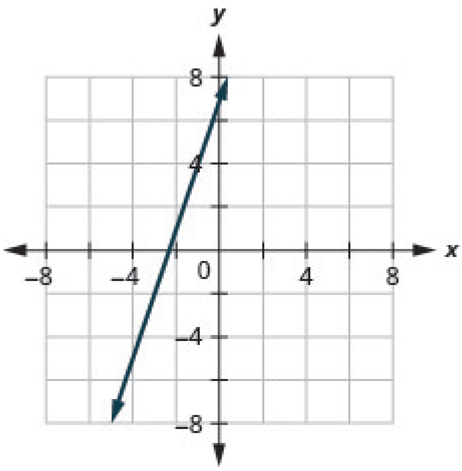

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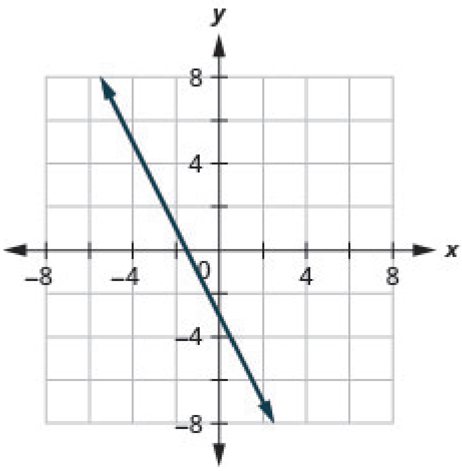

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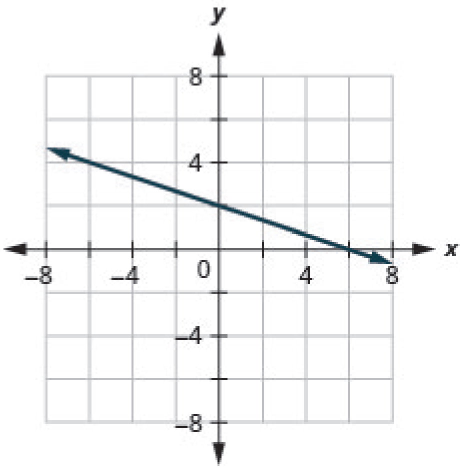

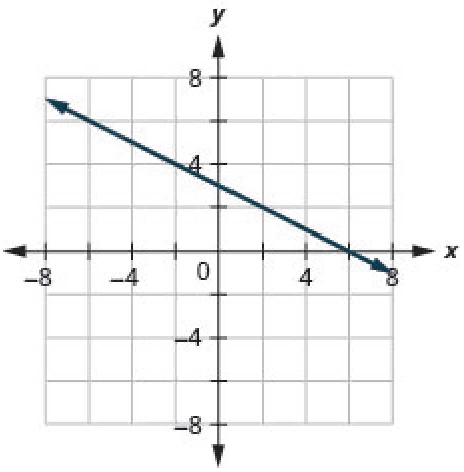

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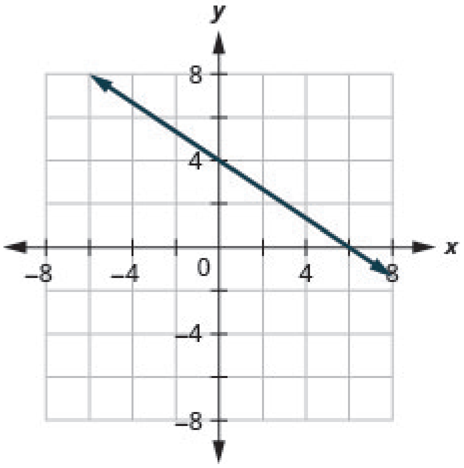

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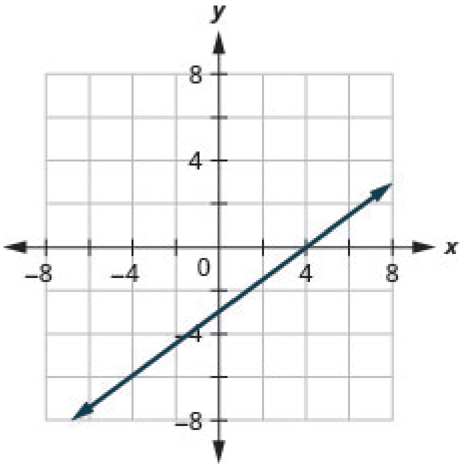

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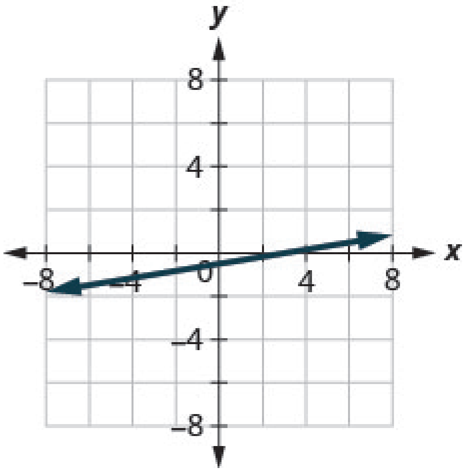

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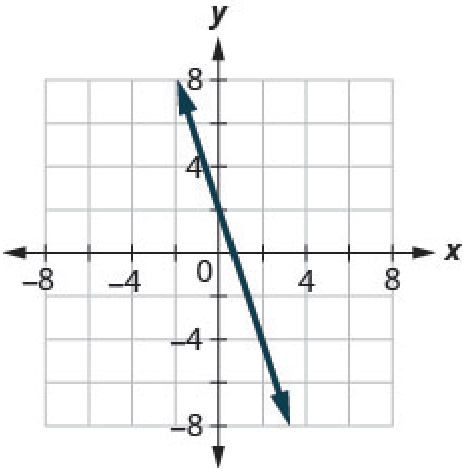

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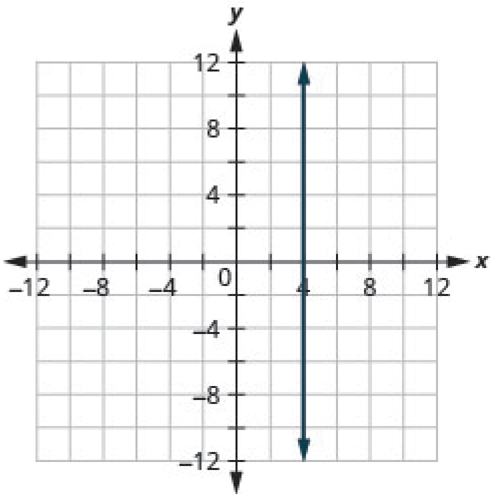

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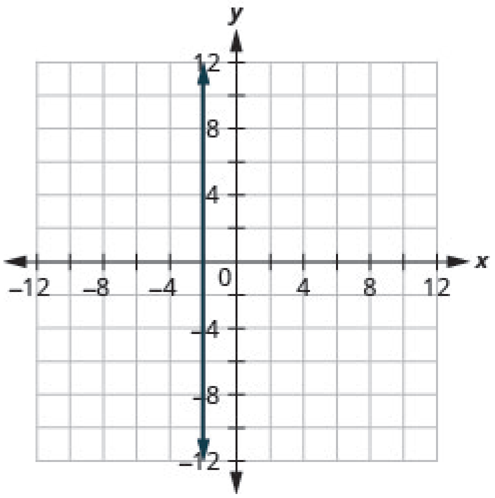

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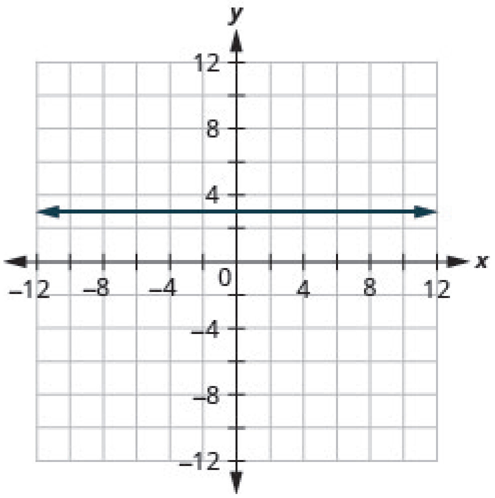

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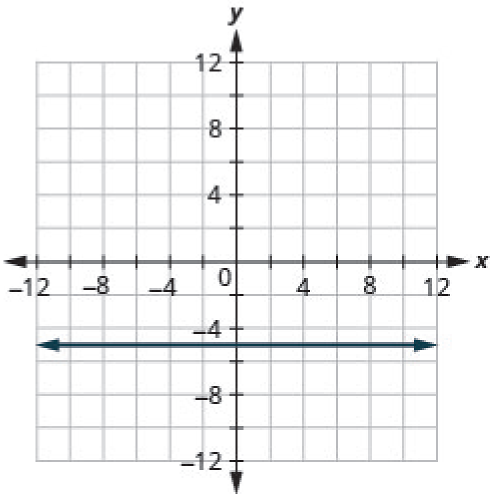
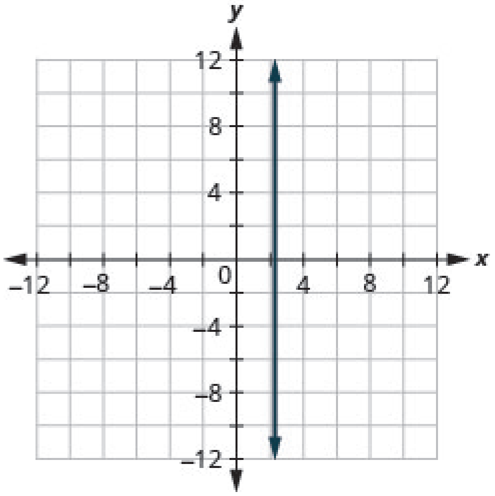
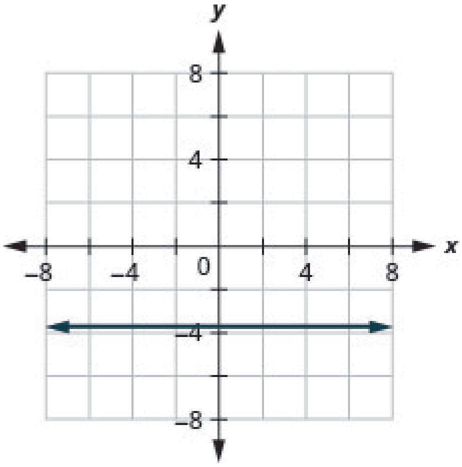
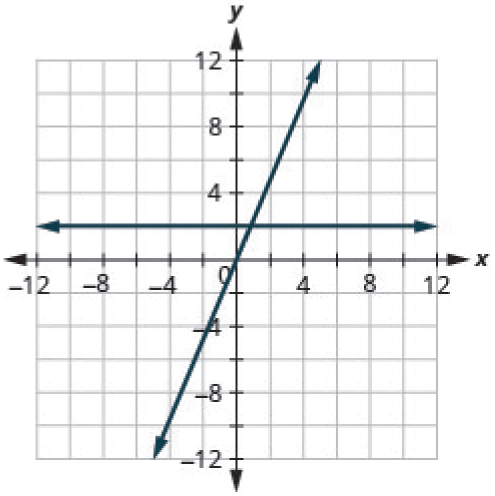
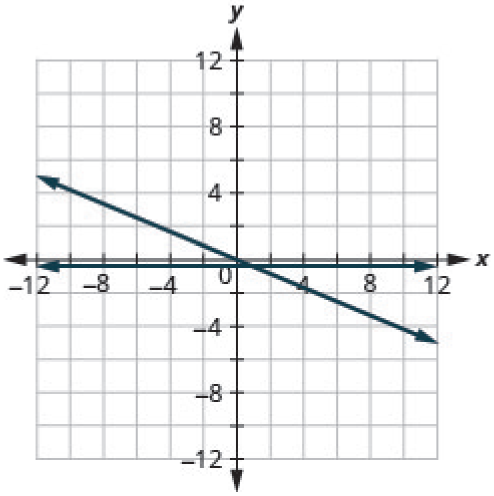
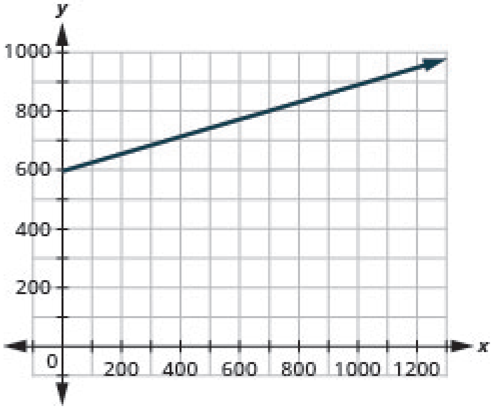


29. 	30. 
31. 	32. 
33. $722, $850, $978 	

3.3 Graph with Intercepts – optional

Learning Objectives

By the end of this section it is expected that you will be able to:

- Identify the x– and y– intercepts on a graph
- Find the x– and y– intercepts from an equation of a line
- Graph a line using the intercepts

Identify the *x*– and *y*– Intercepts on a Graph

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

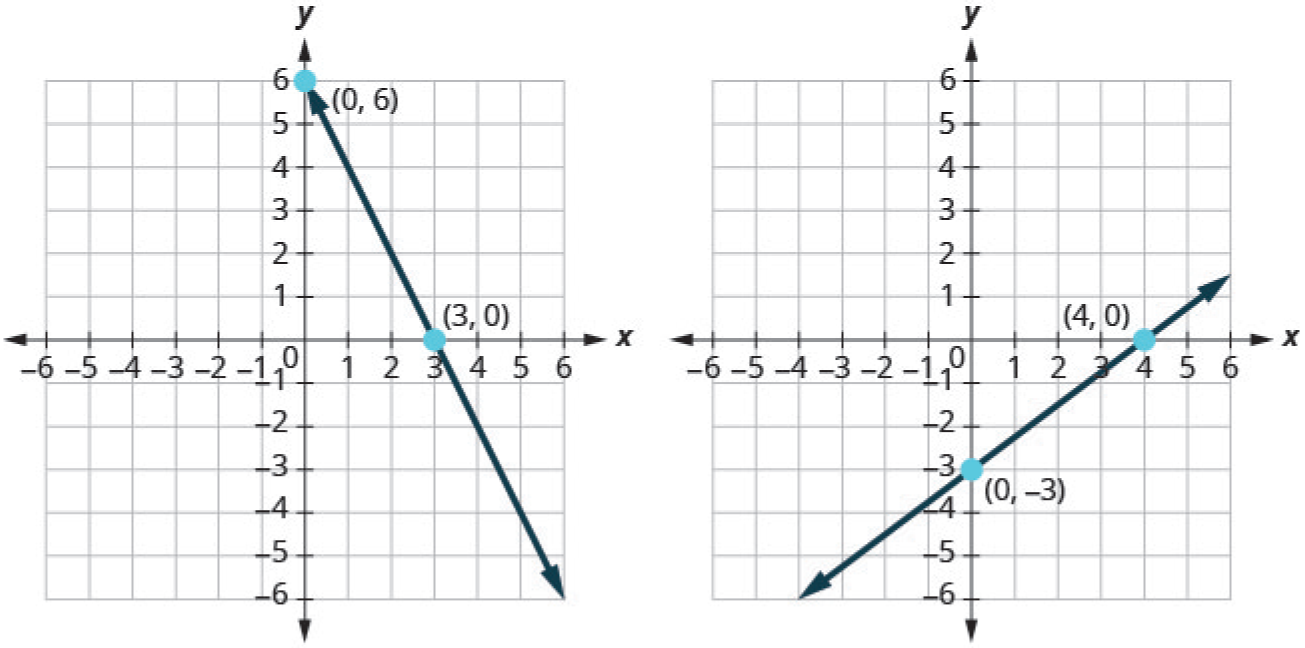
At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*– axis and the *y*– axis. These points are called the *intercepts* of the line.

Intercepts of a line

The points where a line crosses the *x*– axis and the *y*– axis are called the intercepts of a line.

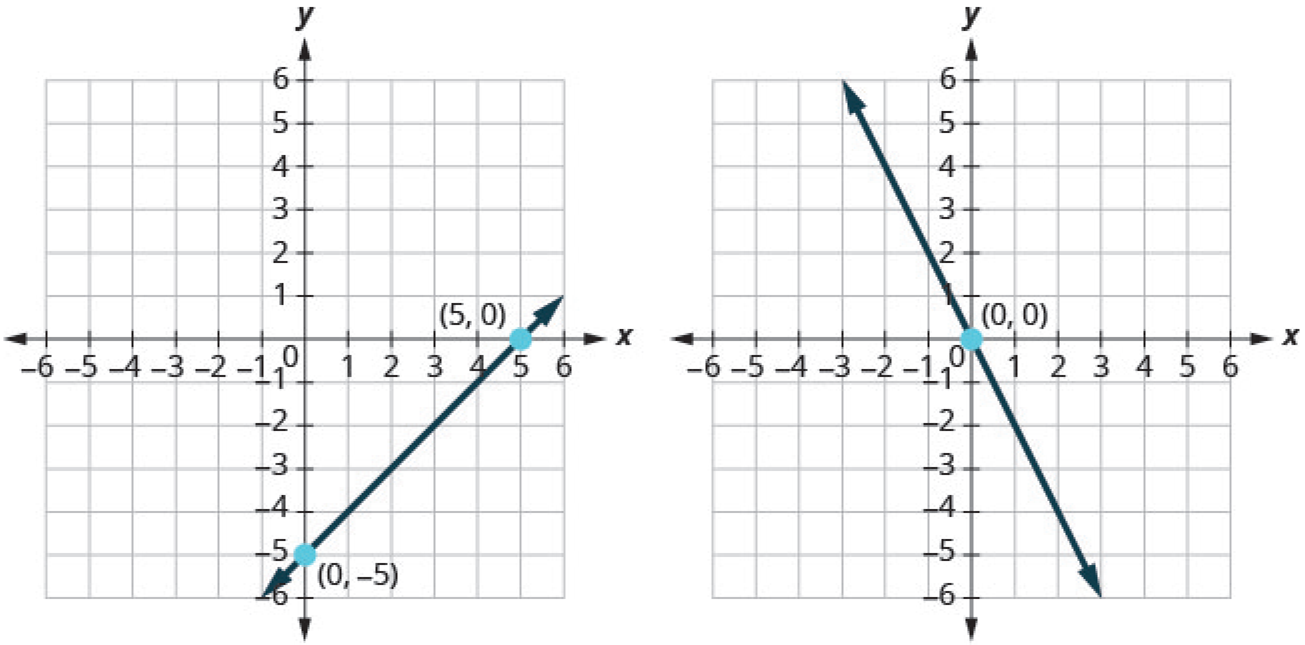
Let's look at the graphs of the lines in (Figure 1).

Examples of graphs crossing the x-negative axis.



(a) $2x + y = 6$

(b) $3x - 4y = 12$



(c) $x - y = 5$

(d) $y = -2x$

Figure .1

First, notice where each of these lines crosses the x negative axis. See (Figure 1).

Figure	**The line crosses the *x*– axis at:**	**Ordered pair of this point**
Figure (a)	3	$(3, 0)$
Figure (b)	4	$(4, 0)$
Figure (c)	5	$(5, 0)$
Figure (d)	0	$(0, 0)$

Do you see a pattern?

For each row, the *y*– coordinate of the point where the line crosses the *x*– axis is zero. The point where the line crosses the *x*– axis has the form $(a, 0)$ and is called the *x*– intercept of a line. The *x*– intercept occurs when y is zero.

Now, let's look at the points where these lines cross the *y*– axis. See the table below.

Figure	**The line crosses the *y*-axis at:**	**Ordered pair for this point**
Figure (a)	6	$(0, 6)$
Figure (b)	-3	$(0, -3)$
Figure (c)	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$

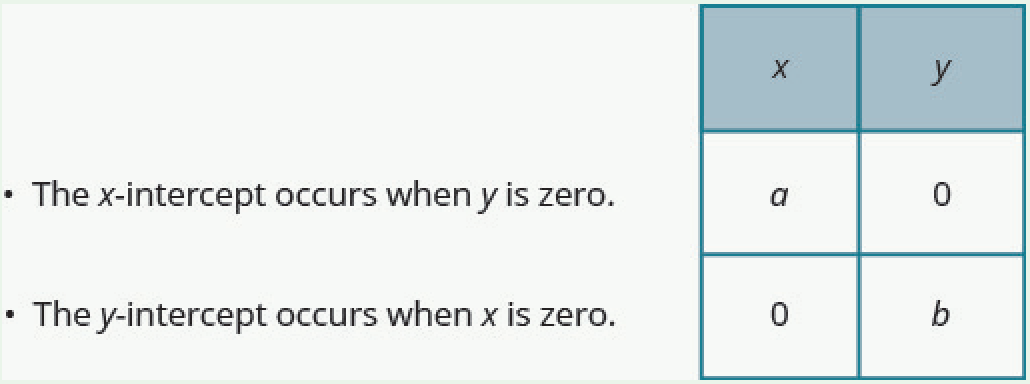
What is the pattern here?

In each row, the *x*– coordinate of the point where the line crosses the *y*– axis is zero. The point where the line crosses the *y*– axis has the form $(0, b)$ and is called the *y*- *intercept* of the line. The *y*– intercept occurs when x is zero.

x– intercept and *y*– intercept of a line

The *x*– intercept is the point $(a, 0)$ where the line crosses the *x*– axis.

The *y*– intercept is the point $(0, b)$ where the line crosses the *y*– axis.

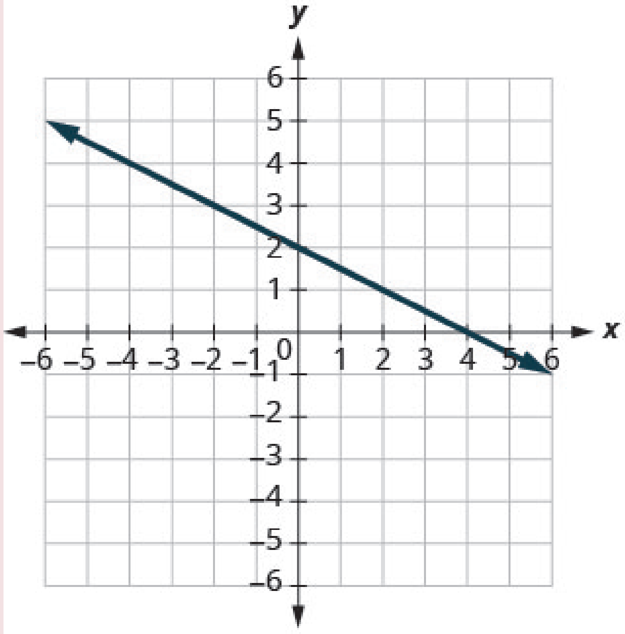


- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

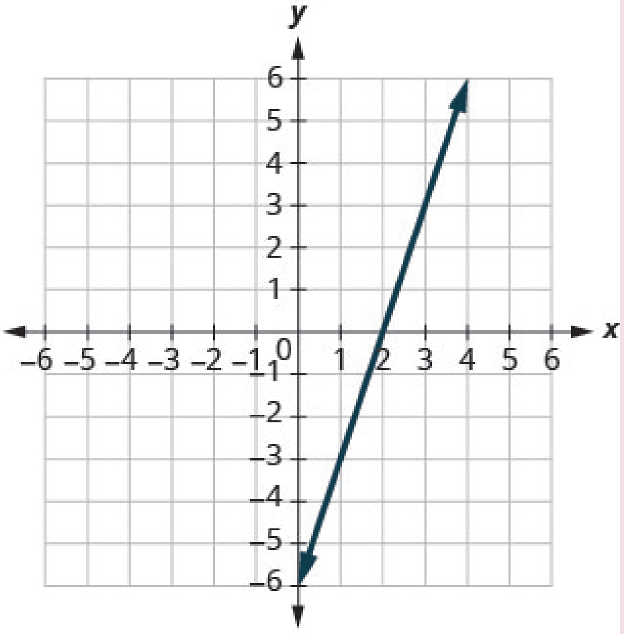
x	y
a	0
0	b

EXAMPLE 1

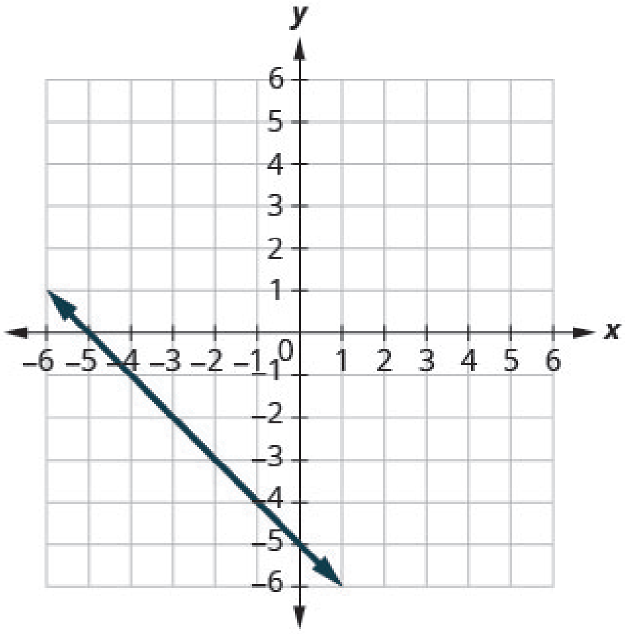
Find the x– and y– intercepts on each graph.



(a)



(b)



(c)

Solution

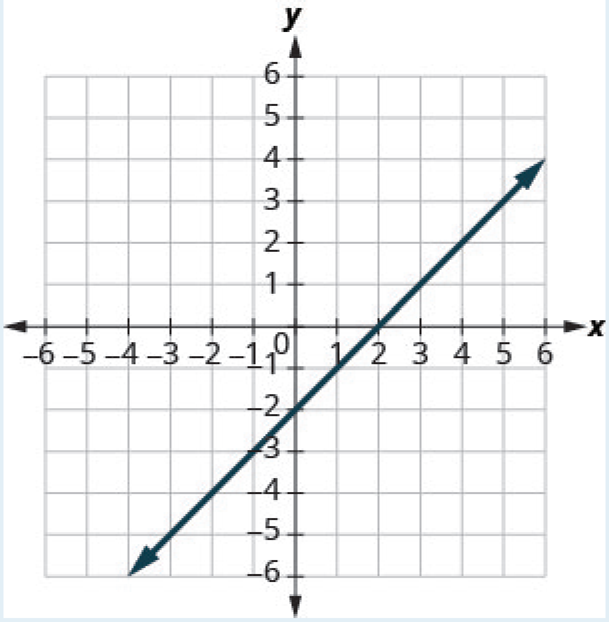
a) The graph crosses the *x*– axis at the point $(4, 0)$. The *x*– intercept is $(4, 0)$.
The graph crosses the *y*– axis at the point $(0, 2)$. The *y*– intercept is $(0, 2)$.

b) The graph crosses the *x*– axis at the point $(2, 0)$. The *x*– intercept is $(2, 0)$
The graph crosses the *y*– axis at the point $(0, -6)$. The *y*– intercept is $(0, -6)$.

c) The graph crosses the *x*– axis at the point $(-5,0)$. The *x*– intercept is $(-5,0)$.
The graph crosses the *y*– axis at the point $(0,-5)$. The *y*– intercept is $(0,-5)$.

TRY IT 1

Find the *x*– and *y*– intercepts on the graph.



Show answer
x– intercept: $(2,0)$; *y*– intercept: $(0,-2)$

Find the *x*– and *y*– Intercepts from an Equation of a Line

Recognizing that the *x*– intercept occurs when *y* is zero and that the *y*– intercept occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the *x*– intercept, let $y=0$ and solve for *x*. To find the *y*– intercept, let $x=0$ and solve for *y*.

Find the *x*– and *y*– intercepts from the equation of a line

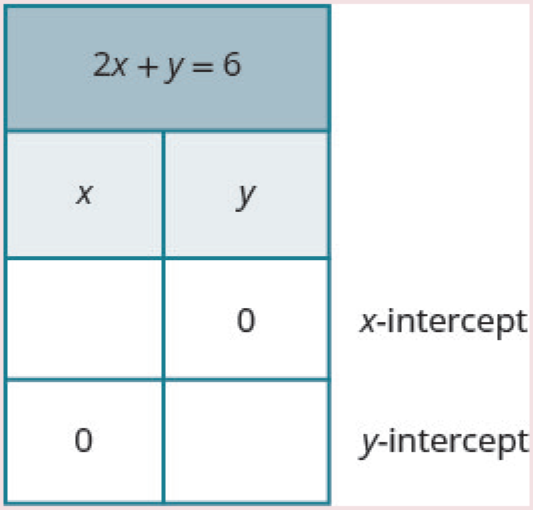
Use the equation of the line. To find:

- the *x*– intercept of the line, let $y=0$ and solve for x.
- the *y*– intercept of the line, let $x=0$ and solve for y.

EXAMPLE 2

Find the intercepts of $2x + y = 6$.

Solution
We will let $y = 0$ to find the *x*– intercept, and let $x = 0$ to find the *y*– intercept. We will fill in the table, which reminds us of what we need to find.



$2x+y=6$		
x	y	
	0	*x*-intercept
0		*y*-intercept

To find the *x*– intercept, let $y = 0$.

	$2x+y=6$
Let *y* = 0.	$2x+0=6$
Simplify.	$2x=6$
	$x=3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let *x* = 0.	
	$2x+y=6$
Let *x* = 0.	$2 \cdot 0+y=6$
Simplify.	$0+y=6$
	$y=6$
The *y*-intercept is	(0, 6)

The intercepts are the points $(3, 0)$ and $(0, 6)$ as shown in the following table.

$$2x + y = 6$$

x	y
3	0
0	6

TRY 2

Find the intercepts of $3x + y = 12$.

Show answer
x– intercept: $(4, 0)$, *y*– intercept: $(0, 12)$

EXAMPLE 3

Find the intercepts of $4x - 3y = 12$.

Solution

To find the *x*-intercept, let *y* = 0.	
	$4x - 3y = 12$
Let *y* = 0.	$4x - 3 \cdot 0 = 12$
Simplify.	$4x - 0 = 12$
	$4x = 12$
	$x = 3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let *x* = 0.	
	$4x - 3y = 12$
Let *x* = 0.	$4 \cdot 0 - 3y = 12$
Simplify.	$0 - 3y = 12$
	$-3y = 12$
	$y = -4$
The *y*-intercept is	(0, −4)

The intercepts are the points (3, 0) and (0, −4) as shown in the following table.

$$4x - 3y = 12$$

x	y
3	0
0	-4

TRY IT 3

Find the intercepts of $3x - 4y = 12$.

Show answer
x– intercept: $(4, 0)$, *y*– intercept: $(0, -3)$

Graph a Line Using the Intercepts

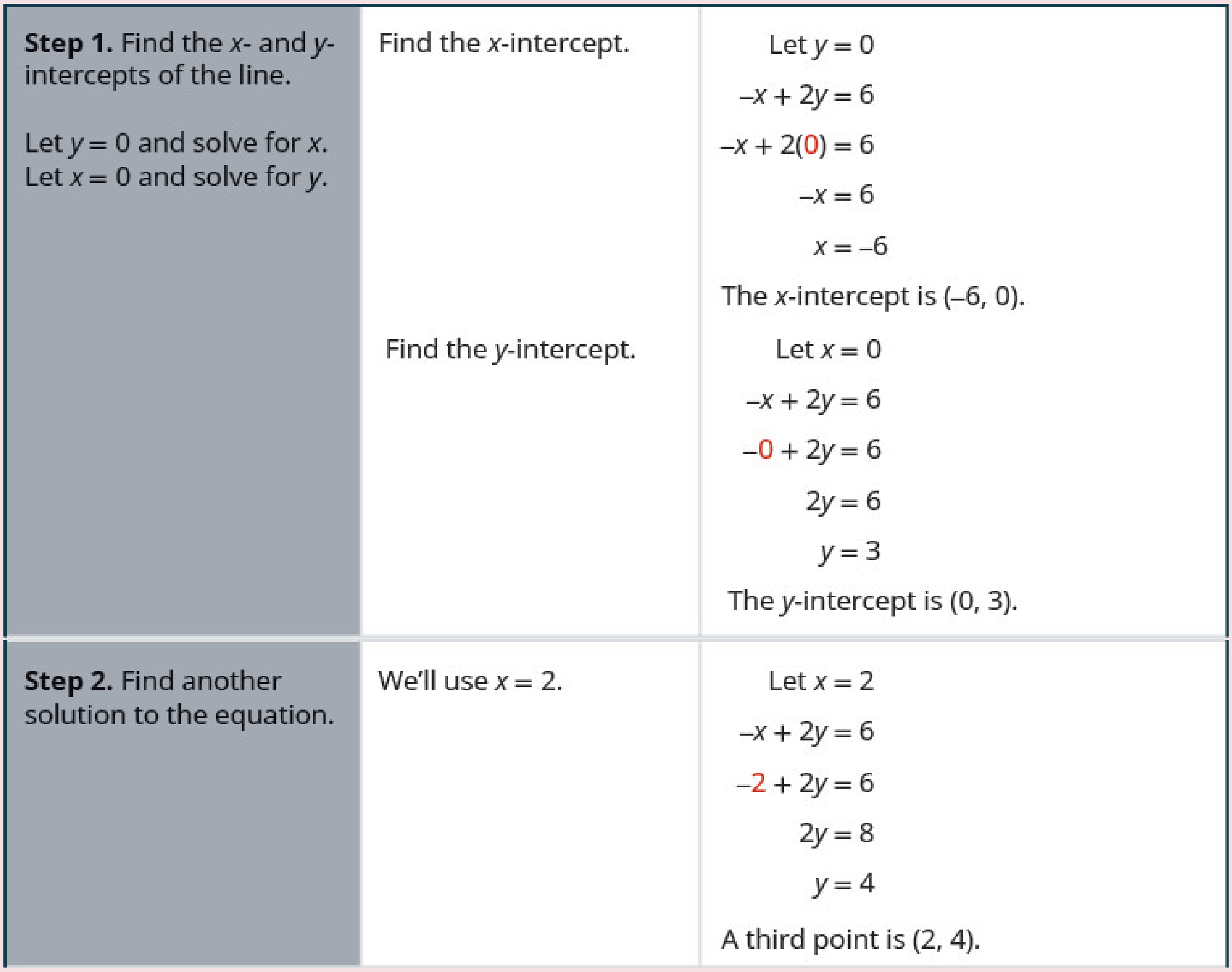
To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*– and *y*– intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

EXAMPLE 4

How to Graph a Line Using Intercepts

Graph $-x+2y=6$ using the intercepts.

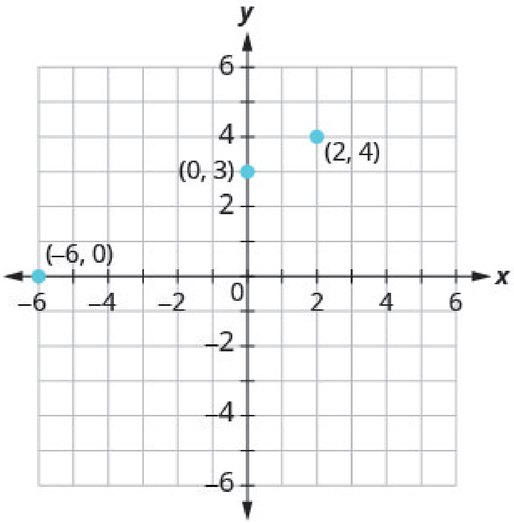
Solution



Step 1. Find the *x*- and *y*-intercepts of the line. Let $y = 0$ and solve for x. Let $x = 0$ and solve for y.	Find the *x*-intercept.	Let $y=0$ $-x+2y=6$ $-x+2(0)=6$ $-x=6$ $x=-6$ The *x*-intercept is $(-6, 0)$.
	Find the *y*-intercept.	Let $x=0$ $-x+2y=6$ $-0+2y=6$ $2y=6$ $y=3$ The *y*-intercept is $(0, 3)$.
Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x=2$ $-x+2y=6$ $-2+2y=6$ $2y=8$ $y=4$ A third point is $(2, 4)$.

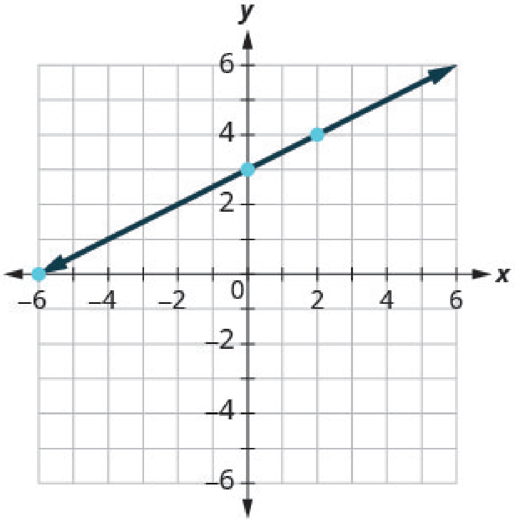
Step 3. Plot the three points. Check that the points line up.

x	y	(x, y)
–6	0	(–6, 0)
0	3	(0, 3)
2	4	(2, 4)



Step 4. Draw the line.

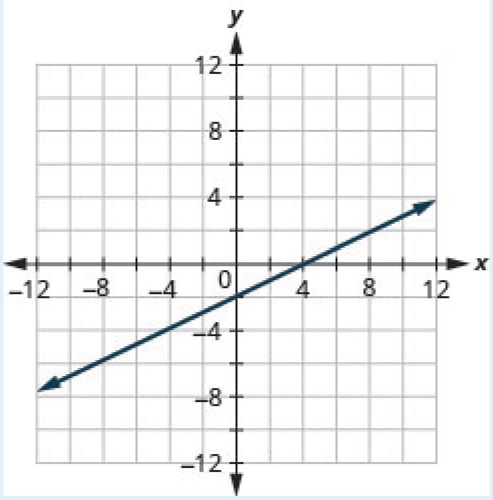
See the graph.



TRY IT 4

Graph $x - 2y = 4$ using the intercepts.

Show answer



HOW TO: Graph a linear equation using the intercepts

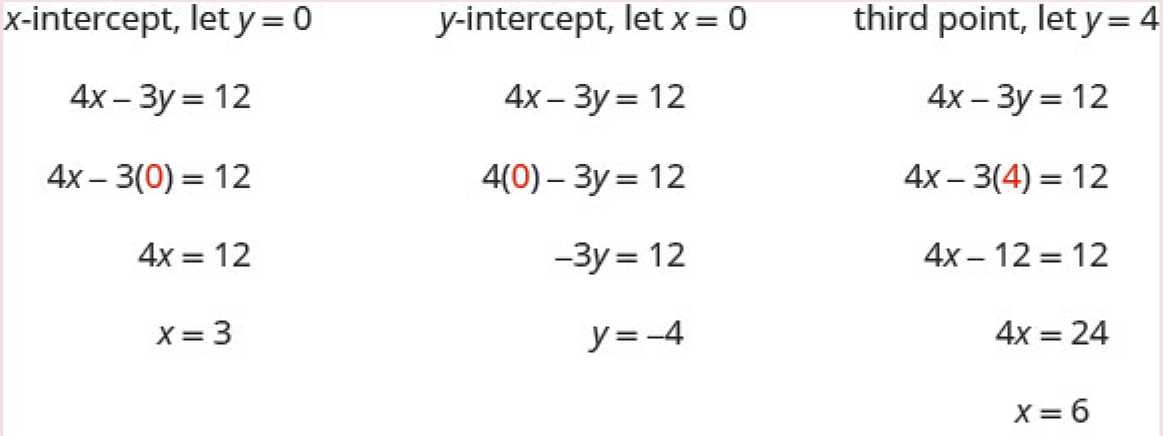
The steps to graph a linear equation using the intercepts are summarized below.

1. Find the *x*– and *y*– intercepts of the line.
    - Let $y = 0$ and solve for x
    - Let $x = 0$ and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

EXAMPLE 5

Graph $4x - 3y = 12$ using the intercepts.

Solution
Find the intercepts and a third point.

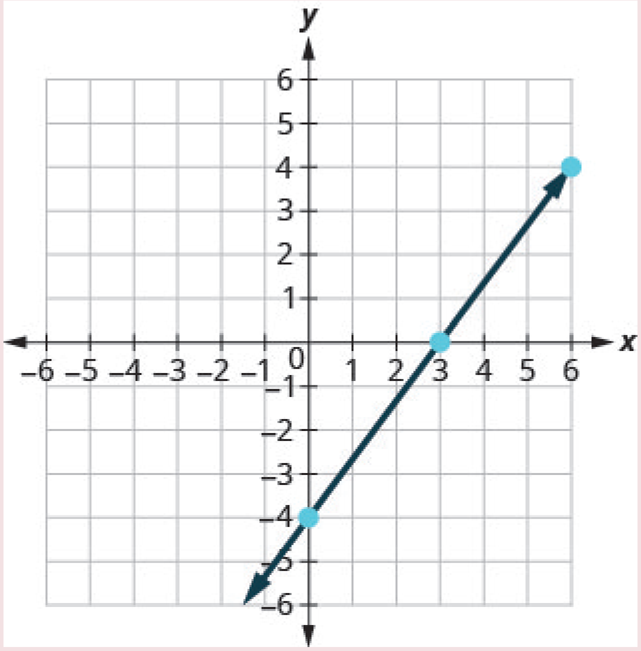


x-intercept, let $y = 0$	y-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in following table and show the graph below.

$$4x - 3y = 12$$

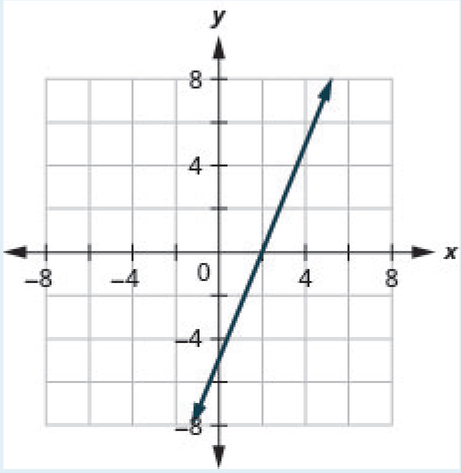
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



TRY IT 5

Graph $5x - 2y = 10$ using the intercepts.

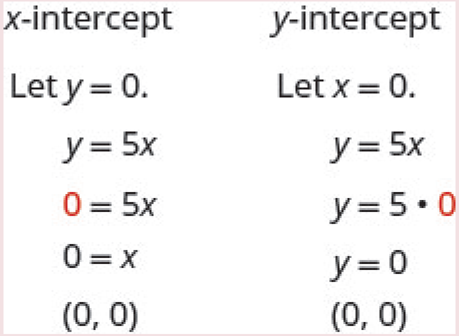
Show answer



EXAMPLE 6

Graph $y = 5x$ using the intercepts.

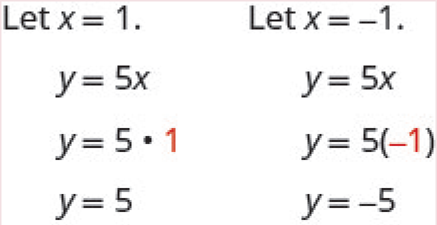
Solution



x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
(0, 0)	(0, 0)

This line has only one intercept. It is the point $(0, 0)$.

To ensure accuracy we need to plot three points. Since the *x*– and *y*– intercepts are the same point, we need *two* more points to graph the line.



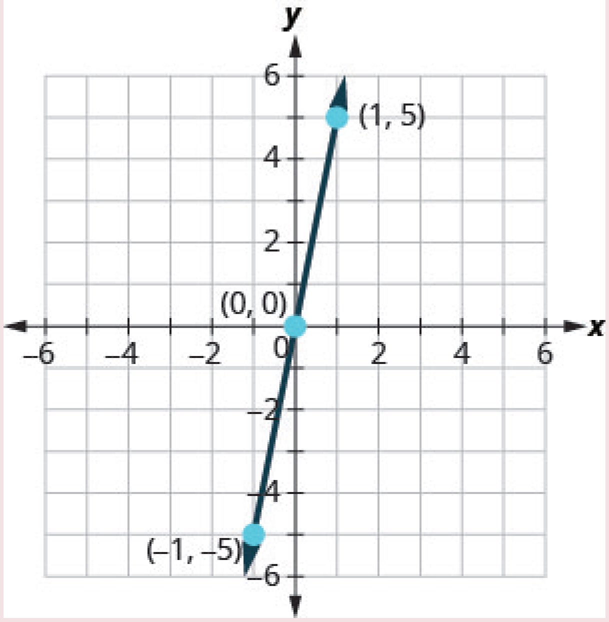
Let $x = 1$.	Let $x = –1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(–1)$
$y = 5$	$y = –5$

See following table..

$$y = 5x$$

x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

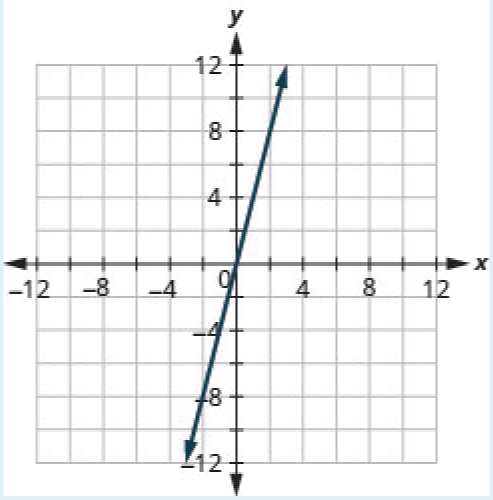
Plot the three points, check that they line up, and draw the line.



TRY IT 6

Graph $y = 4x$ using the intercepts.

Show answer



Key Concepts

- **Find the *x*– and *y*– Intercepts from the Equation of a Line**
    - Use the equation of the line to find the *x*– intercept of the line, let $y = 0$ and solve for *x*.
    - Use the equation of the line to find the *y*– intercept of the line, let $x = 0$ and solve for *y*.
- **Graph a Linear Equation using the Intercepts**
    1. Find the *x*– and *y*– intercepts of the line.
       Let $y = 0$ and solve for *x*.
       Let $x = 0$ and solve for *y*.
    2. Find a third solution to the equation.
    3. Plot the three points and then check that they line up.
    4. Draw the line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:**
    - Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the *x*– axis at a
      $y = b$ is a horizontal line passing through the *y*– axis at b.
    - If *y* is isolated on one side of the equation, graph by plotting points.
    - Choose any three values for *x* and then solve for the corresponding *y*– values.
    - If the equation is of the form $ax + by = c$, find the intercepts. Find the *x*– and *y*– intercepts and then a third point.

Glossary

intercepts of a line

The points where a line crosses the *x*– axis and the *y*– axis are called the intercepts of the line.

***x*– intercept**

The point $(a, 0)$ where the line crosses the *x*– axis; the *x*– intercept occurs when y is zero.

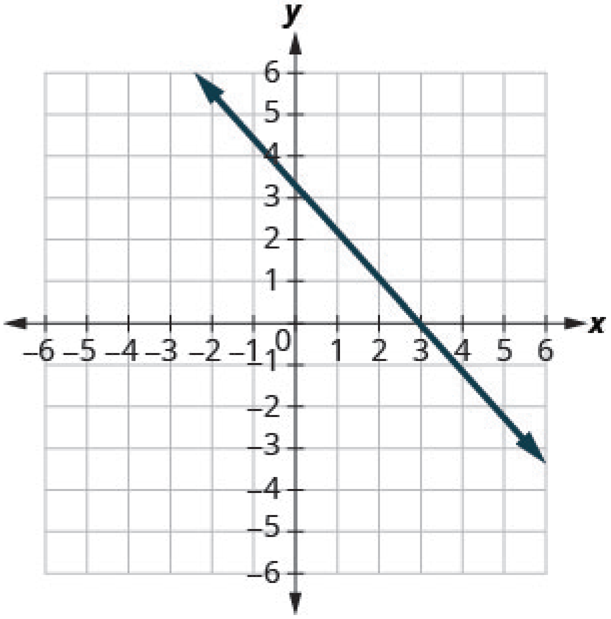
y-intercept

The point $(0, b)$ where the line crosses the *y*– axis; the *y*– intercept occurs when x is zero.

3.3 Exercise Set

In the following exercises, find the *x*– and *y*– intercepts on each graph.

1.


y
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6
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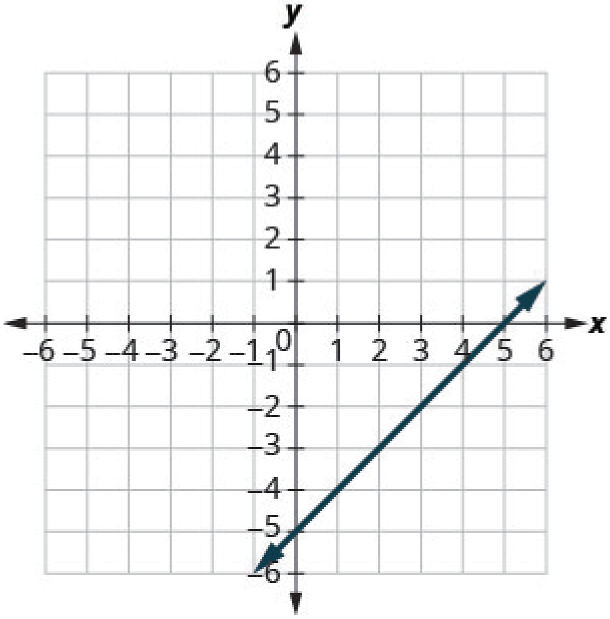
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

2.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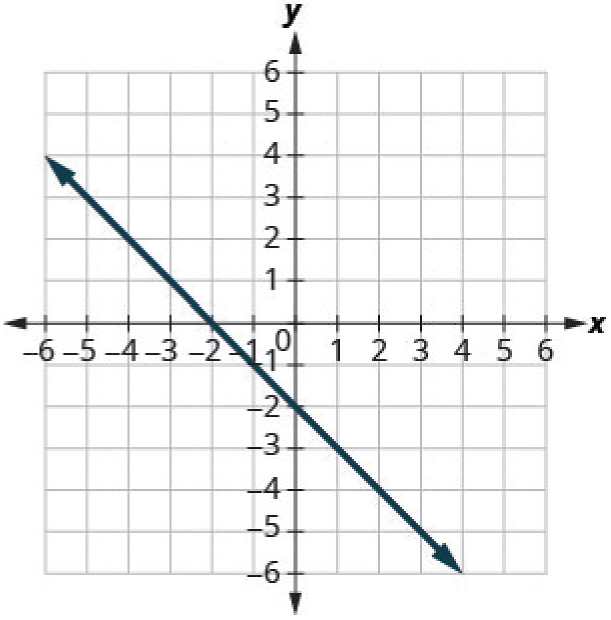
3.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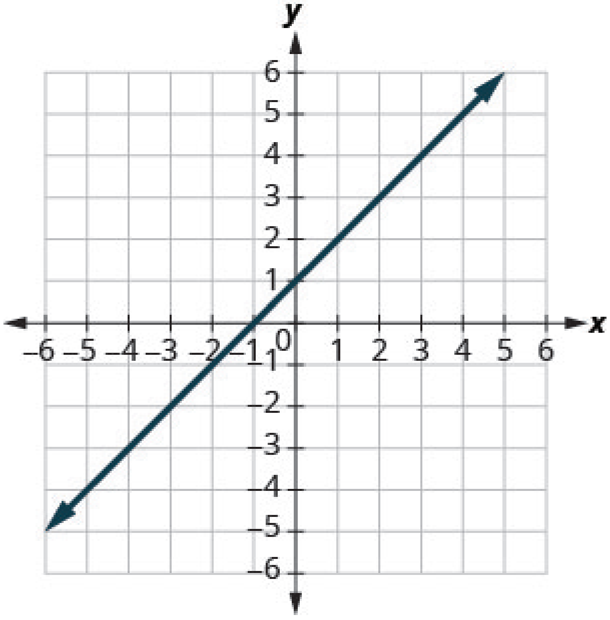
4.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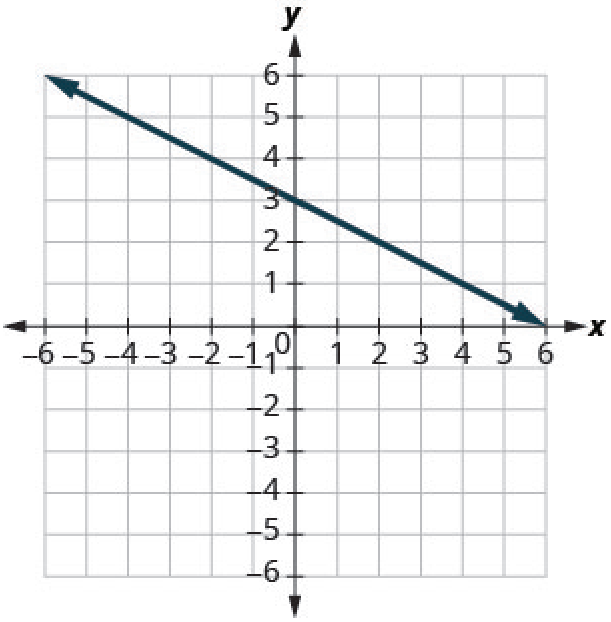
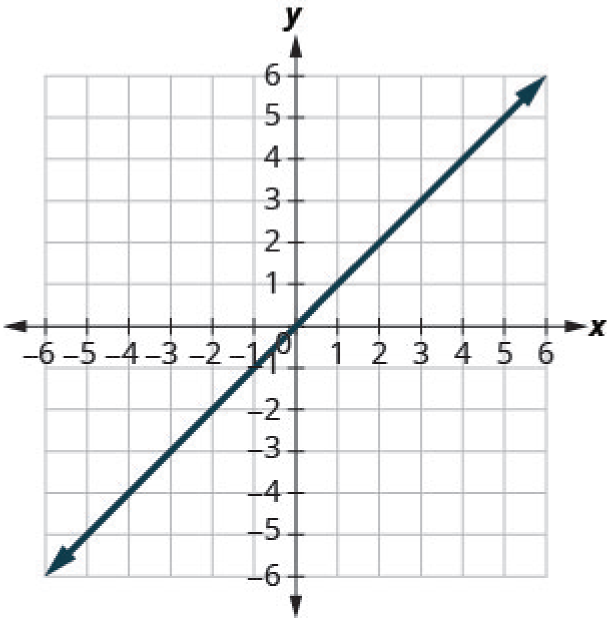
5.



6.



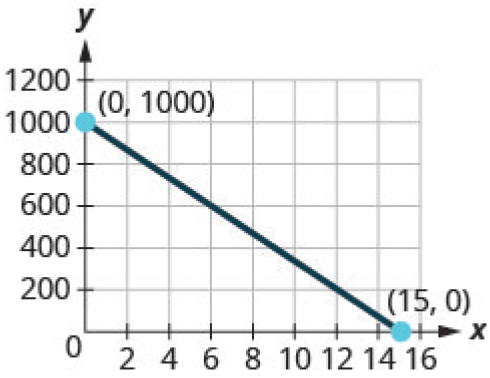
In the following exercises, find the intercepts for each equation.

7. $x + y = 4$
8. $x + y = -2$
9. $x - y = 5$
10. $x - y = -3$
11. $x + 2y = 8$
12. $3x + y = 6$
13. $x - 3y = 12$
14. $4x - y = 8$
15. $3x - 2y = 12$
16. $y = \frac{1}{3}x + 1$
17. $y = \frac{1}{5}x + 2$
18. $y = 3x$
19. $y = -4x$

In the following exercises, graph using the intercepts.

20. $-x + 5y = 10$
21. $x + 2y = 4$
22. $x + y = 2$
23. $x + y = -3$
24. $x - y = 1$
25. $x - y = -4$
26. $4x + y = 4$
27. $2x + 4y = 12$
28. $3x - 2y = 6$
29. $2x - 5y = -20$
30. $3x - y = -6$
31. $y = \frac{3}{2}x$
32. $y = x$

33. Damien is driving from Thunder Bay to Montreal, a distance of 1000 miles. The *x*– axis on the graph below shows the time in hours since Damien left Thunder Bay. The *y*– axis represents the distance he has left to drive.



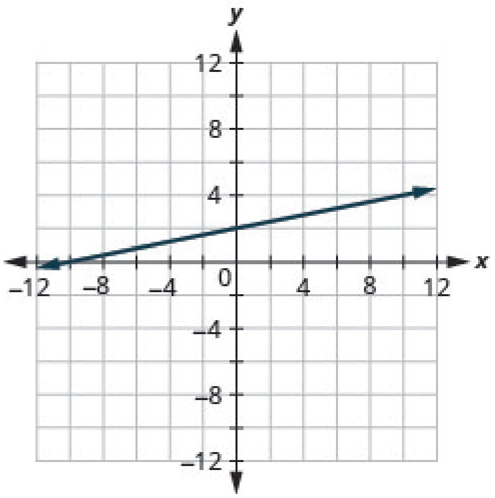
a) Find the *x*– and *y*– intercepts.

b) Explain what the *x*– and *y*– intercepts mean for Damien.

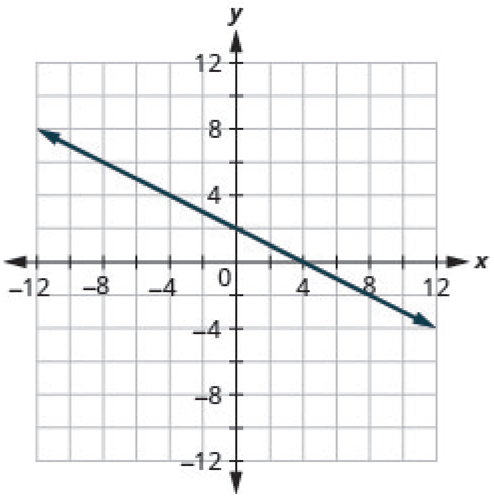
Answers

1. $(3,0)\,,(0,3)$
2. $(5,0)\,,(0,-5)$
3. $(-2,0)\,,(0,-2)$
4. $(-1,0)\,,(0,1)$
5. $(6,0)\,,(0,3)$
6. $(0,0)$
7. $(4,0)\,,(0,4)$
8. $(-2,0)\,,(0,-2)$
9. $(5,0)\,,(0,-5)$
10. $(-3,0)\,,\quad(0,3)$
11. $(8,0)\,,(0,4)$
12. $(2,0)\,,(0,6)$
13. $(12,0)\,,(0,-4)$
14. $(2,0)\,,(0,-8)$
15. $(4,0)\,,(0,-6)$
16. $(-3,0)\,,(0,1)$
17. $(-10,0)\,,(0,2)$
18. $(0,0)$
19. $(0,0)$

20.



21.



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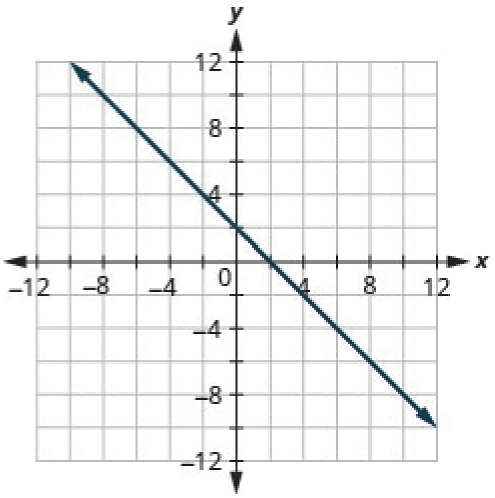

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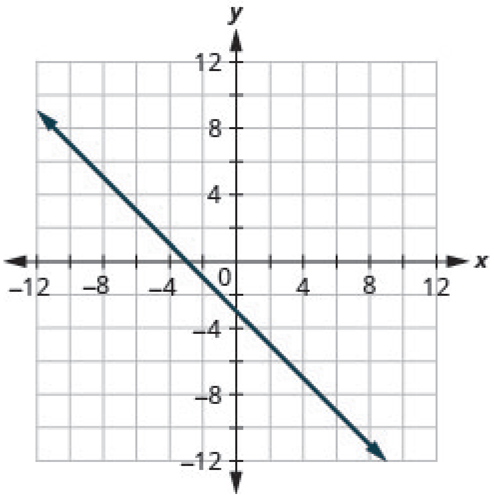

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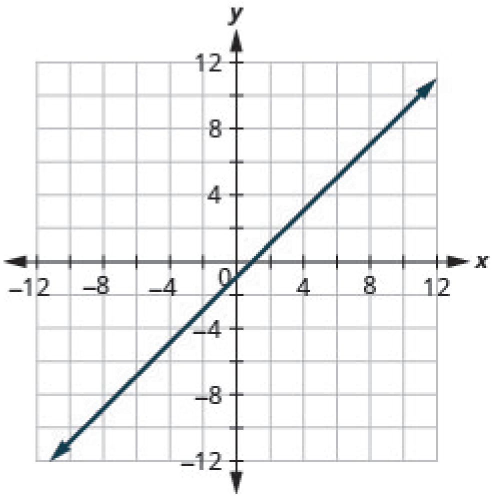

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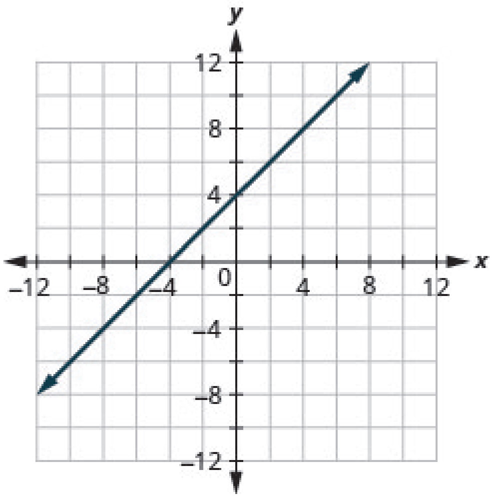

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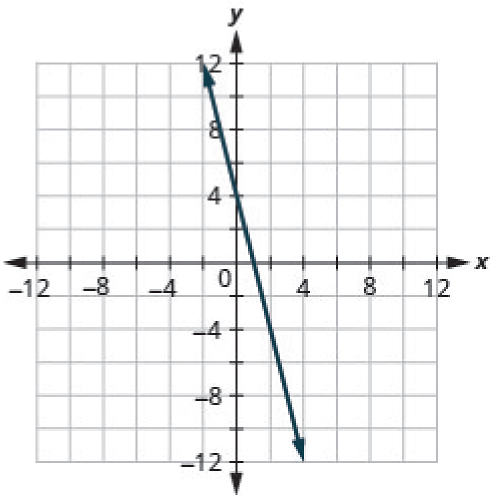
y
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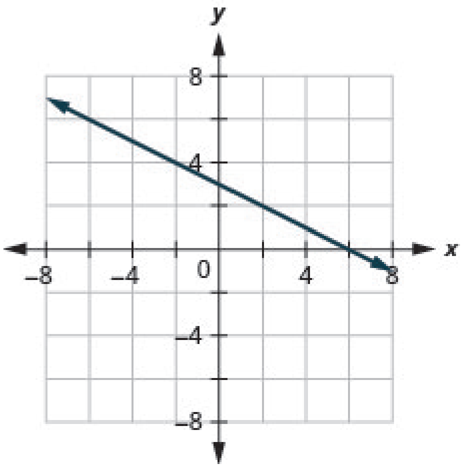
27.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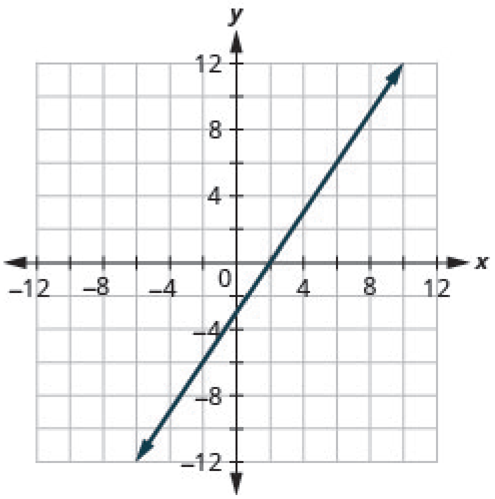
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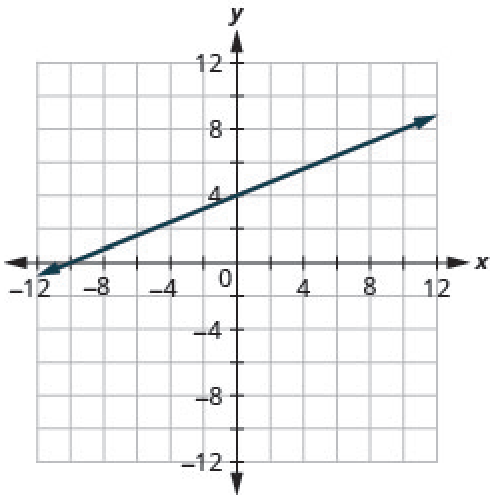
29.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x
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–8
–4
4
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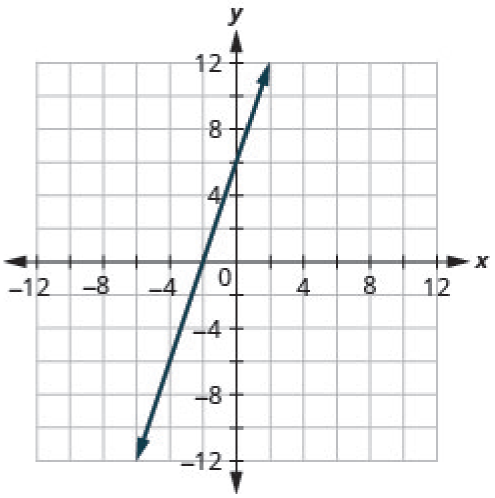
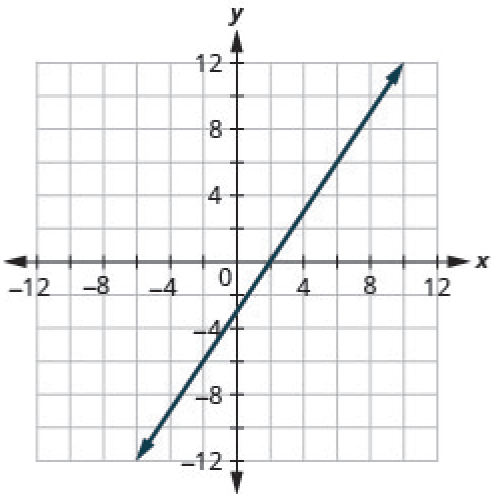
12

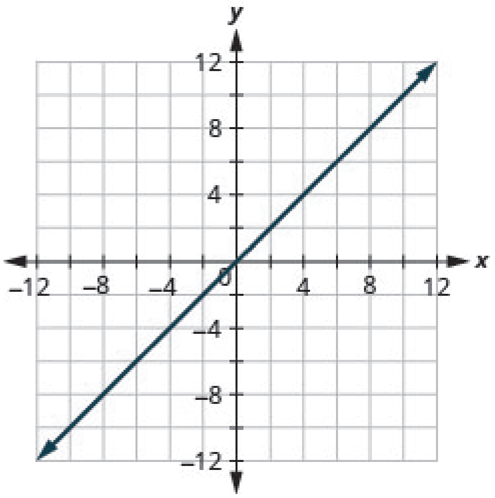
30.



31.



32.



33.

a) $(0, 1000)$, $(15, 0)$
b) At $(0, 1000)$, he has been gone 0 hours and has 1000 miles left. At $(15, 0)$, he has been gone 15 hours and has 0 miles left to go.

Attributions

This chapter has been adapted from “Graph with Intercepts” in *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Adaptation Statement for more information.

3.4 Understand Slope of a Line – optional

Learning Objectives

By the end of this section it is expected that you will be able to:

- Use $m = \frac{rise}{run}$ to find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. What determines whether a line tilts up or down or if it is steep or flat?

In mathematics, the 'tilt' of a line is called the *slope* of the line. The concept of slope has many applications in the real world. The pitch of a roof, grade of a highway, and a ramp for a wheelchair are some examples where you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

In this section, we will explore the concept of slope.

The slope of a line is the ratio of the rise to the run. In mathematics, it is always referred to with the letter m.

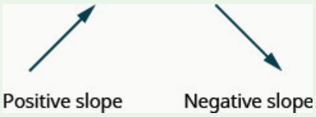
Slope of a line

The slope of a line of a line is $m = \frac{\text{rise}}{\text{run}}$.

The rise measures the vertical change and the run measures the horizontal change between two points on the line.

Positive and negative slopes

We ‘read’ a line from left to right just like we read words in English. As you read from left to right, the line is going up; it has positive slope. The line is going down; it has negative slope.



Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

We’ll look at some graphs on the xy-coordinate plane and see how to find their slopes.

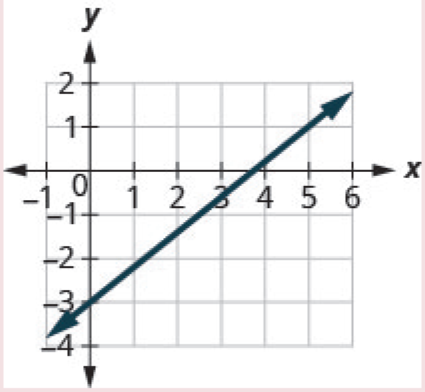
To find the slope, we must count out the rise and the run. But where do we start?

We locate two points on the line whose coordinates are integers. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

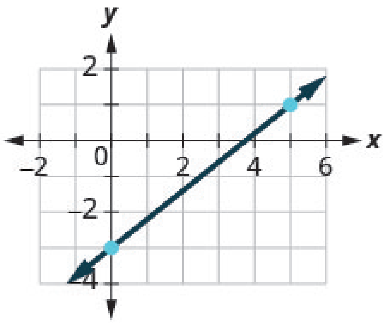
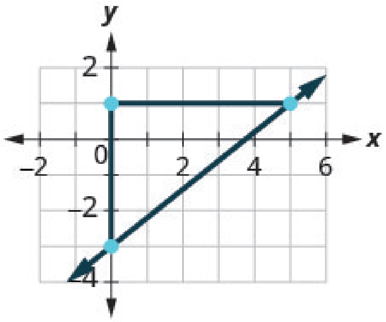
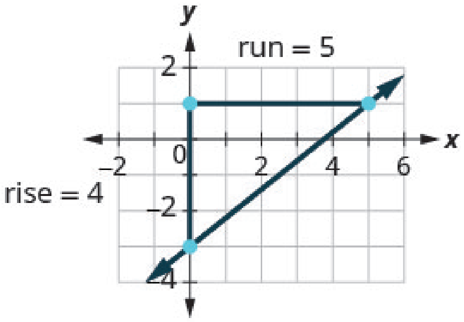
EXAMPLE 1

How to Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

Find the slope of the line shown.

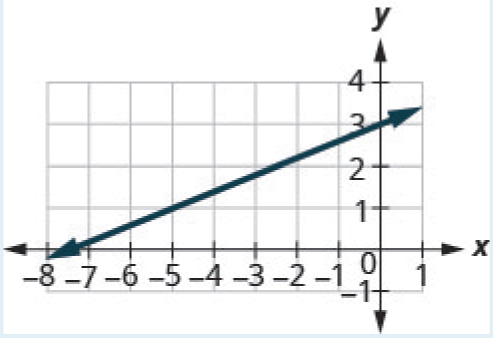


Solution

Step 1. Locate two points on the graph whose coordinates are integers.	Mark (0, –3) and (5, 1).	
Step 2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.	Starting at (0, –3), sketch a right triangle to (5, 1).	
Step 3. Count the rise and the run on the legs of the triangle.	Count the rise. Count the run.	 The rise is 4. The run is 5.
Step 4. Take the ratio of rise to run to find the slope. $m = \frac{\text{rise}}{\text{run}}$	Use the slope formula. Substitute the values of the rise and run.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{4}{5}$ The slope of the line is $\frac{4}{5}$. This means that y increases 4 units as x increases 5 units.

TRY IT 1

Find the slope of the line shown.



Show answer

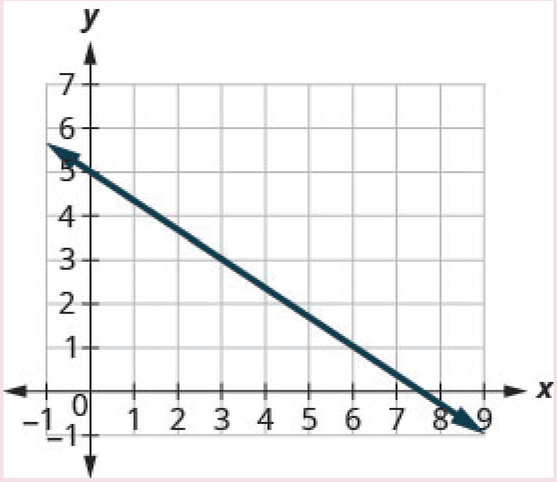
$\frac{2}{5}$

HOW TO: Find the slope of a line from its graph using m = rise / run.

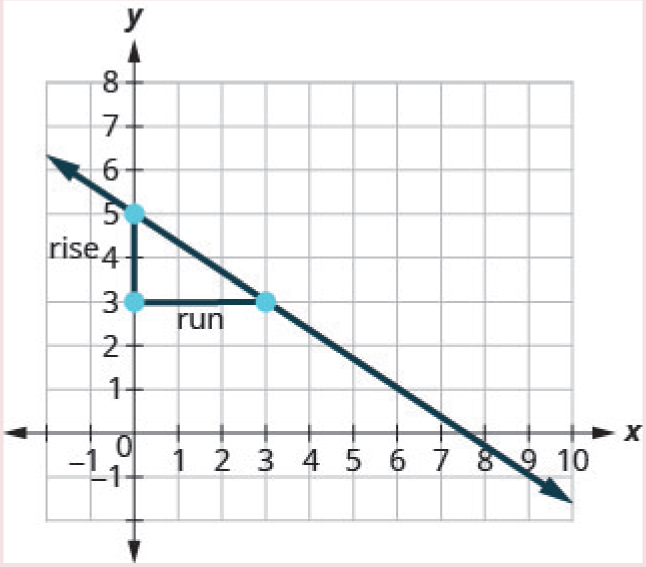
1. Locate two points on the line whose coordinates are integers.
2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope, $m = \frac{\text{rise}}{\text{run}}$.

EXAMPLE 2

Find the slope of the line shown.

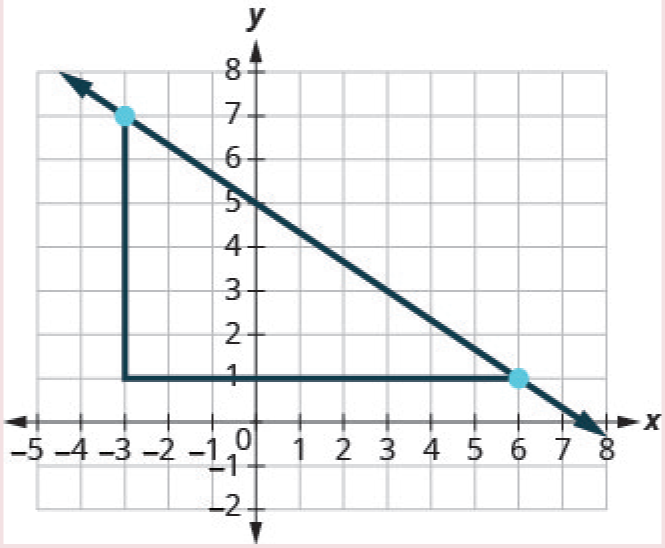


Solution

Locate two points on the graph whose coordinates are integers.	$(0,5)$ and $(3,3)$
Which point is on the left?	$(0,5)$
Starting at $(0,5)$, sketch a right triangle to $(3,3)$.	
Count the rise—it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m=\frac{-2}{3}$
Simplify.	$m=-\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.

So y increases by 3 units as x decreases by 2 units.

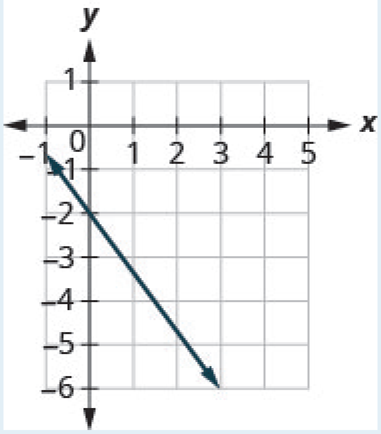
What if we used the points $(-3,7)$ and $(6,1)$ to find the slope of the line?



The rise would be -6 and the run would be 9. Then $m = \frac{-6}{9}$, and that simplifies to $m = -\frac{2}{3}$. Remember, it does not matter which points you use—the slope of the line is always the same.

TRY IT 2

Find the slope of the line shown.



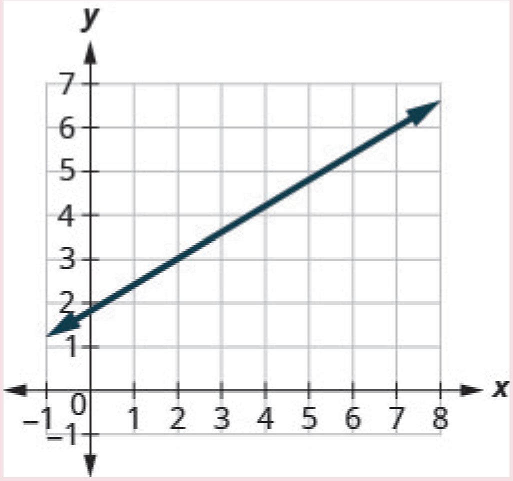
Show answer

$-\frac{4}{3}$

In the last two examples, the lines had *y*-intercepts with integer values, so it was convenient to use the *y*-intercept as one of the points to find the slope. In the next example, the *y*-intercept is a fraction. Instead of using that point, we'll look for two other points whose coordinates are integers. This will make the slope calculations easier.

EXAMPLE 3

Find the slope of the line shown.



Solution

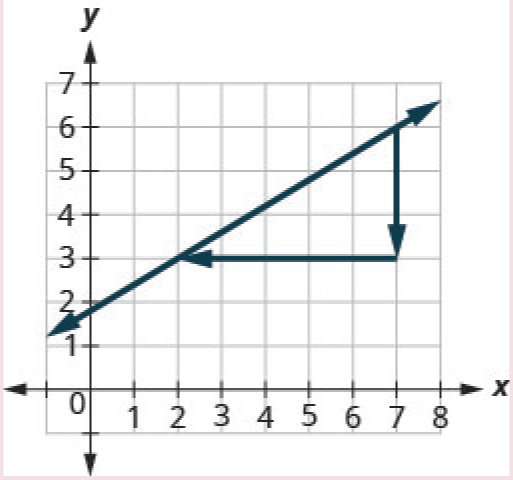
Locate two points on the graph whose coordinates are integers.	$(2, 3)$ and $(7, 6)$
Which point is on the left?	$(2, 3)$
Starting at $(2, 3)$, sketch a right triangle to $(7, 6)$.	
Count the rise.	The rise is 3.
Count the run.	The run is 5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{3}{5}$
	The slope of the line is $\frac{3}{5}$.

This means that y increases 5 units as x increases 3 units.

When we used geoboards to introduce the concept of slope, we said that we would always start with the point on the left and count the rise and the run to get to the point on the right. That way the run was always positive and the rise determined whether the slope was positive or negative.

What would happen if we started with the point on the right?

Let's use the points $(2, 3)$ and $(7, 6)$ again, but now we'll start at $(7, 6)$.

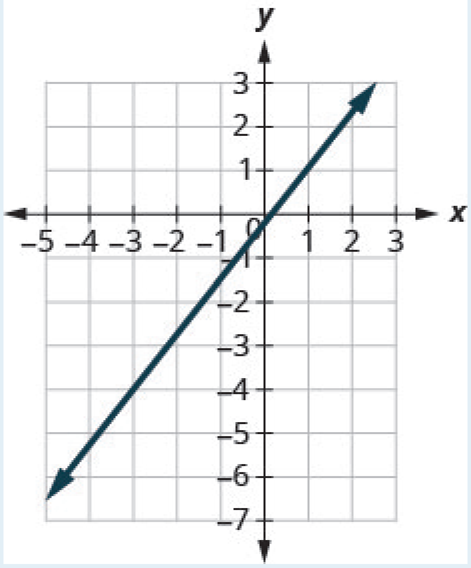


Count the rise.	The rise is -3.
Count the run. It goes from right to left, so it is negative.	The run is -5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-3}{-5}$
	The slope of the line is $\frac{-3}{-5}$.

It does not matter where you start—the slope of the line is always the same.

TRY IT 3

Find the slope of the line shown.



Show answer
$\frac{5}{4}$

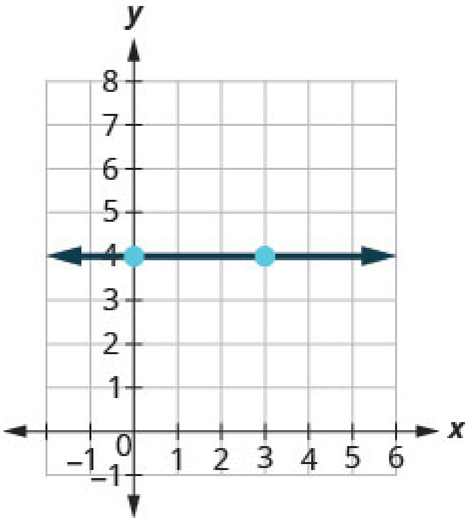
Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.

$$\textbf{Horizontal line} \quad \mathbf{y = b} \qquad \textbf{Vertical line} \quad \mathbf{x = a}$$

$$\text{y-coordinates are the same.} \quad \text{x-coordinates are the same.}$$

So how do we find the slope of the horizontal line $y = 4$? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let’s see what happens when we do this.



What is the rise?	The rise is 0.
Count the run.	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{0}{3}$ $m = 0$
	The slope of the horizontal line $y = 4$ is 0.

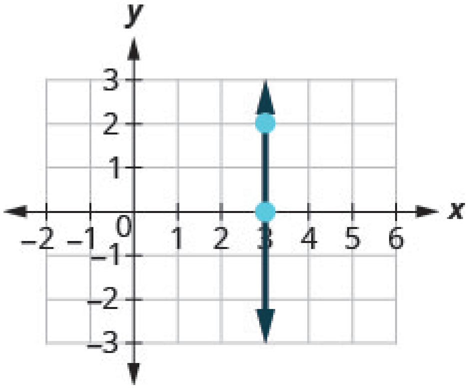
All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

Slope of a horizontal line

The slope of a horizontal line, $y = b$, is 0.

The floor of your room is horizontal. Its slope is 0. If you carefully placed a ball on the floor, it would not roll away.

Now, we'll consider a vertical line, the line.



What is the rise?	The rise is 2.
Count the run.	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{2}{0}$

But we can't divide by 0. Division by 0 is not defined. So we say that the slope of the vertical line $x = 3$ is undefined.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Slope of a vertical line

The slope of a vertical line, $x = a$, is undefined.

EXAMPLE 4

Find the slope of each line:

a) $x = 8$ b) $y = -5$.

Solution
a) $x = 8$
This is a vertical line.
Its slope is undefined.

b) $y = -5$

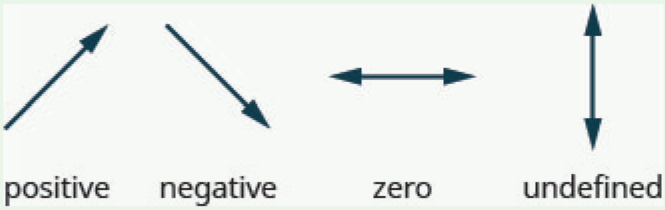
This is a horizontal line.
It has slope 0.

TRY IT 4

Find the slope of the line: $x = -4$.

Show answer
undefined

Quick guide to the slopes of lines



Remember, we ‘read’ a line from left to right, just like we read written words in English.

Use the Slope Formula to find the Slope of a Line Between Two Points

Sometimes we’ll need to find the slope of a line between two points when we don’t have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we’ll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

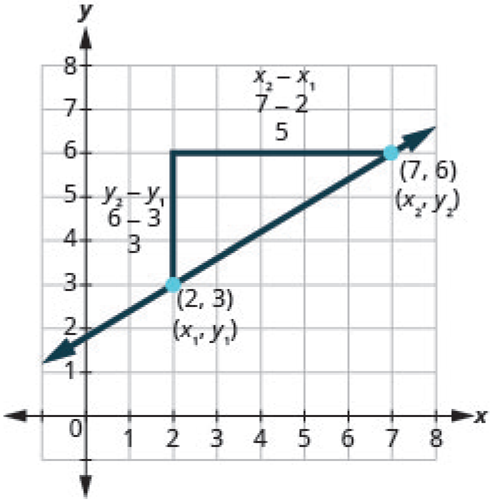
$$\begin{array}{ll} (x_1, y_1) & \text{read ' } x \text{ sub } 1, \; y \text{ sub } 1\text{'} \\ (x_2, y_2) & \text{read ' } x \text{ sub } 2, \; y \text{ sub } 2\text{'} \end{array}$$

The use of subscripts in math is very much like the use of last name initials in elementary school. Maybe you remember Laura C. and Laura M. in your third grade class?

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let’s see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$.



Since we have two points, we will use subscript notation, $\begin{pmatrix} x_1, & y_1 \\ 2, & 3 \end{pmatrix}\begin{pmatrix} x_2, & y_2 \\ 7, & 6 \end{pmatrix}$.

On the graph, we counted the rise of 3 and the run of 5.

Notice that the rise of 3 can be found by subtracting the *y*-coordinates 6 and 3.

$3 = 6 - 3$

And the run of 5 can be found by subtracting the *x*-coordinates 7 and 2.

$5 = 7 - 2$

We know $m = \frac{\text{rise}}{\text{run}}$. So $m = \frac{3}{5}$.

We rewrite the rise and run by putting in the coordinates $m = \frac{6-3}{7-2}$.

But 6 is y_2, the *y*-coordinate of the second point and 3 is y_1, the *y*-coordinate of the first point.

So we can rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{7 - 2}$

Also, 7 is x_2, the *x*-coordinate of the second point and 2 is x_1, the *x*-coordinate of the first point.

So, again, we rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{x_2 - x_1}$

We’ve shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Slope formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the slope formula.

The slope is:

$$\begin{array}{c}\text{y of the second point minus y of the first point} \\ \text{over} \\ \text{x of the second point minus x of the first point.}\end{array}$$

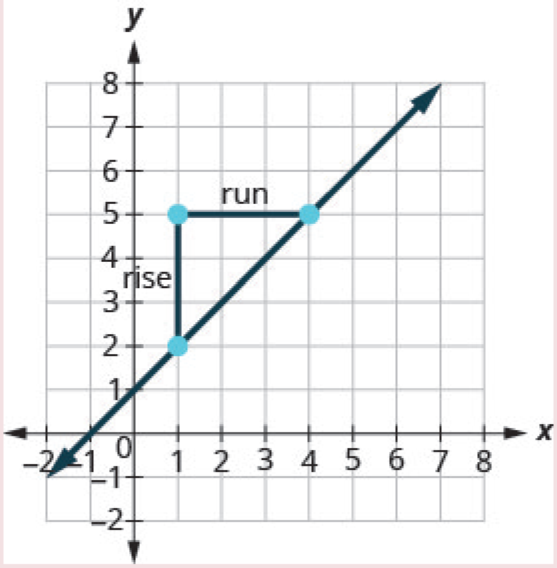
EXAMPLE 5

Use the slope formula to find the slope of the line between the points $(1, 2)$ and $(4, 5)$.

Solution

We'll call $(1, 2)$ point #1 and $(4, 5)$ point #2.	$\begin{pmatrix} x_1, & y_1 \\ 1, & 2 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ 4, & 5 \end{pmatrix}.$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}.$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{5 - 2}{x_2 - x_1}.$
x of the second point minus x of the first point	$m = \frac{5 - 2}{4 - 1}.$
Simplify the numerator and the denominator.	$m = \frac{3}{3}.$
Simplify.	$m = 1.$

Let's confirm this by counting out the slope on a graph using $m = \frac{\text{rise}}{\text{run}}$.



It doesn’t matter which point you call point #1 and which one you call point #2. The slope will be the same. Try the calculation yourself.

TRY IT 5

Use the slope formula to find the slope of the line through the points: $(8, 5)$ and $(6, 3)$.

Show answer
1

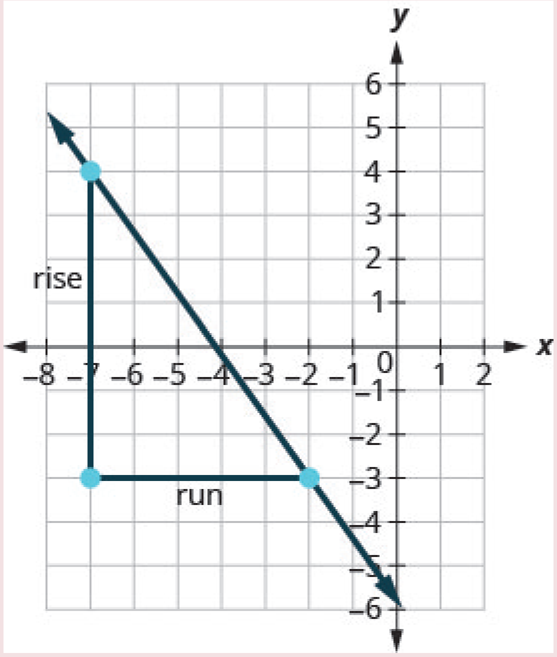
EXAMPLE 6

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

We’ll call $(-2,-3)$ point #1 and $(-7,4)$ point #2.	$\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix}\begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}.$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}.$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{4-(-3)}{x_2 - x_1}.$
x of the second point minus x of the first point	$m = \frac{4-(-3)}{-7-(-2)}.$
Simplify.	$m = \frac{7}{-5}$ $m = -\frac{7}{5}$

Let’s verify this slope on the graph shown.



$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{-7}{5} \\ m &= -\frac{7}{5} \end{aligned}$$

TRY IT 6

Use the slope formula to find the slope of the line through the points: $(-3, 4)$ and $(2, -1)$.

Show answer

-1

Graph a Line Given a Point and the Slope

Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

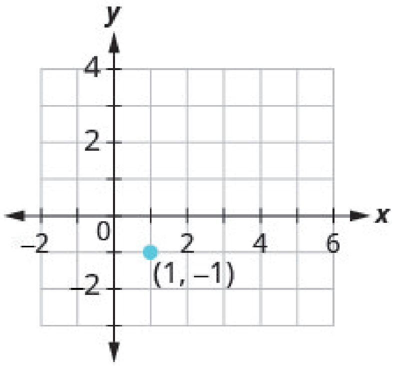
One other method we can use to graph lines is called the point–slope method. We will use this method when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

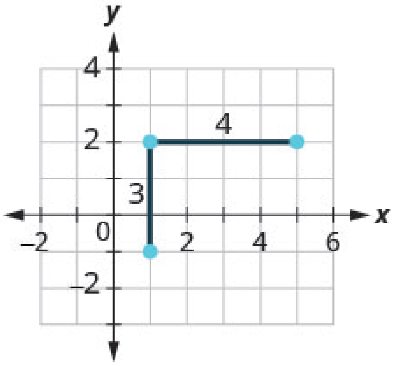
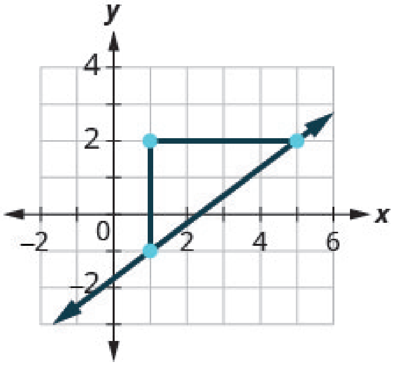
EXAMPLE 7

How To Graph a Line Given a Point and The Slope

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

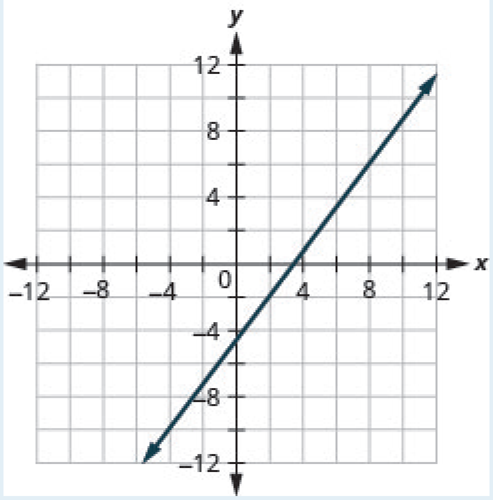
Step 1. Plot the given point.	Plot (1, –1).	

Step 2. Use the slope formula $m=\frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m=\frac{3}{4}$ $\frac{\text{rise}}{\text{run}}=\frac{3}{4}$ $\text{rise}=3$ $\text{run}=4$
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

EXAMPLE 7

Graph the line passing through the point $(2,-2)$ with the slope $m=\frac{4}{3}$.

Show answer



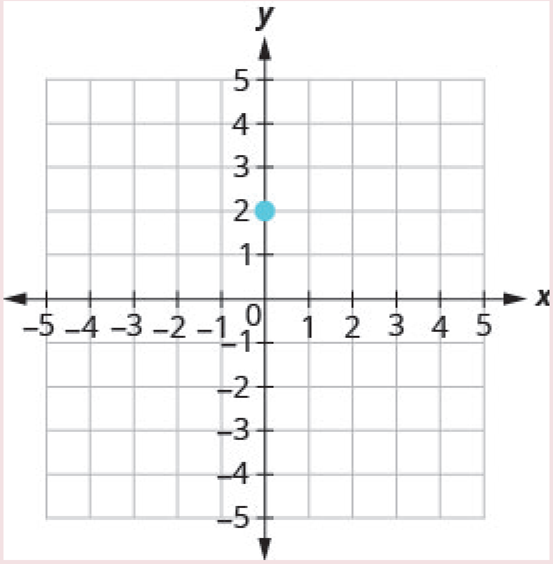
Graph a line given a point and the slope.

1. Plot the given point.
2. Use the slope formula $m = \dfrac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

EXAMPLE 8

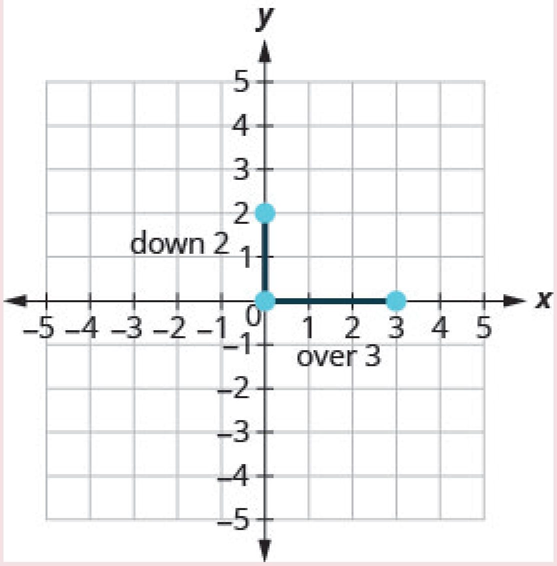
Graph the line with *y*-intercept 2 whose slope is $m = -\dfrac{2}{3}$.

Solution
Plot the given point, the *y*-intercept, $(0, 2)$.

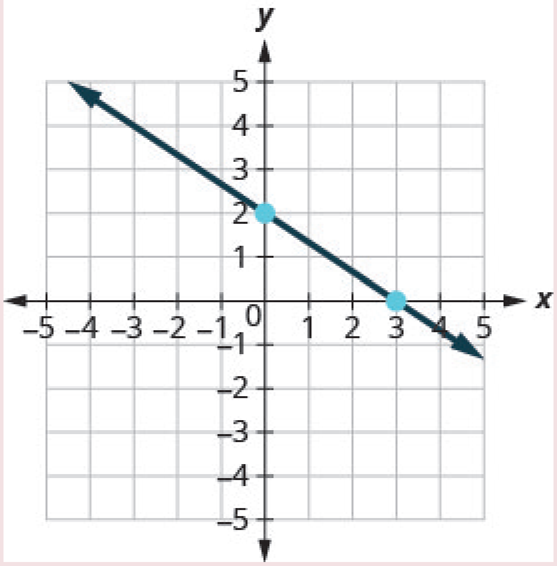


Identify the rise and the run.	$m = -\frac{2}{3}$
	$\frac{\text{rise}}{\text{run}} = \frac{-2}{3}$
	$\text{rise} = -2$
	$\text{run} = 3$

Count the rise and the run. Mark the second point.



Connect the two points with a line.

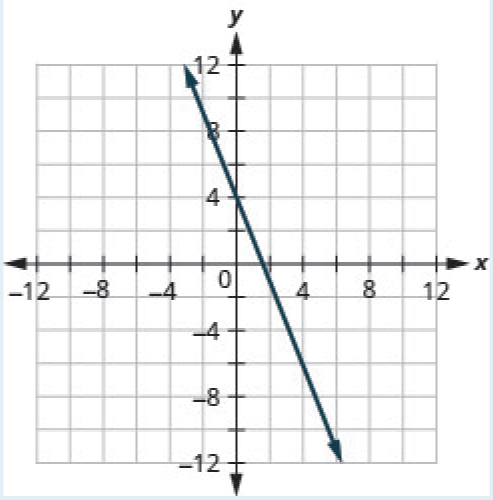


You can check your work by finding a third point. Since the slope is $m = -\frac{2}{3}$, it can be written as $m = \frac{2}{-3}$. Go back to $(0, 2)$ and count out the rise, 2, and the run, -3.

TRY IT 8

Graph the line with the *y*-intercept 4 and slope $m = -\frac{5}{2}$.

Show answer

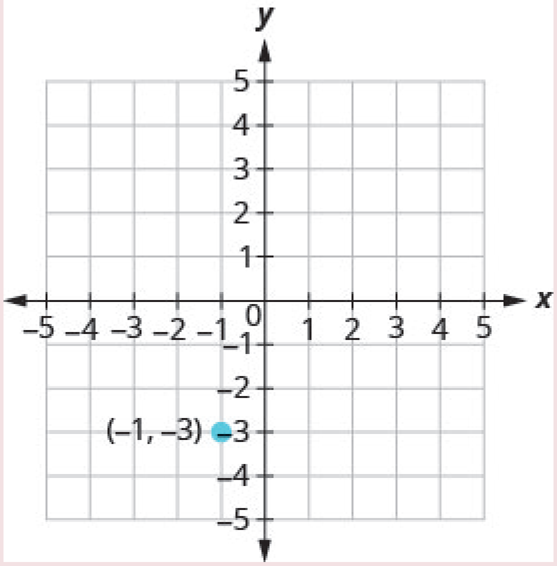


EXAMPLE 9

Graph the line passing through the point $(-1, -3)$ whose slope is $m = 4$.

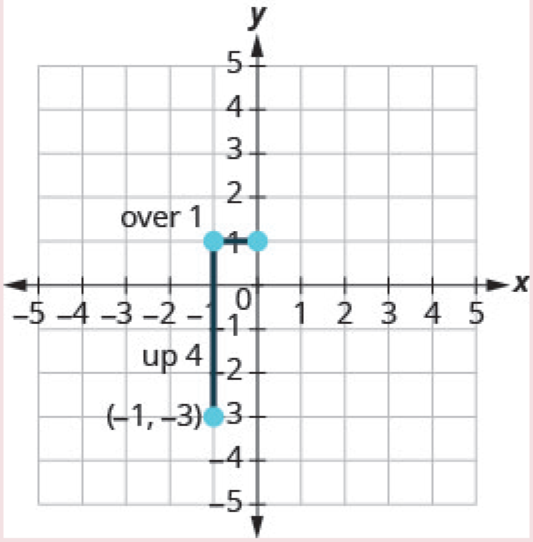
Solution

Plot the given point.

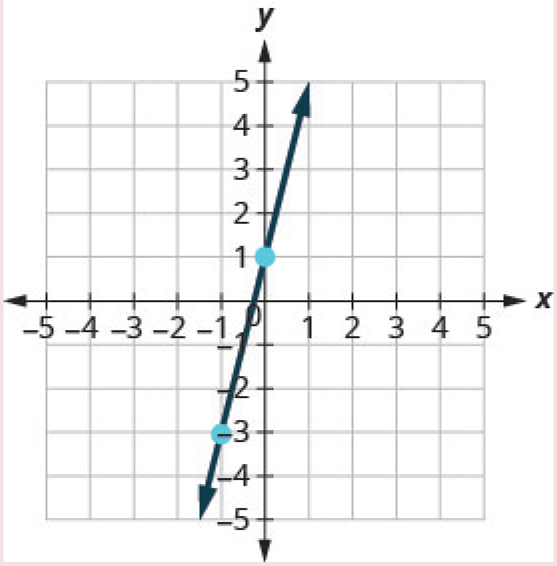


Identify the rise and the run.	$m = 4$
Write 4 as a fraction.	$\frac{\text{rise}}{\text{run}} = \frac{4}{1}$
	$\text{rise} = 4, \text{run} = 1$

Count the rise and run and mark the second point.



Connect the two points with a line.

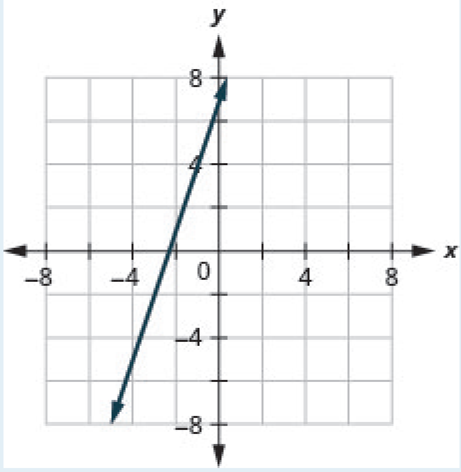


You can check your work by finding a third point. Since the slope is $m = 4$, it can be written as $m = \frac{-4}{-1}$. Go back to $(-1, -3)$ and count out the rise, -4, and the run, -1.

TRY IT 9

Graph the line with the point $(-2, 1)$ and slope $m = 3$.

Show answer

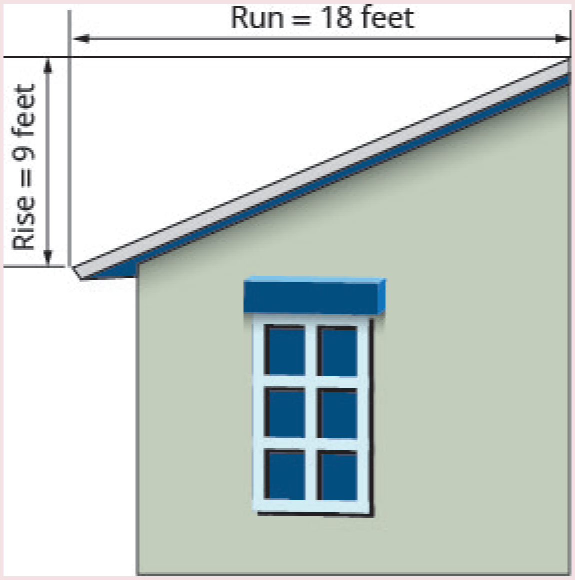


Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let's look at a few now.

EXAMPLE 10

The 'pitch' of a building's roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values for rise and run.	$m = \frac{9}{18}$
Simplify.	$m = \frac{1}{2}$
The slope of the roof is $\frac{1}{2}$.	
	The roof rises 1 foot for every 2 feet of horizontal run.

TRY IT 10

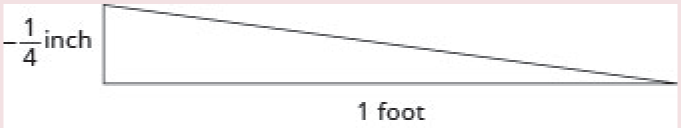
Use (Example 10), substituting the rise = 14 and run = 24.

Show answer

$$\frac{7}{12}$$

EXAMPLE 11

Have you ever thought about the sewage pipes going from your house to the street? They must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{-\frac{1}{4}\text{inch}}{1\ \text{foot}}$ $m = \frac{-\frac{1}{4}\text{inch}}{12\ \text{inches}}$
Simplify.	$m = -\frac{1}{48}$
	The slope of the pipe is $-\frac{1}{48}$.

The pipe drops 1 inch for every 48 inches of horizontal run.

TRY IT 11

Find the slope of a pipe that slopes down $\frac{1}{3}$ inch per foot.

Show answer

$$-\frac{1}{36}$$

Key Concepts

- **Find the Slope of a Line from its Graph using** $m = \frac{\text{rise}}{\text{run}}$
    1. Locate two points on the line whose coordinates are integers.
    2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
    3. Count the rise and the run on the legs of the triangle.
    4. Take the ratio of rise to run to find the slope.
- **Graph a Line Given a Point and the Slope**
    1. Plot the given point.
    2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
    3. Starting at the given point, count out the rise and run to mark the second point.
    4. Connect the points with a line.
- **Slope of a Horizontal Line**
    - The slope of a horizontal line, $y = b$, is 0.
- **Slope of a vertical line**
    - The slope of a vertical line, $x = a$, is undefined

Glossar

negative slope

A negative slope of a line goes down as you read from left to right.

positive slope

A positive slope of a line goes up as you read from left to right.

rise

The rise of a line is its vertical change.

run

The run of a line is its horizontal change.

slope formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

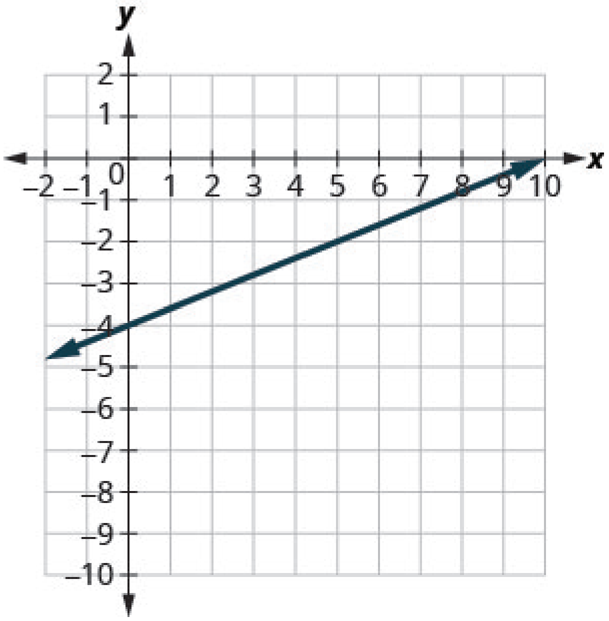
slope of a line

The slope of a line is $m = \frac{\text{rise}}{\text{run}}$. The rise measures the vertical change and the run measures the horizontal change.

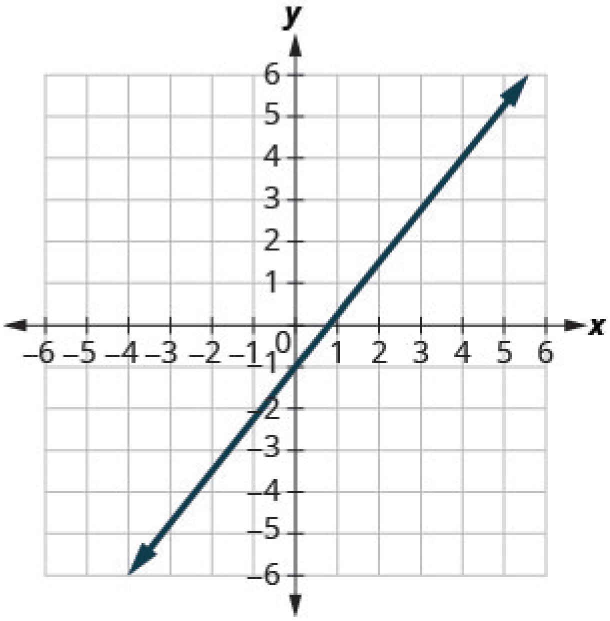
3.4 Exercise Set

In the following exercises, find the slope of each line shown.

1.


y
x
2
1
0
–2 –1
1 2 3 4 5 6 7 8 9 10
–1
–2
–3
–4
–5
–6
–7
–8
–9
–10

2.


y
x
6
5
4
3
2
1
0
–6 –5 –4 –3 –2 –1
1 2 3 4 5 6
–1
–2
–3
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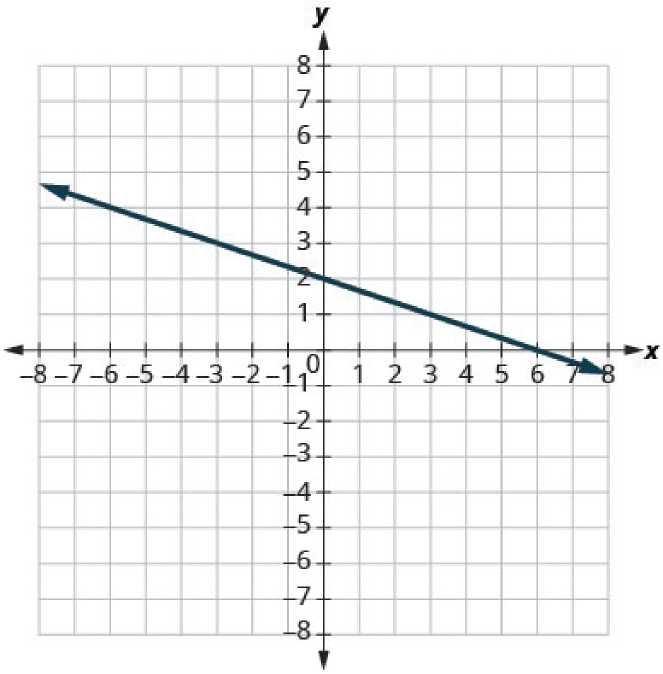
3.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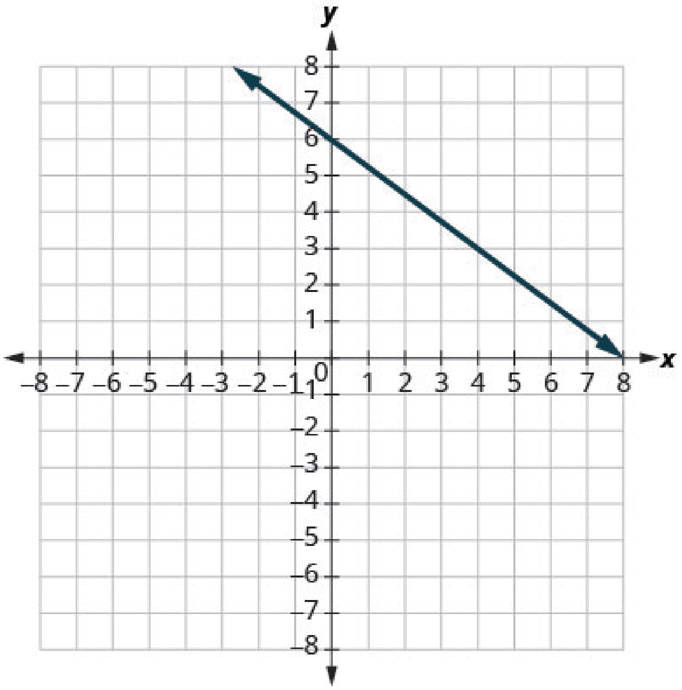
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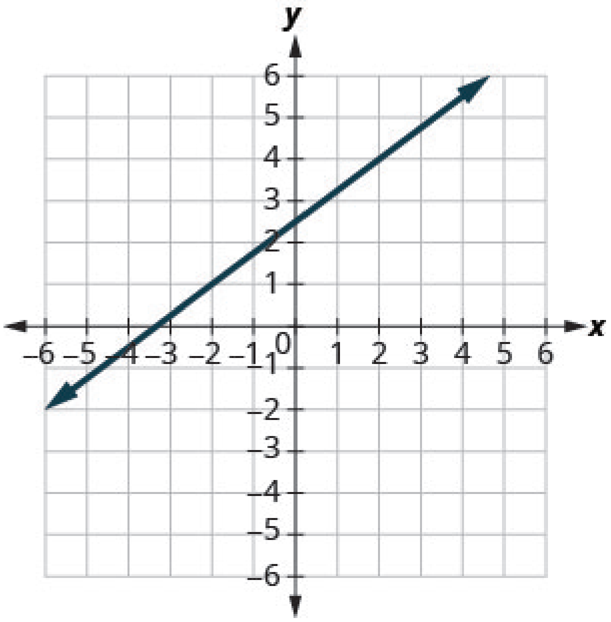

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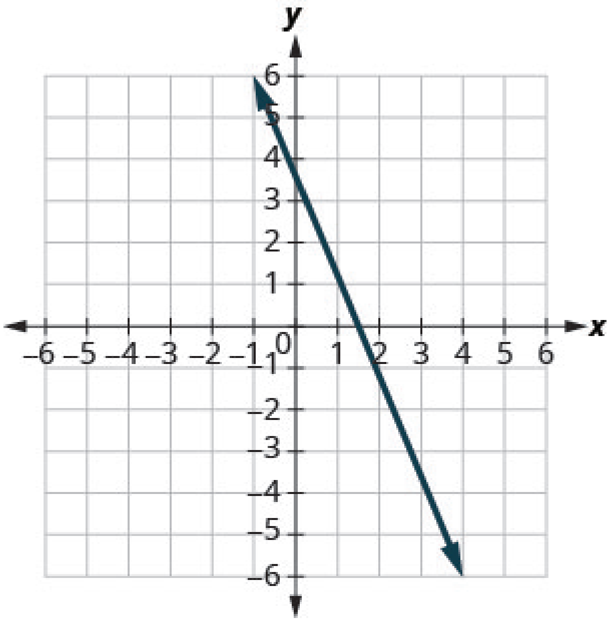

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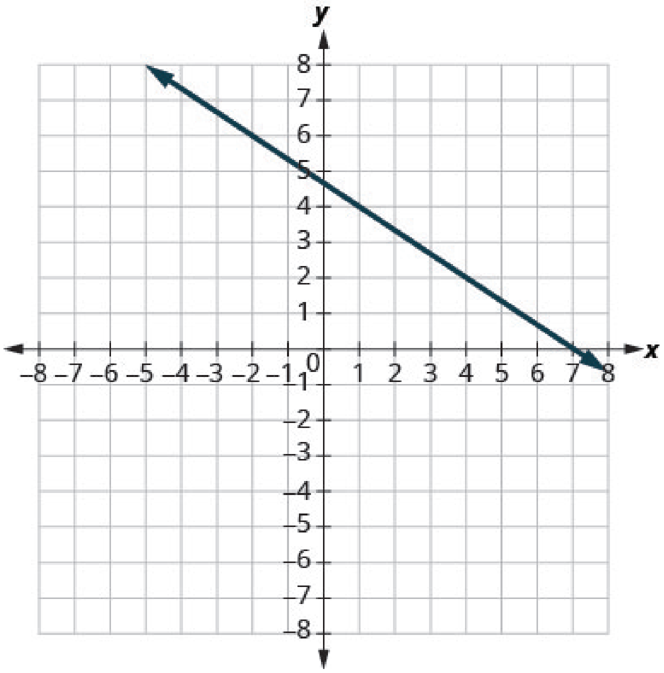

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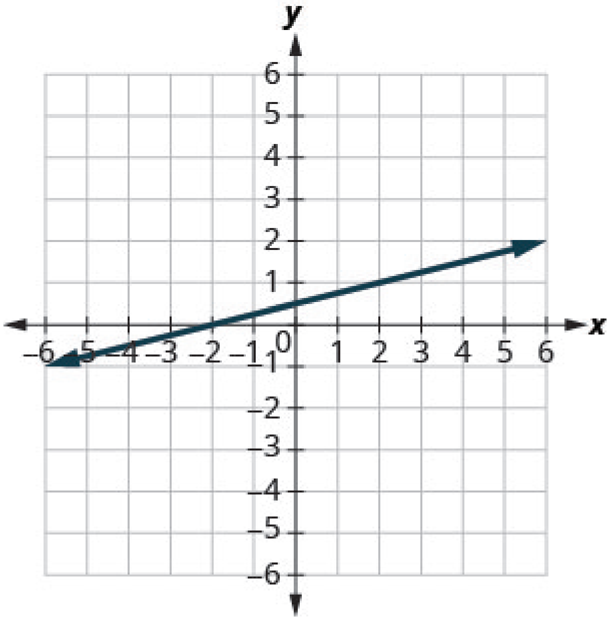


In the following exercises, find the slope of each line.

9. $y = 3$

10. $x = 4$

11. $y = -2$

12. $x = -5$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

13. $(1, 4), (3, 9)$

14. $(0, 3), (4, 6)$

15. $(2,5),(4,0)$
16. $(-3,3),(4,-5)$
17. $(-1,-2),(2,5)$
18. $(4,-5),(1,-2)$

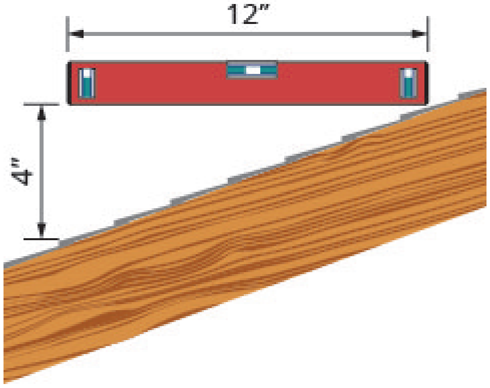
In the following exercises, graph each line with the given point and slope.

19. $(1,-2); m=\frac{3}{4}$
20. $(2,5); m=-\frac{1}{3}$
21. $(-3,4); m=-\frac{3}{2}$
22. $(-1,-4); m=\frac{4}{3}$
23. *y*-intercept 3; $m=-\frac{2}{5}$
24. *x*-intercept -2; $m=\frac{3}{4}$
25. $(-3,3); m=2$
26. $(1,5); m=-3$

27. An easy way to determine the slope of a roof is to set one end of a 12 inch level on the roof surface and hold it level. Then take a tape measure or ruler and measure from the other end of the level down to the roof surface. This will give you the slope of the roof. Builders, sometimes, refer to this as pitch and state it as an "x 12 pitch" meaning $\frac{x}{12}$, where x is the measurement from the roof to the level—the rise. It is also sometimes stated as an " x-in-12 pitch".

a) What is the slope of the roof in this picture?

b) What is the pitch in construction terms?

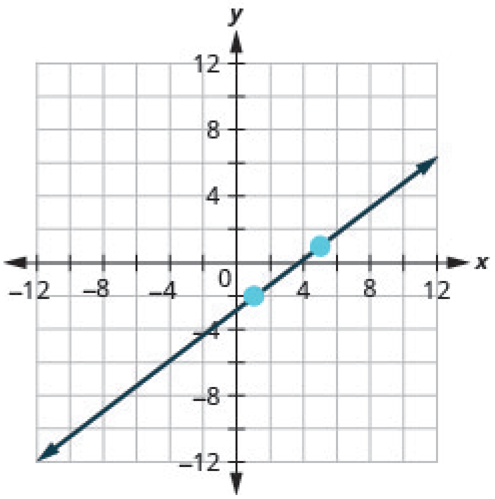


28. A local road has a grade of 6%. The grade of a road is its slope expressed as a percent. Find the slope of the road as a fraction and then simplify. What rise and run would reflect this slope or grade?
29. The rules for wheelchair ramps require a maximum 1-inch rise for a 12-inch run.
    a. How long must the ramp be to accommodate a 24-inch rise to the door?
    b. Create a model of this ramp.

Answers

1. $\frac{2}{5}$
2. $\frac{5}{4}$
3. $-\frac{1}{3}$
4. $-\frac{3}{4}$
5. $\frac{3}{4}$
6. $-\frac{5}{2}$
7. $-\frac{2}{3}$
8. $\frac{1}{4}$
9. 0
10. undefined
11. 0
12. undefined
13. $\frac{5}{2}$
14. $\frac{3}{4}$
15. $-\frac{5}{2}$
16. $-\frac{8}{7}$
17. $\frac{7}{3}$
18. -1

19.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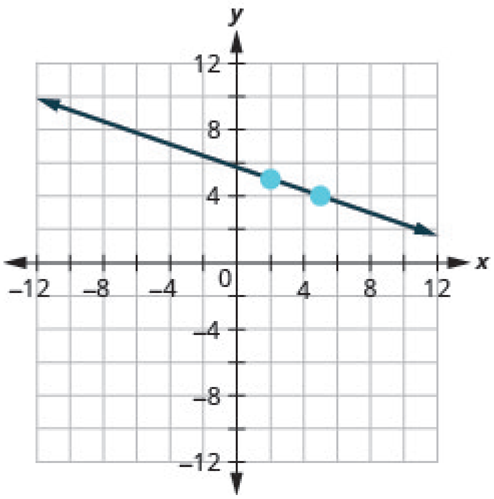
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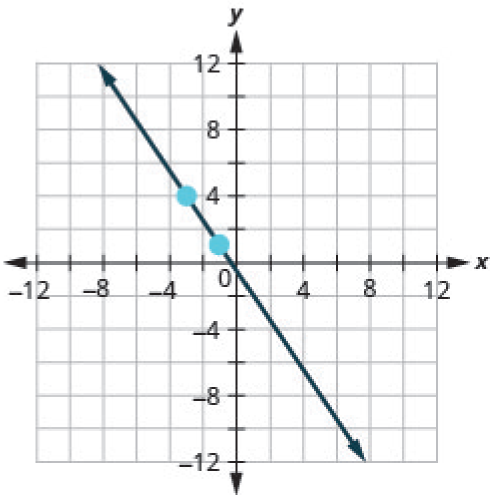
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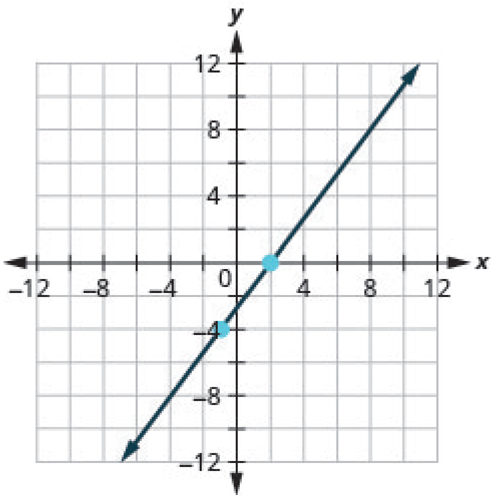
22.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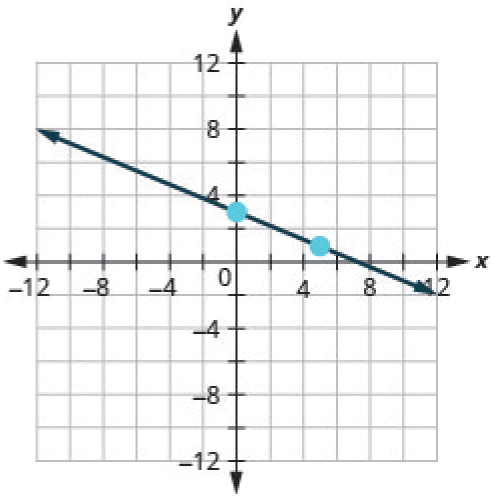

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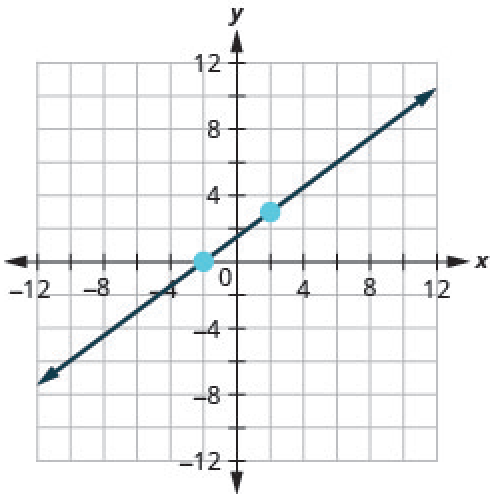

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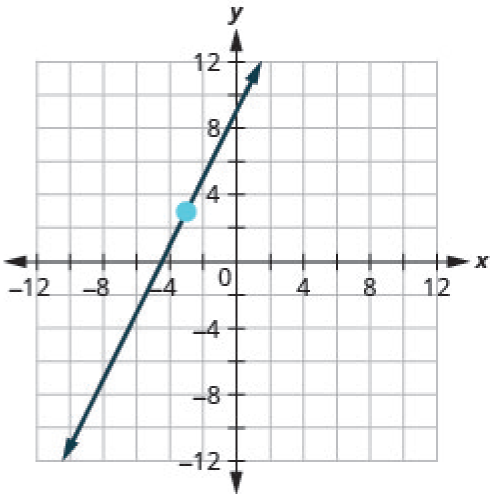

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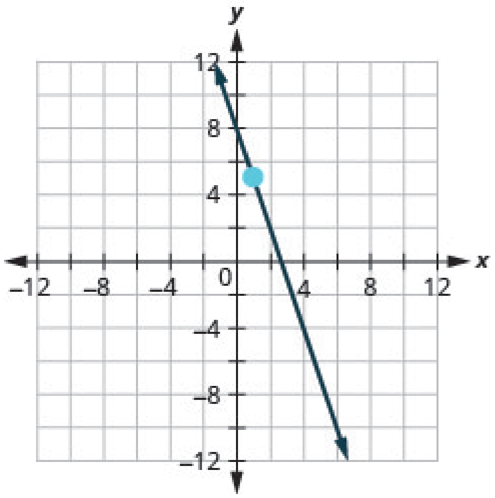


27. a) $\frac{1}{3}$ b) 4 12 pitch or 4-in-12 pitch

28. $\frac{3}{50}$; rise = 3, run = 50

29. a) 288 inches (24 feet) b) Models will vary.

Attributions

This chapter has been adapted from "Understand Slope of a Line" in *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Adaptation Statement for more information.

3.5 Use the Slope–Intercept Form of an Equation of a Line -optional

Learning Objectives

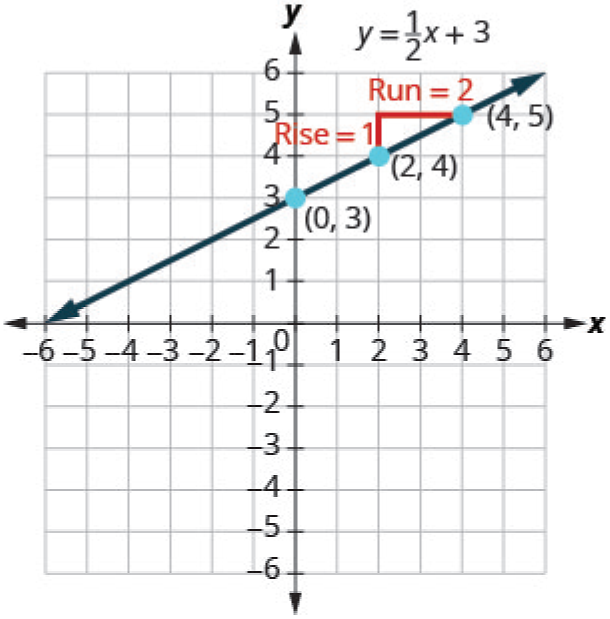
By the end of this section it is expected that you will be able to:

- Recognize the relation between the graph and the slope–intercept form of an equation of a line
- Identify the slope and y-intercept form of an equation of a line
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel lines
- Use slopes to identify perpendicular lines

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using the point–slope method. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

See (Figure). Let's find the slope of this line.



The red lines show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{1}{2} \end{aligned}$$

What is the *y*-intercept of the line? The *y*-intercept is where the line crosses the *y*-axis, so *y*-intercept is $(0, 3)$. The equation of this line is:

$$y = \frac{1}{2}x + 3$$

Notice, the line has:

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form.

$m = \frac{1}{2}$; y-intercept is $(0, 3)$

$$y = \frac{1}{2}x + 3$$

$$y = mx + b$$

Slope-intercept form of an equation of a line

The slope–intercept form of an equation of a line with slope m and *y*-intercept, $(0, b)$ is,

$y = mx + b$

Sometimes the slope–intercept form is called the “*y*-form.”

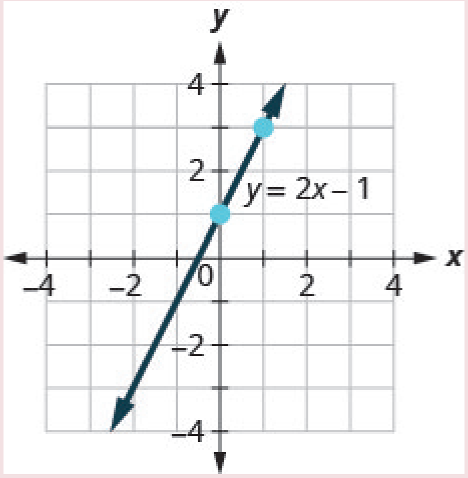
EXAMPLE 1

Use the graph to find the slope and *y*-intercept of the line, $y = 2x + 1$.

Compare these values to the equation$y = mx + b$.

Solution

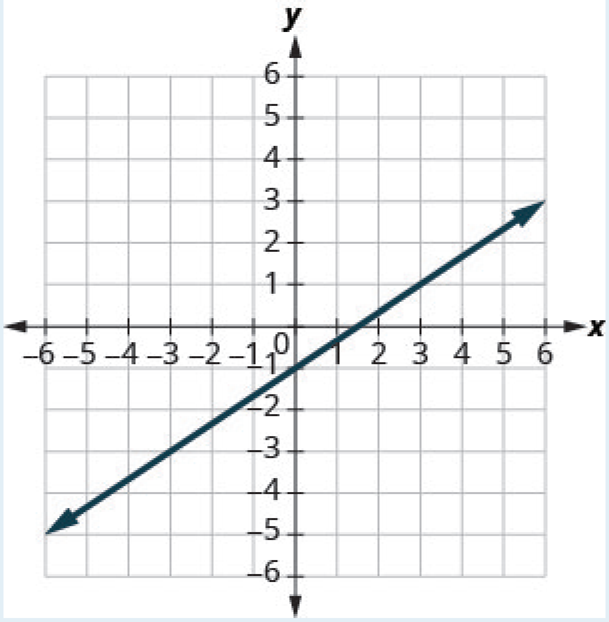
To find the slope of the line, we need to choose two points on the line. We’ll use the points $(0, 1)$ and $(1, 3)$.

	
Find the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{1}$
	$m = 2$
Find the *y*-intercept of the line.	The *y*-intercept is the point (0, 1).
We found slope $m = 2$ and *y*-intercept (0, 1).	$y = 2x + 1$ $y = mx + b$

The slope is the same as the coefficient of x and the *y*-coordinate of the *y*-intercept is the same as the constant term.

TRY IT 1

Use the graph to find the slope and *y*-intercept of the line $y = \frac{2}{3}x - 1$. Compare these values to the equation $y = mx + b$.



Show answer

slope $m = \frac{2}{3}$ and *y*-intercept $(0, -1)$

Identify the Slope and *y*-Intercept From an Equation of a Line

In the last sub-chapter, we graphed a line using the slope and a point. When we are given an equation in slope–intercept form, we can use the *y*-intercept as the point, and then count out the slope from there. Let’s practice finding the values of the slope and *y*-intercept from the equation of a line.

EXAMPLE 2

Identify the slope and *y*-intercept of the line with equation $y = -3x + 5$.

Solution
We compare our equation to the slope–intercept form of the equation.

	$y = mx + b$
Write the equation of the line.	$y = -3x + 5$
Identify the slope.	$m = -3$
Identify the *y*-intercept.	*y*-intercept is (0, 5)

TRY IT 2

Identify the slope and *y*-intercept of the line $y = \frac{2}{5}x - 1$.

Show answer

$\frac{2}{5}; (0, -1)$

When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

EXAMPLE 3

Identify the slope and *y*-intercept of the line with equation $x + 2y = 6$.

Solution
This equation is not in slope–intercept form. In order to compare it to the slope–intercept form we must first solve the equation for$y$.

Solve for *y*.	$x + 2y = 6$
Subtract *x* from each side.	$2y = -x + 6$
Divide both sides by 2.	$\frac{2y}{2} = \frac{-x+6}{2}$
Simplify.	$\frac{2y}{2} = \frac{-x}{2} + \frac{6}{2}$
(Remember:$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$)	
Simplify.	$y = -\frac{1}{2}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{2}x + 3$
Identify the slope.	$m = -\frac{1}{2}$
Identify the *y*-intercept.	*y*-intercept is (0, 3)

TRY IT 3

Identify the slope and *y*-intercept of the line $x + 4y = 8$.

Show answer

$-\frac{1}{4}; (0, 2)$

Graph a Line Using its Slope and Intercept

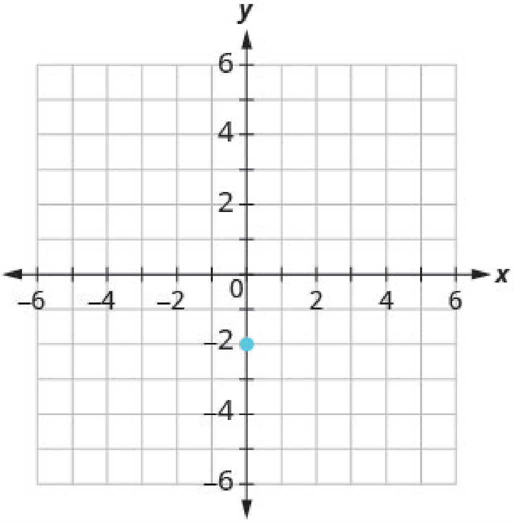
Now that we know how to find the slope and *y*-intercept of a line from its equation, we can graph the line by plotting the *y*-intercept and then using the slope to find another point.

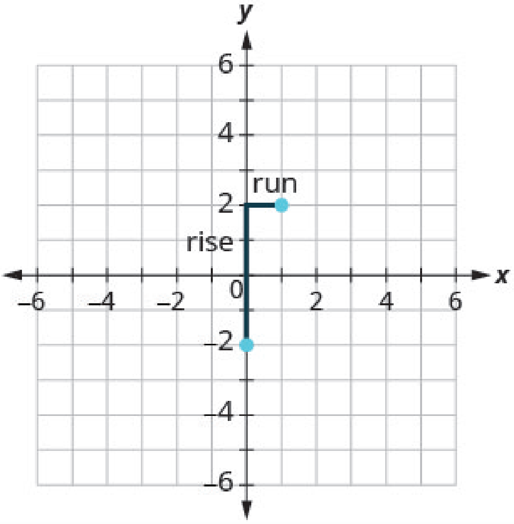
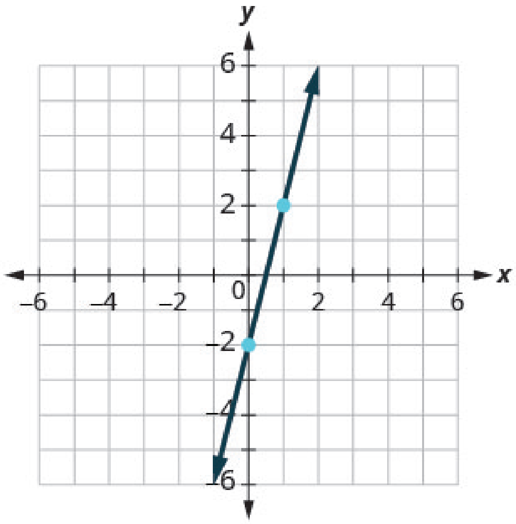
EXAMPLE 4

How to Graph a Line Using its Slope and Intercept

Graph the line of the equation $y = 4x - 2$ using its slope and y-intercept.

Solution

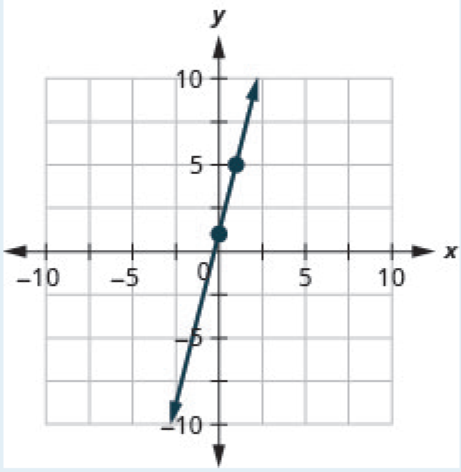
Step 1. Find the slope–intercept form of the equation.	This equation is in slope–intercept form.	$y = 4x - 2$
Step 2. Identify the slope and y-intercept.	Use $y = mx + b$ Find the slope. Find the y-intercept.	$y = mx + b$ $y = 4x + (-2)$ $m = 4$ $b = -2$, $(0, -2)$
Step 3. Plot the y-intercept.	Plot $(0, -2)$.	
Step 4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = 4$ $\frac{\text{rise}}{\text{run}} = \frac{4}{1}$ rise $= 4$ run $= 1$

Step 5. Starting at the *y*-intercept, count out the rise and run to mark the second point.	Start at (0, –2) and count the rise and the run. Up 4, right 1.	
Step 6. Connect the points with a line.	Connect the two points with a line.	

TRY IT 4

Graph the line of the equation $y = 4x + 1$ using its slope and *y*-intercept.

Show answer



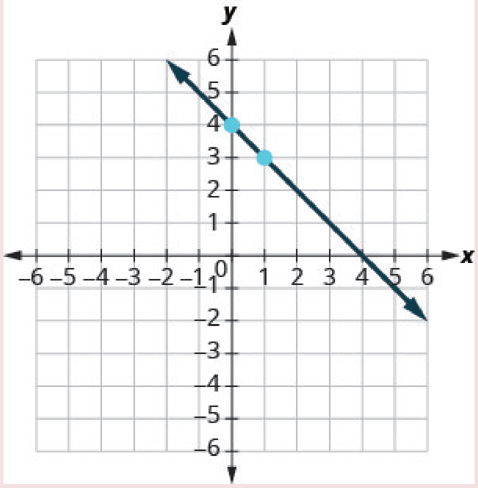
HOW TO: Graph a line using its slope and *y*-intercept

1. Find the slope-intercept form of the equation of the line.
2. Identify the slope and *y*-intercept.
3. Plot the *y*-intercept.
4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
5. Starting at the *y*-intercept, count out the rise and run to mark the second point.
6. Connect the points with a line.

EXAMPLE 5

Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

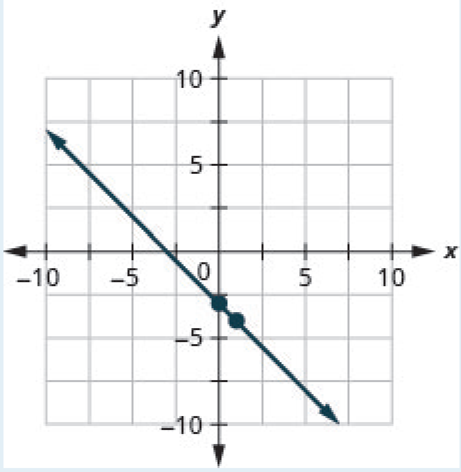
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$
	y-intercept is (0, 4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise −1, run 1
Draw the line.	
To check your work, you can find another point on the line and make sure it is a solution of the equation. In the graph we see the line goes through (4, 0).	
Check. $y = -x + 4$ $0 \overset{?}{=} -4 + 4$ $0 = 0✓$	

TRY IT 5

Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

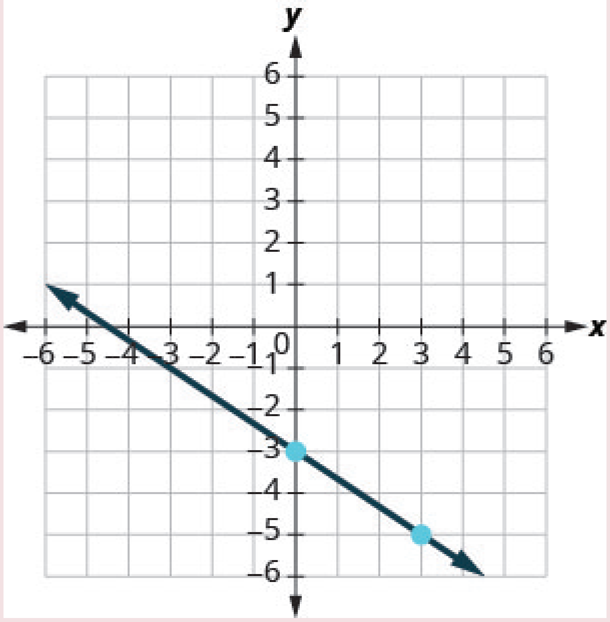
Show answer



EXAMPLE 6

Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and *y*-intercept.

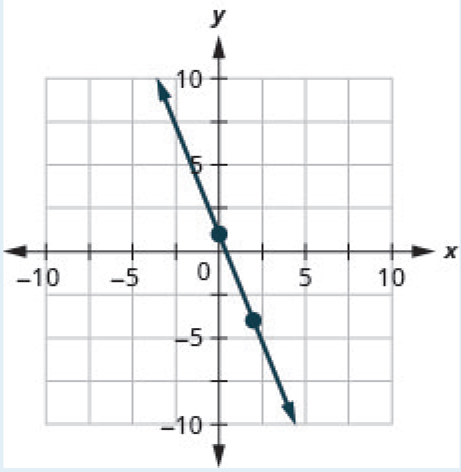
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -\frac{2}{3}x - 3$
Identify the slope and *y*-intercept.	$m = -\frac{2}{3}$; *y*-intercept is (0, −3)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	
Count out the rise and run to mark the second point.	
Draw the line.	

TRY IT 6

Graph the line of the equation $y = -\frac{5}{2}x + 1$ using its slope and *y*-intercept.

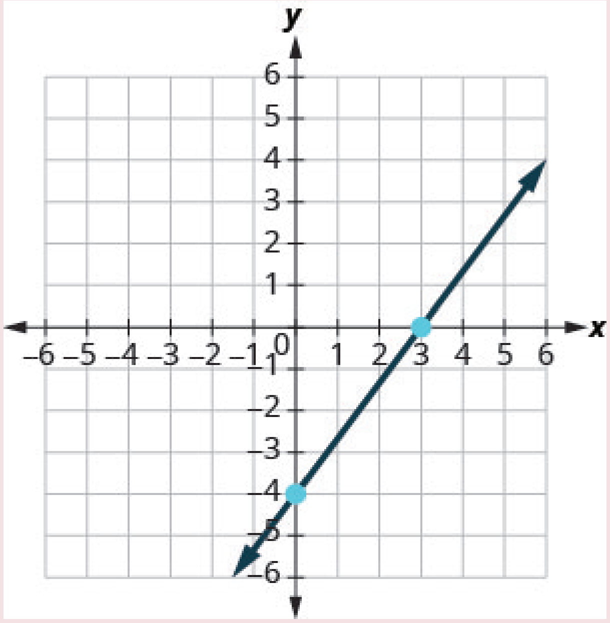
Show answer



EXAMPLE 7

Graph the line of the equation $4x - 3y = 12$ using its slope and *y*-intercept.

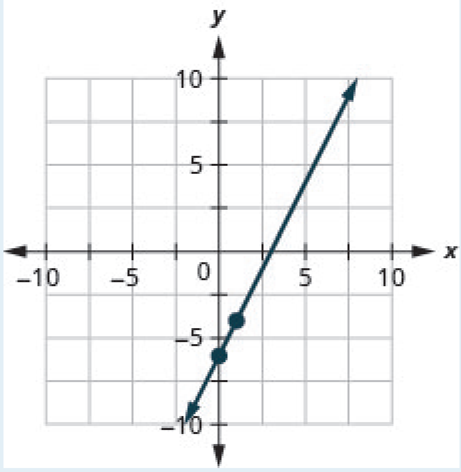
Solution

	$4x - 3y = 12$
Find the slope–intercept form of the equation.	$-3y = -4x + 12$
	$-\frac{3y}{3} = \frac{-4x + 12}{-3}$
The equation is now in slope–intercept form.	$y = \frac{4}{3}x - 4$
Identify the slope and *y*-intercept.	$m = \frac{4}{3}$
	y-intercept is (0, −4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run; count out the rise and run to mark the second point.	
Draw the line.	

TRY IT 7

Graph the line of the equation $2x - y = 6$ using its slope and *y*-intercept.

Show answer



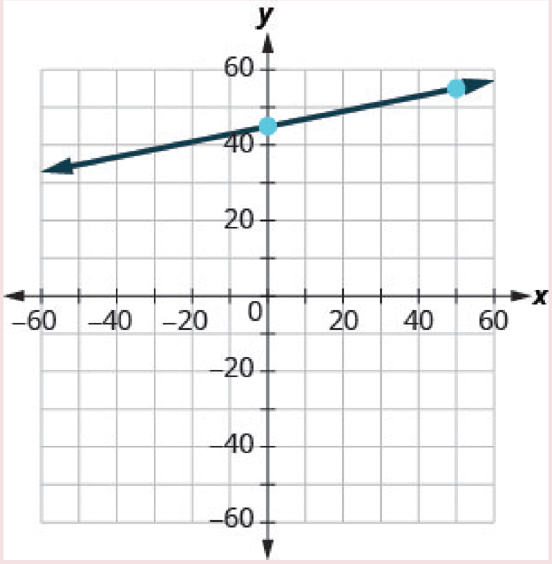
We have used a grid with x and y both going from about -10 to 10 for all the equations we've graphed so far. Not all linear equations can be graphed on this small grid. Often, especially in applications with real-world data, we'll need to extend the axes to bigger positive or smaller negative numbers.

EXAMPLE 8

Graph the line of the equation $y = 0.2x + 45$ using its slope and *y*-intercept.

Solution

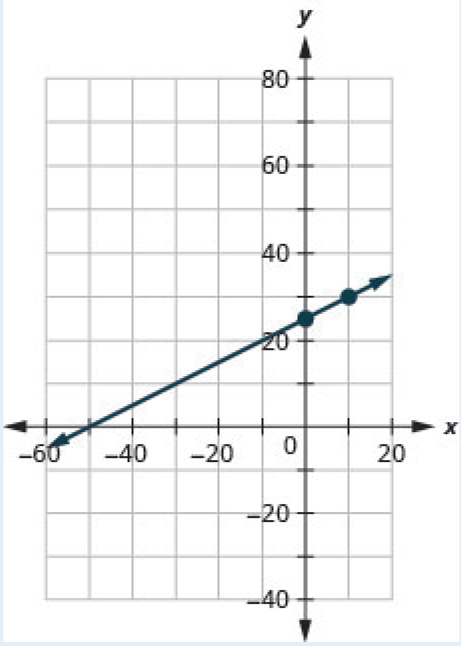
We'll use a grid with the axes going from about -80 to 80.

	$y = mx + b$
The equation is in slope–intercept form.	$y = 0.2x + 45$
Identify the slope and *y*-intercept.	$m = 0.2$
	The *y*-intercept is (0, 45)
Plot the *y*-intercept.	See graph below.
Count out the rise and run to mark the second point. The slope is $m = 0.2$; in fraction form this means $m = \frac{2}{10}$. Given the scale of our graph, it would be easier to use the equivalent fraction $m = \frac{10}{50}$.	
Draw the line.	

TRY IT 8

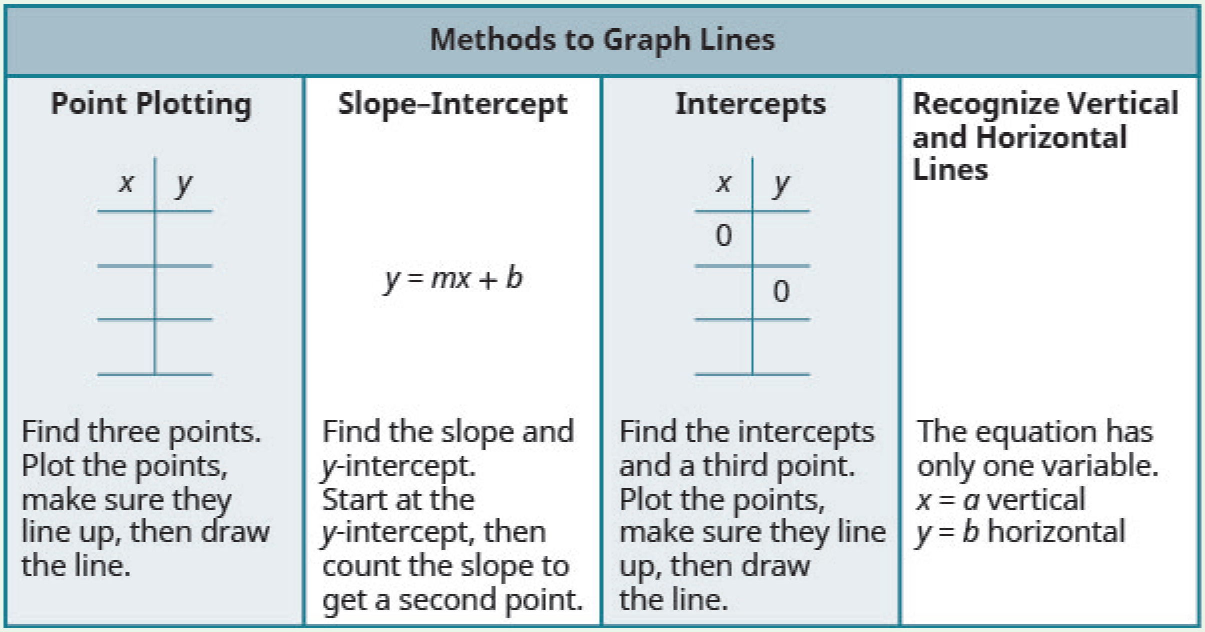
Graph the line of the equation $y = 0.5x + 25$ using its slope and *y*-intercept.

Show answer



Now that we have graphed lines by using the slope and y-intercept, let’s summarize all the methods we have used to graph lines. See (Figure).

Methods to graph lines



Methods to Graph Lines

Point Plotting	Slope–Intercept	Intercepts	Recognize Vertical and Horizontal Lines
x y	$y = mx + b$	x y 0 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier. Generally, plotting points is not the most efficient way to graph a line. We saw better methods in sections 4.3, 4.4, and earlier in this section. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are six equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = 4$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = 4x - 2$	Slope-intercept
#6	$y = -x + 4$	Slope-intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*-intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for x or y.

Equations #5 and #6 are written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for choosing the most convenient method to graph a line

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
    - $x = a$ is a vertical line passing through the *x*-axis at a.
    - $y = b$ is a horizontal line passing through the *y*-axis at b.
- If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.

    - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
    - Find the *x*– and *y*-intercepts, a third point, and then graph.

EXAMPLE 9

Determine the most convenient method to graph each line.

a) $y = -6$ b) $5x - 3y = 15$ c) $x = 7$ d) $y = \frac{2}{5}x - 1$.

Solution

a) $y = -6$

This equation has only one variable,y. Its graph is a horizontal line crossing the *y*-axis at -6.

b) $5x - 3y = 15$

This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

c) $x = 7$

There is only one variable, x. The graph is a vertical line crossing the *x*-axis at 7.

d) $y = \frac{2}{5}x - 1$

Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercept.

TRY IT 9

Determine the most convenient method to graph each line: a) $3x + 2y = 12$ b) $y = 4$ c) $y = \frac{1}{5}x - 4$ d) $x = -7$.

Show answer

a) intercepts b) horizontal line c) slope–intercept d) vertical line

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real-world situations.

Usually when a linear equation models a real-world situation, different letters are used for the variables, instead of *x* and *y*. The variable names remind us of what quantities are being measured.

EXAMPLE 10

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

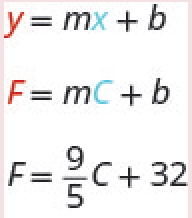
a) Find the Fahrenheit temperature for a Celsius temperature of 0.
b) Find the Fahrenheit temperature for a Celsius temperature of 20.
c) Interpret the slope and *F*-intercept of the equation.
d) Graph the equation.

Solution

a) Find the Fahrenheit temperature for a Celsius temperature of 0. Find F when $C = 0$. Simplify.	$F = \frac{9}{5}C + 32$ $F = \frac{9}{5}(0) + 32$ $F = 32$
b) Find the Fahrenheit temperature for a Celsius temperature of 20. Find F when $C = 20$. Simplify. Simplify.	$F = \frac{9}{5}C + 32$ $F = \frac{9}{5}(20) + 32$ $F = 36 + 32$ $F = 68$

c) Interpret the slope and *F*-intercept of the equation.

Even though this equation uses $F$and C, it is still in slope–intercept form.



$y = mx + b$

$F = mC + b$

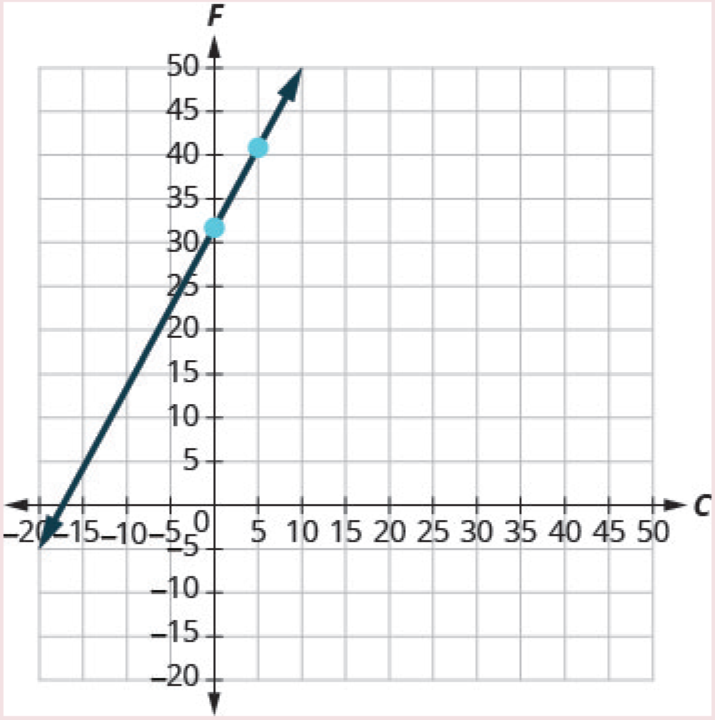
$F = \frac{9}{5}C + 32$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (*F*) increases 9 degrees when the temperature Celsius (*C*) increases 5 degrees.

The *F*-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

d) Graph the equation.

We'll need to use a larger scale than our usual. Start at the *F*-intercept $(0, 32)$ then count out the rise of 9 and the run of 5 to get a second point. See (Figure).



TRY IT 10

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, *h*, based on her shoe size, *s*.

a) Estimate the height of a child who wears women's shoe size 0.
b) Estimate the height of a woman with shoe size 8.
c) Interpret the slope and *h*-intercept of the equation.
d) Graph the equation.

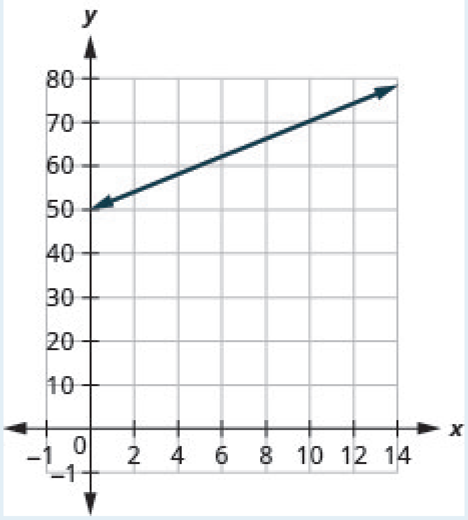
Show answer

a) 50 inches

b) 66 inches

c) The slope, 2, means that the height, *h*, increases by 2 inches when the shoe size, *s*, increases by 1. The *h*-intercept means that when the shoe size is 0, the height is 50 inches.

d)



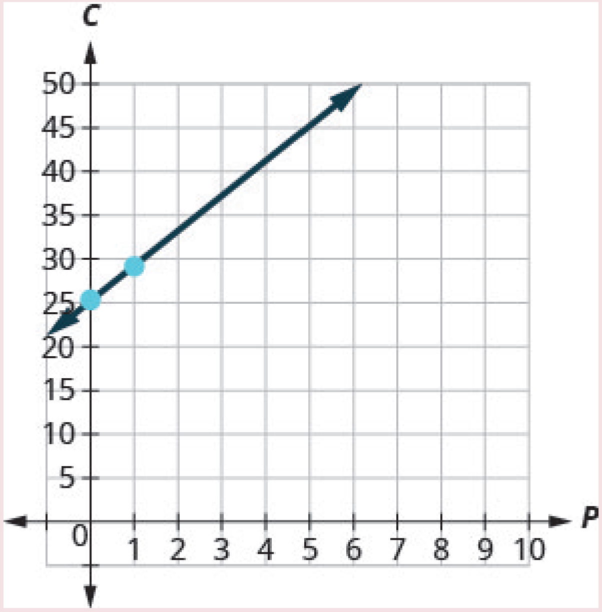
The cost of running some types business has two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

EXAMPLE 11

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

a) Find Stella's cost for a week when she sells no pizzas.
b) Find the cost for a week when she sells 15 pizzas.
c) Interpret the slope and *C*-intercept of the equation.
d) Graph the equation.

Solution

a) Find Stella’s cost for a week when she sells no pizzas.	$C = 4p + 25$
Find *C* when $p = 0$.	$C = 4(0) + 25$
Simplify.	$C = 25$
	Stella’s fixed cost is $25 when she sells no pizzas.
b) Find the cost for a week when she sells 15 pizzas.	$C = 4p + 25$
Find *C* when $p = 15$.	$C = 4(15) + 25$
Simplify.	$C = 60 + 25$
	$C = 85$
	Stella’s costs are $85 when she sells 15 pizzas.
c) Interpret the slope and *C*-intercept of the equation.	$y = mx + b$ $C = 4p + 25$
	The slope, 4, means that the cost increases by $4 for each pizza Stella sells. The *C*-intercept means that even when Stella sells no pizzas, her costs for the week are $25.
d) Graph the equation. We’ll need to use a larger scale than our usual. Start at the *C*-intercept (0, 25) then count out the rise of 4 and the run of 1 to get a second point.	

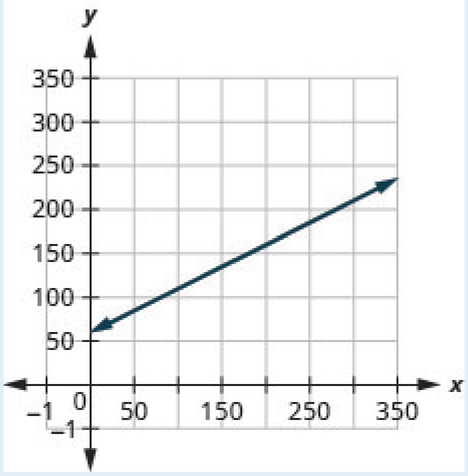
TRY IT 11

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

? Find Sam's cost for a week when he drives 0 miles.
? Find the cost for a week when he drives 250 miles.
? Interpret the slope and *C*-intercept of the equation.
? Graph the equation.

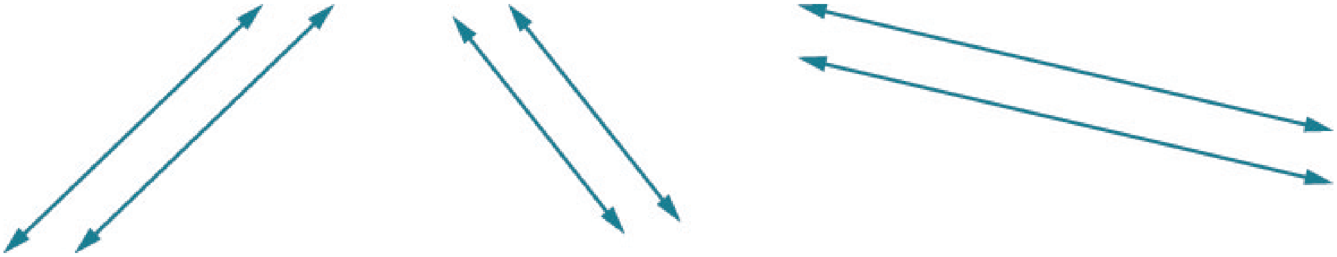
Show answer

1. ? $60
2. ? $185
3. ? The slope, 0.5, means that the weekly cost, *C*, increases by $0.50 when the number of miles driven, *n*, increases by 1. The *C*-intercept means that when the number of miles driven is 0, the weekly cost is $60
4. ?



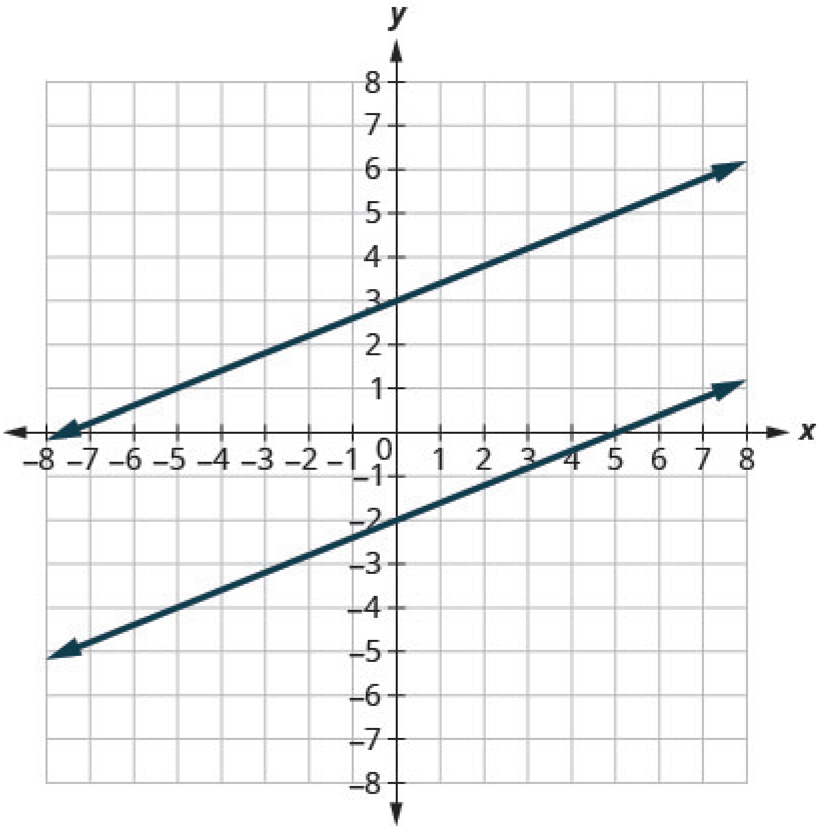
Use Slopes to Identify Parallel Lines

The slope of a line indicates how steep the line is and whether it rises or falls as we read it from left to right. Two lines that have the same slope are called parallel lines. Parallel lines never intersect.



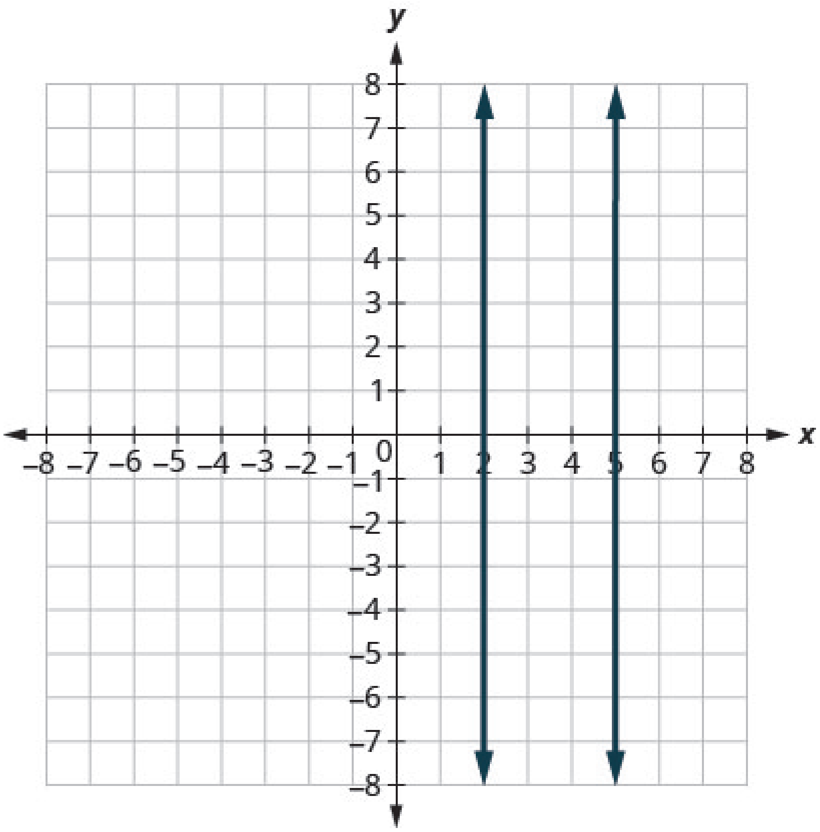
We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See (Figure).

Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.



What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel. See (Figure).

Vertical lines with different *x*-intercepts are parallel.



Parallel lines

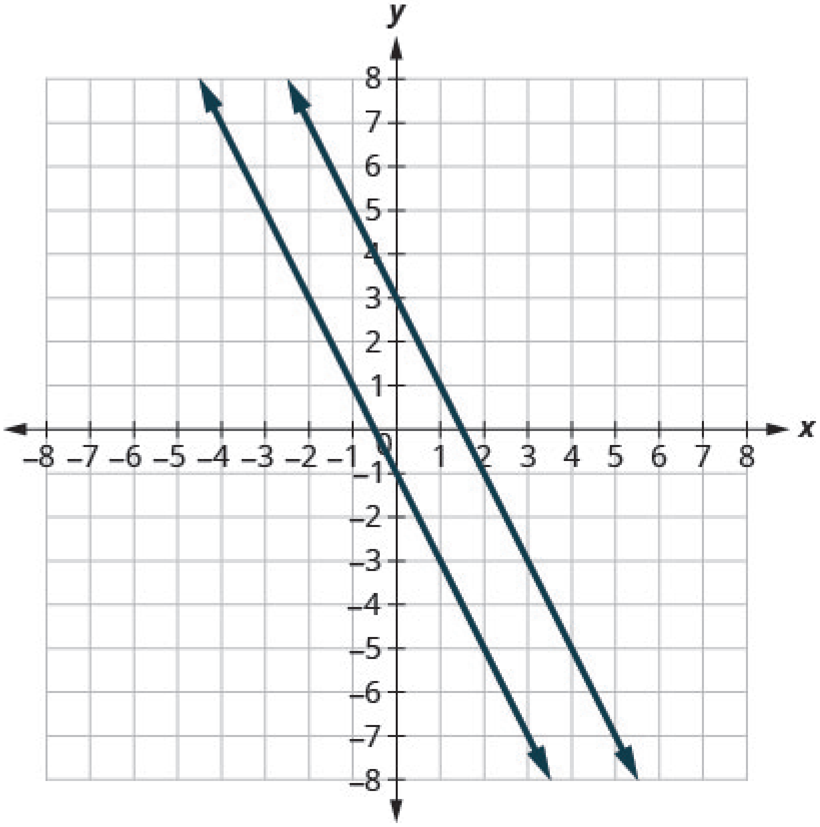
Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then$m_1 = m_2$.
- Parallel vertical lines have different *x*-intercepts.

Let’s graph the equations $y = -2x + 3$ and $2x + y = -1$ on the same grid. The first equation is already in slope–intercept form: $y = -2x + 3$. We solve the second equation for y:

$$\begin{aligned} 2x + y &= -1 \\ y &= -2x - 1 \end{aligned}$$

Graph the lines.



Notice the lines look parallel. What is the slope of each line? What is the *y*-intercept of each line?

$$\begin{aligned} y &= mx+b \\ y &= -2x+3 \\ m &= -2 \\ b &= 3, (0,\ 3) \end{aligned} \qquad \begin{aligned} y &= mx+b \\ y &= -2x-1 \\ m &= -2 \\ b &= -1, (0,\ 1) \end{aligned}$$

The slopes of the lines are the same and the *y*-intercept of each line is different. So we know these lines are parallel.

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

EXAMPLE 12

Use slopes and *y*-intercepts to determine if the lines $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$ are parallel.

Solution

Solve the first equation for y.	$\begin{aligned} 3x-2y &= 6 \\ -2y &= -3x+6 \\ \frac{-2y}{-2} &= \frac{-3x+6}{-2} \end{aligned}$	and	$y=\frac{3}{2}x+1$
The equation is now in slope-intercept form.	$y=\frac{3}{2}x-3$		
The equation of the second line is already in slope-intercept form.			$y=\frac{3}{2}x+1$
Identify the slope and y-intercept of both lines.	$y=\frac{3}{2}x-3$ $y=mx+b$ $m=\frac{3}{2}$		$y=\frac{3}{2}x+1$ $y=mx+b$ $m=\frac{3}{2}$
	y-intercept is (0, −3)		y-intercept is (0, 1)

The lines have the same slope and different *y*-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.

TRY IT 12

Use slopes and *y*-intercepts to determine if the lines $2x+5y=5$ and $y=-\frac{2}{5}x-4$ are parallel.

Show answer
parallel

EXAMPLE 13

Use slopes and *y*-intercepts to determine if the lines $y=-4$ and $y=3$ are parallel.

Solution

	$y = -4$ $y = 0x - 4$	and	$y = 3$ $y = 0x + 3$
Write each equation in slope-intercept form.	$y = 0x - 4$		$y = 0x + 3$
Since there is no x term we write $0x$.	$y = mx + b$		$y = mx + b$
Identify the slope and y-intercept of both lines.	$m = 0$		$m = 0$
	y-intercept is (0, 4)		y-intercept is (0, 3)

The lines have the same slope and different *y*-intercepts and so they are parallel.

There is another way you can look at this example. If you recognize right away from the equations that these are horizontal lines, you know their slopes are both 0. Since the horizontal lines cross the *y*-axis at $y = -4$ and at $y = 3$, we know the *y*-intercepts are $(0, -4)$ and $(0, 3)$. The lines have the same slope and different *y*-intercepts and so they are parallel.

TRY IT 13

Use slopes and *y*-intercepts to determine if the lines $y = 8$ and $y = -6$ are parallel.

Show answer
parallel

EXAMPLE 14

Use slopes and *y*-intercepts to determine if the lines $x = -2$ and $x = -5$ are parallel.

Solution
$x = -2$ and $x = -5$
Since there is no$y$, the equations cannot be put in slope–intercept form. But we recognize them as equations of vertical lines. Their *x*-intercepts are -2 and -5. Since their *x*-intercepts are different, the vertical lines are parallel.

TRY IT 14

Use slopes and *y*-intercepts to determine if the lines $x = 1$ and $x = -5$ are parallel.

Show answer

parallel

EXAMPLE 15

Use slopes and *y*-intercepts to determine if the lines $y = 2x - 3$ and $-6x + 3y = -9$ are parallel. You may want to graph these lines, too, to see what they look like.

Solution

	$y = 2x - 3$	and	$-6x + 3y = -9$
The first equation is already in slope-intercept form.	$y = 2x - 3$		
Solve the second equation for y.			$\begin{aligned} -6x + 3y &= -9 \\ 3y &= 6x - 9 \\ \frac{3y}{3} &= \frac{6x - 9}{3} \\ y &= 2x - 3 \end{aligned}$
The second equation is now in slope-intercept form.	$y = 2x - 3$		
Identify the slope and y-intercept of both lines.	$y = 2x - 3$ $y = mx + b$ $m = 2$		$y = 2x - 3$ $y = mx + b$ $m = 2$
	$y\text{-intercept is } (0,\ 3)$		$y\text{-intercept is } (0,\ 3)$

The lines have the same slope, but they also have the same *y*-intercepts. Their equations represent the same line. They are not parallel; they are the same line.

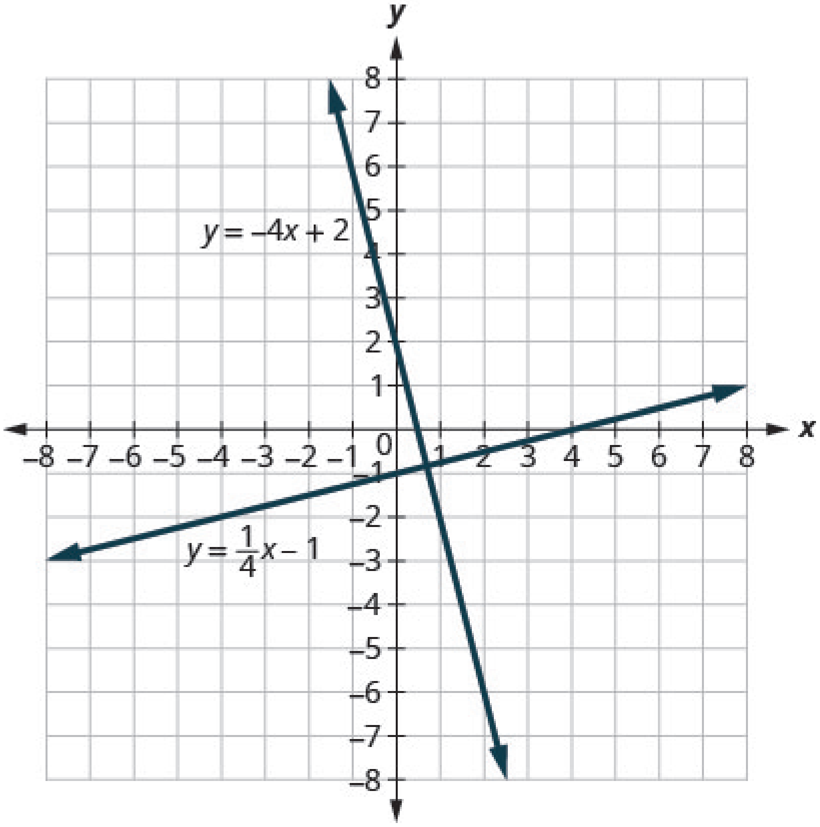
TRY IT 15

Use slopes and *y*-intercepts to determine if the lines $y = -\frac{1}{2}x - 1$ and $x + 2y = 2$ are parallel.

Show answer
not parallel; same line

Use Slopes to Identify Perpendicular Lines

Let’s look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$, shown in (Figure).



These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

What do you notice about the slopes of these two lines? As we read from left to right, the line $y=\frac{1}{4}x-1$ rises, so its slope is positive. The line$y=-4x+2$ drops from left to right, so it has a negative slope. Does it make sense to you that the slopes of two perpendicular lines will have opposite signs?

If we look at the slope of the first line, $m_1=\frac{1}{4}$, and the slope of the second line, $m_2=-4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$\begin{array}{l} m_1\cdot m_2 \\ \frac{1}{4}(-4) \\ -1 \end{array}$$

This is always true for perpendicular lines and leads us to this definition.

Perpendicular lines

Perpendicular lines are lines in the same plane that form a right angle.

If m_1 and m_2 are the slopes of two perpendicular lines, then:

$m_1 \cdot m_2 = -1$ and $m_1 = \dfrac{-1}{m_2}$

Vertical lines and horizontal lines are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are negative reciprocals. If the product of the slopes is -1, the lines are perpendicular. Perpendicular lines may have the same *y*-intercepts.

EXAMPLE 16

Use slopes to determine if the lines, $y = -5x - 4$ and $x - 5y = 5$ are perpendicular.

Solution

The first equation is already in slope-intercept form.	$y = -5x - 4$	
Solve the second equation for y.	$\begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$	
Identify the slope of each line.	$\begin{aligned} y &= -5x - 4 \\ y &= mx + b \\ m_1 &= -5 \end{aligned}$	$\begin{aligned} y &= \frac{1}{5}x - 1 \\ y &= mx + b \\ m_2 &= \frac{1}{5} \end{aligned}$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes,

$m_1 \cdot m_2$

$-5\left(\frac{1}{5}\right)$

$-1\checkmark$

TRY IT 16

Use slopes to determine if the lines $y = -3x + 2$ and $x - 3y = 4$ are perpendicular.

Show answer
perpendicular

EXAMPLE 17

Use slopes to determine if the lines, $7x + 2y = 3$ and $2x + 7y = 5$ are perpendicular.

Solution

Solve the equations for y.	$\begin{aligned} 7x+2y &= 3 \\ 2y &= -7x+3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= -\frac{7}{2}x+\frac{3}{2} \end{aligned}$	$\begin{aligned} 2x+7y &= 5 \\ 7y &= -2x+5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= -\frac{2}{7}x+\frac{5}{7} \end{aligned}$
Identify the slope of each line.	$\begin{aligned} y &= mx+b \\ m_1 &= -\frac{7}{2} \end{aligned}$	$\begin{aligned} y &= mx+b \\ m_2 &= -\frac{2}{7} \end{aligned}$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

TRY IT 17

Use slopes to determine if the lines $5x + 4y = 1$ and $4x + 5y = 3$ are perpendicular.

Show answer
not perpendicular

Access this online resource for additional instruction and practice with graphs.

- Explore the Relation Between a Graph and the Slope–Intercept Form of an Equation of a Line

Key Concepts

- **The slope–intercept form of an equation of a line with slope** m and *y*-intercept, $(0, b)$ is, $y = mx + b$.
- **Graph a Line Using its Slope and *y*-Intercept**
    1. Find the slope-intercept form of the equation of the line.
    2. Identify the slope and *y*-intercept.
    3. Plot the *y*-intercept.
    4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
    5. Starting at the *y*-intercept, count out the rise and run to mark the second point.
    6. Connect the points with a line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:** Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the *x*-axis at a.
      $y = b$ is a horizontal line passing through the *y*-axis at b.
    - If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
      Identify the slope and *y*-intercept and then graph.
    - If the equation is of the form $Ax + By = C$, find the intercepts.
      Find the *x*– and *y*-intercepts, a third point, and then graph.
- **Parallel lines are lines in the same plane that do not intersect.**
    - Parallel lines have the same slope and different *y*-intercepts.
    - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
    - Parallel vertical lines have different *x*-intercepts.
- **Perpendicular lines are lines in the same plane that form a right angle.**
    - If m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 m_2 = -1$ and $m_1 = \frac{-1}{m_2}$.
    - Vertical lines and horizontal lines are always perpendicular to each other.

Glossary

parallel lines
Lines in the same plane that do not intersect.

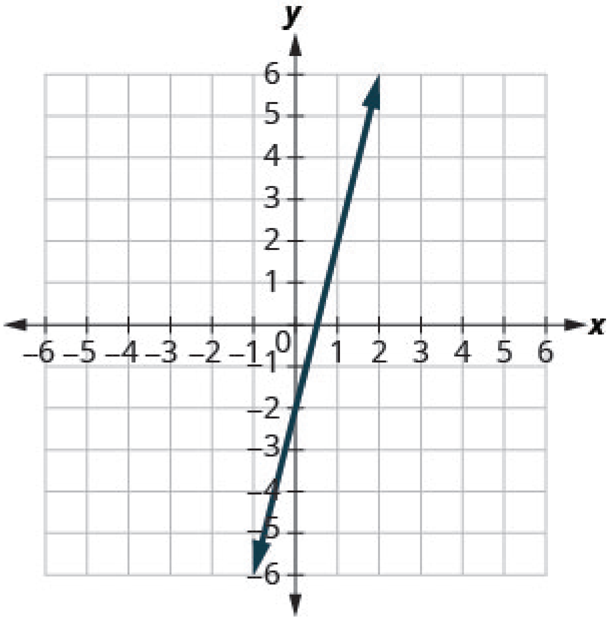
perpendicular lines
Lines in the same plane that form a right angle.

slope-intercept form of an equation of a line
The slope–intercept form of an equation of a line with slope and *y*-intercept, is,
.

3.5 Exercise Set

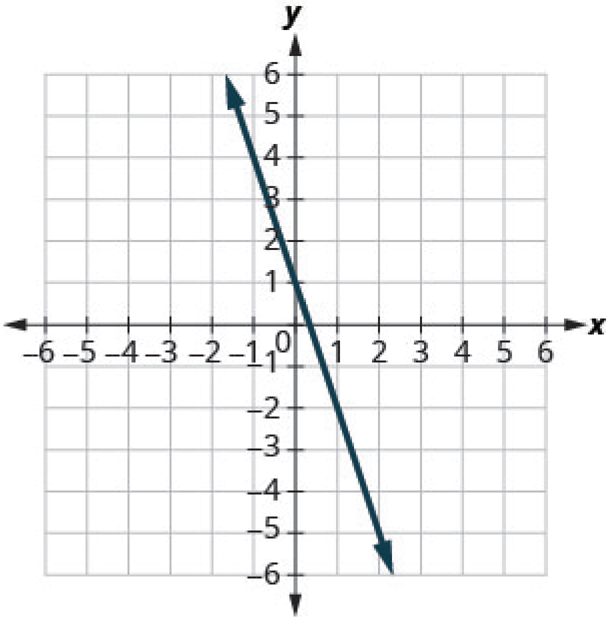
In the following exercises, use the graph to find the slope and y-intercept of each line. Compare the values to the equation $y = mx + b$.

1.

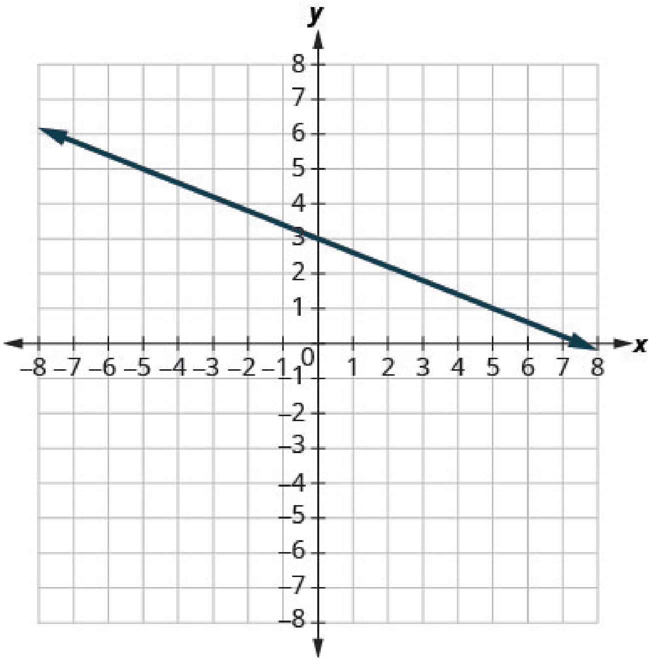


$y = 4x - 2$

2.



3.



$y = -\frac{2}{5}x + 3$

In the following exercises, identify the slope and y-intercept of each line.

4. $y = -9x + 7$

5. $y = 4x - 10$

6. $4x + y = 8$
7. $8x + 3y = 12$
8. $7x - 3y = 9$

In the following exercises, graph the line of each equation using its slope and y-intercept.

9. $y = x + 4$
10. $y = 2x - 3$
11. $y = x + 3$
12. $y = x - 2$
13. $y = -\frac{2}{5} - 3$
14. $y = -\frac{2}{3} + 1$
15. $4x - 3y = 6$
16. $y = 0.1x + 15$

In the following exercises, determine the most convenient method to graph each line.

17. $y = 4$
18. $x = -3$
19. $y = -3x + 4$
20. $x - y = 1$
21. $y = \frac{4}{5}x - 3$
22. $y = -1$
23. $2x - 5y = -10$
24. $y = -\frac{1}{3}x + 5$

In the following exercises, graph and interpret applications of slope and intercept.

25. The equation $P = 28 + 2.54w$ models the relation between the amount of Randy's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.
    a. Find the payment for a month when Randy used 0 units of water
    b. Find the payment for a month when Randy used 15 units of water.
    c. Interpret the slope and *P*-intercept of the equation.
    d. Graph the equation.
26. Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, *C*, per day and the number of miles, *m*, she drives in one day.
    a. Find the cost if Janelle drives the car 0 miles one day.
    b. Find the cost on a day when Janelle drives the car 400 miles.
    c. Interpret the slope and *C*–intercept of the equation.
    d. Graph the equation.
27. Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.
    a. Find Patel's salary for a week when his sales were 0.
    b. Find Patel's salary for a week when his sales were 18,540.

c. Interpret the slope and *S*-intercept of the equation.

d. Graph the equation.

28. Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, *C* of the banquet and the number of guests, *g*.

    a. Find the cost if the number of guests is 50.

    b. Find the cost if the number of guests is 100.

    c. Interpret the slope and *C*–intercept of the equation.

    d. Graph the equation.

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

29. $y = \frac{2}{3}x - 1; \;\; 2x - 3y = -2$
30. $3x - 4y = -2; \;\; y = \frac{3}{4}x - 3$
31. $6x - 3y = 9; \;\; 2x - y = 3$
32. $8x + 6y = 6; \;\; 12x + 9y = 12$
33. $x = 7; \;\; x = -8$
34. $x = -3; \;\; x = -2$
35. $y = 5; \;\; y = 1$
36. $y = -1; \;\; y = 2$
37. $4x + 4y = 8; \;\; x + y = 2$
38. $5x - 2y = 11; \;\; 5x - y = 7$
39. $4x - 8y = 16; \;\; x - 2y = 4$
40. $x - 5y = 10; \;\; 5x - y = -10$
41. $9x - 5y = 4; \;\; 5x + 9y = -1$

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

42. $x - 4y = 8; 4x + y = 2$
43. $2x + 3y = 5; 3x - 2y = 7$
44. $3x - 4y = 8; 4x - 3y = 6$
45. $2x + 4y = 3; 6x + 3y = 2$
46. $2x - 6y = 4; 12x + 4y = 9$
47. $8x - 2y = 7; 3x + 12y = 9$
48. The equation $n = 4T - 160$ is used to estimate the number of cricket chirps, *n*, in one minute based on the temperature in degrees Fahrenheit, *T*.

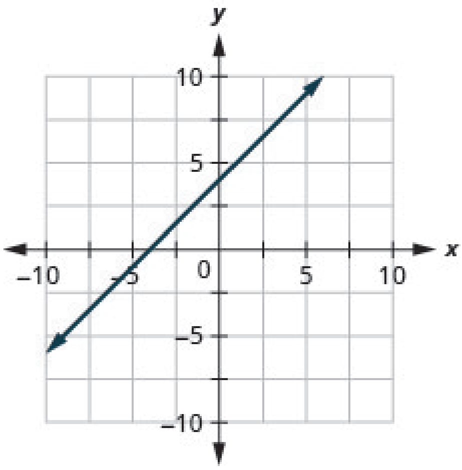
    a. Explain what the slope of the equation means.

    b. Explain what the *n*–intercept of the equation means. Is this a realistic situation?

Answers

1. slope $m = 4$ and *y*-intercept $(0, -2)$
2. slope $m = -3$ and *y*-intercept $(0, 1)$
3. slope $m = -\frac{2}{5}$ and *y*-intercept $(0, 3)$
4. $-9; (0, 7)$
5. $4; (0, -10)$
6. $-4; (0, 8)$
7. $-\frac{8}{3}; (0, 4)$
8. $\frac{7}{3}; (0, -3)$

9.


y
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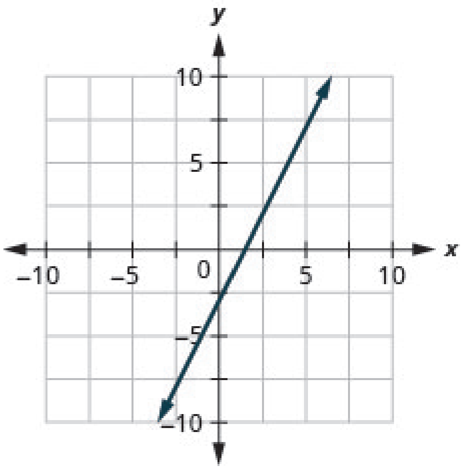
5
10

10.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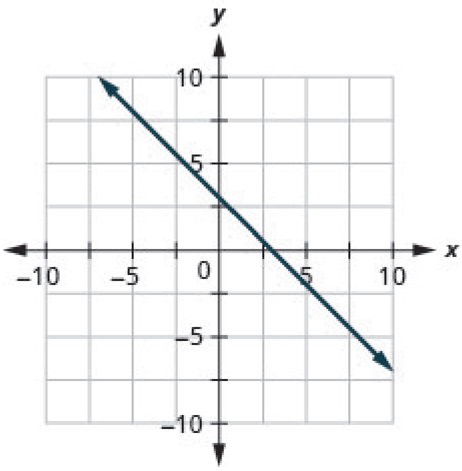
11.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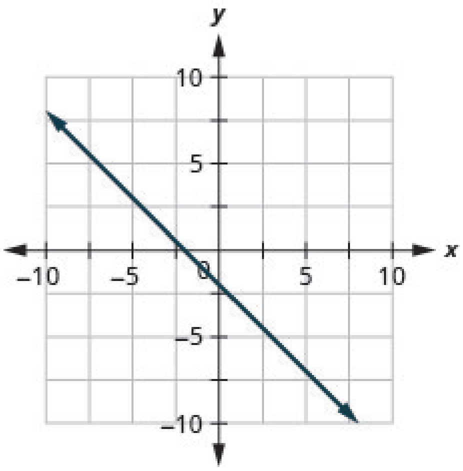
12.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−5
5
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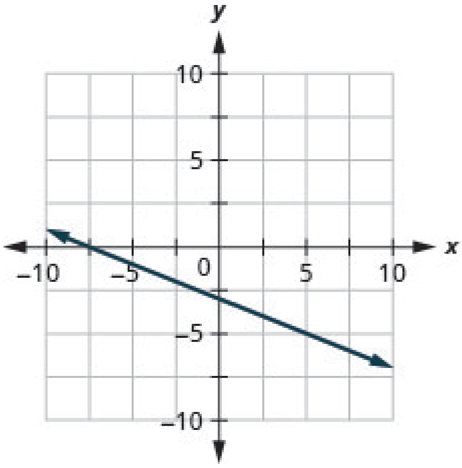
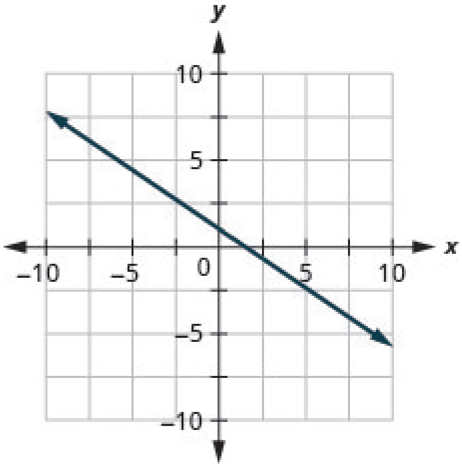
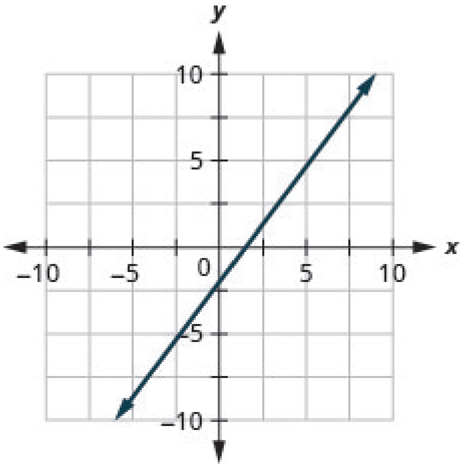
13.



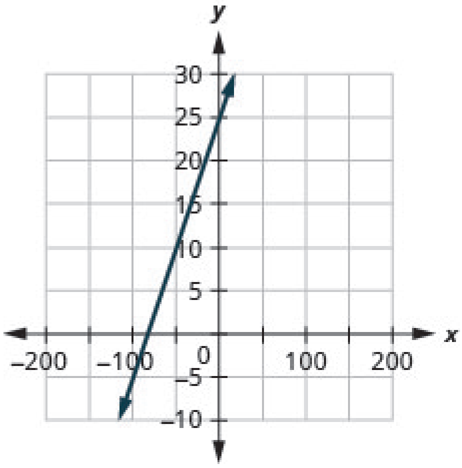
14.



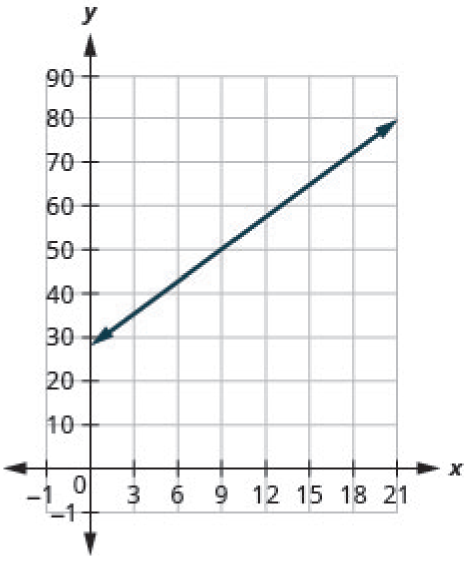
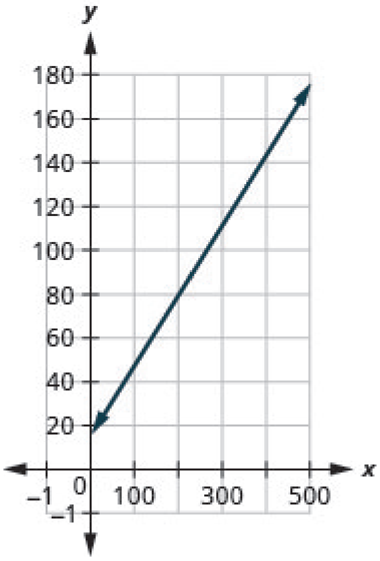
15.



16.



17. horizontal line
18. vertical line
19. slope–intercept
20. intercepts
21. slope–intercept
22. horizontal line
23. intercepts
24. slope–intercept

25. a) $28 b) $66.10 c) The slope, 2.54, means that Randy’s payment, *P*, increases by $2.54 when the number of units of water he used, *w*, increases by 1. The *P*–intercept means that if the number units of water Randy used was 0, the payment would be $28. d)  	26. a) $15 b) $143 c) The slope, 0.32, means that the cost, *C*, increases by $0.32 when the number of miles driven, *m*, increases by 1. The *C*-intercept means that if Janelle drives 0 miles one day, the cost would be $15. d) 

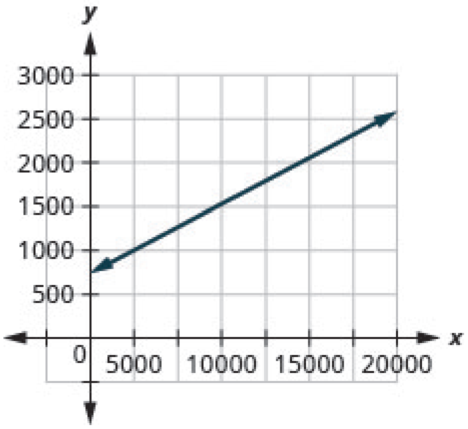
27.

a) $750

b) $2418.60

c) The slope, 0.09, means that Patel’s salary, *S*, increases by $0.09 for every $1 increase in his sales. The *S*-intercept means that when his sales are $0, his salary is $750.

d)



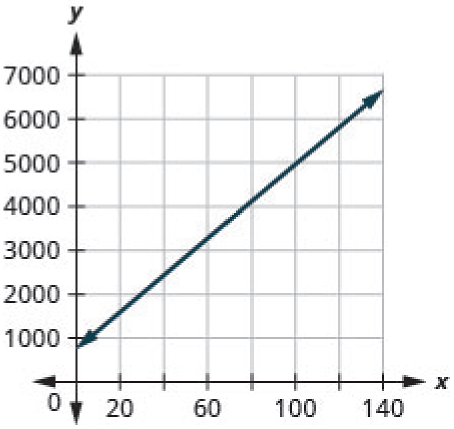
28.

a) $2850

b) $4950

c) The slope, 42, means that the cost, *C*, increases by $42 for when the number of guests increases by 1. The *C*-intercept means that when the number of guests is 0, the cost would be $750.

d)



29. parallel
30. parallel
31. parallel
32. parallel
33. parallel
34. parallel
35. parallel
36. parallel
37. not parallel
38. not parallel
39. not parallel
40. not parallel
41. not parallel
42. perpendicular
43. perpendicular
44. not perpendicular
45. not perpendicular
46. perpendicular
47. perpendicular
48.
    a. For every increase of one degree Fahrenheit, the number of chirps increases by four.
    b. There would be -160 chirps when the Fahrenheit temperature is 0. (Notice that this does not make sense; this model cannot be used for all possible temperatures.)

Attributions

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4. Systems of Equations

In this chapter, you will use systems of equations to solve applications with two unknowns.

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4.1 Solve Systems of Equations by Graphing

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Determine Whether an Ordered Pair is a Solution of a System of Equations

We learned before how to solve linear equations with one variable. Now we will work with systems of linear equations, two or more linear equations grouped together, witch is known as a system of linear equations.

System of Linear Equations

When two or more linear equations are grouped together, they form a system of linear equations.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (*x*, *y*) that make both equations true. These are called the solutions to a system of equations.

Solutions of a System of Equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (*x*, *y*).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let’s consider the system below:

$$\begin{cases} 3x - y = 7 \\ x - 2y = 4 \end{cases}$$

Is the ordered pair $(2, -1)$ a solution?

We substitute $x = 2$ and $y = -1$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(2) - (-1) \overset{?}{=} 7 & 2 - 2(-1) \overset{?}{=} 4 \\ 7 = 7 \text{ true} & 4 = 4 \text{ true} \end{array}$$

The ordered pair (2, −1) made both equations true. Therefore (2, −1) is a solution to this system.

Let’s try another ordered pair. Is the ordered pair (3, 2) a solution?

We substitute $x = 3$ and $y = 2$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(3) - 2 \overset{?}{=} 7 & 2 - 2(2) \overset{?}{=} 4 \\ 7 = 7 \text{ true} & -2 = 4 \text{ false} \end{array}$$

The ordered pair (3, 2) made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

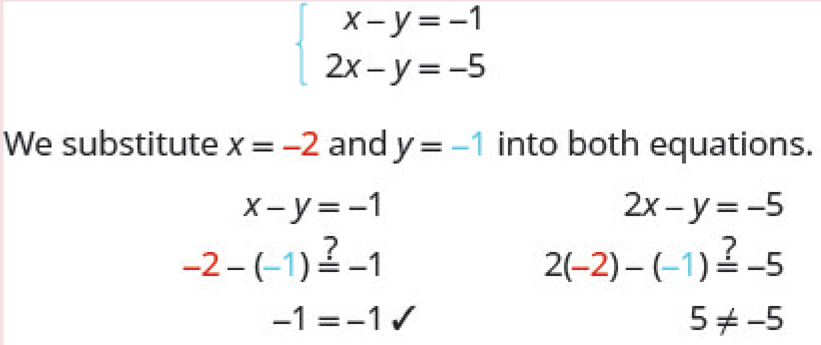
EXAMPLE 1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$

a) $(-2, -1)$ b) $(-4, -3)$

Solution

a)



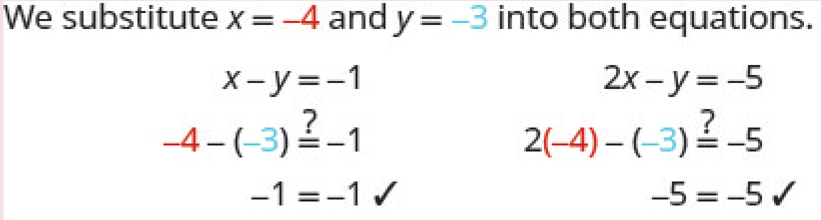
$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$$\begin{array}{rr} x - y = -1 & 2x - y = -5 \\ -2 - (-1) \overset{?}{=} -1 & 2(-2) - (-1) \overset{?}{=} -5 \\ -1 = -1 \checkmark & 5 \neq -5 \end{array}$$

(−2, −1) does not make both equations true. (−2, −1) is not a solution.

b)



We substitute $x = -4$ and $y = -3$ into both equations.

$$\begin{array}{rr} x - y = -1 & 2x - y = -5 \\ -4 - (-3) \overset{?}{=} -1 & 2(-4) - (-3) \overset{?}{=} -5 \\ -1 = -1 \checkmark & -5 = -5 \checkmark \end{array}$$

(−4, −3) does not make both equations true. (−4, −3) is a solution.

TRY IT 1

Determine whether the ordered pair is a solution to the system: $\begin{cases} 3x + y = 0 \\ x + 2y = -5 \end{cases}$.

a) $(1, -3)$ b) $(0, 0)$

Show answer

a) yes b) no

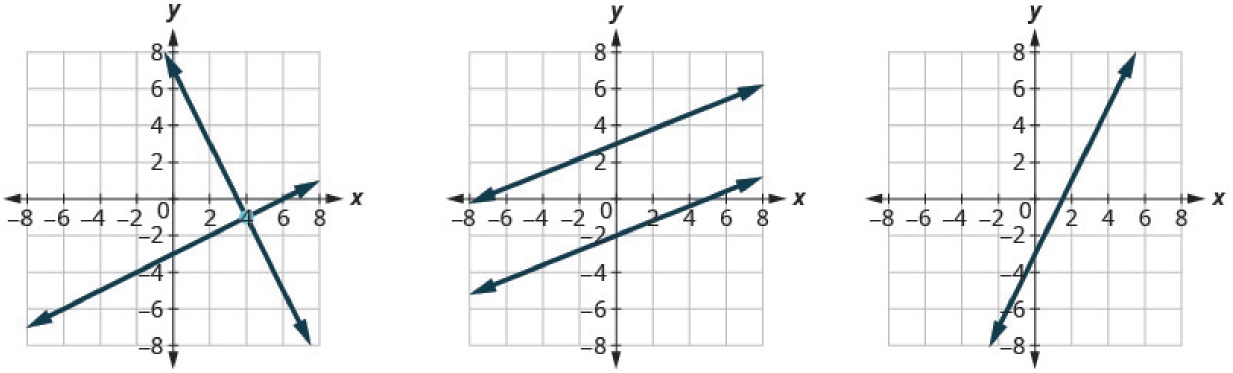
Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in (Figure 1):



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Figure 1

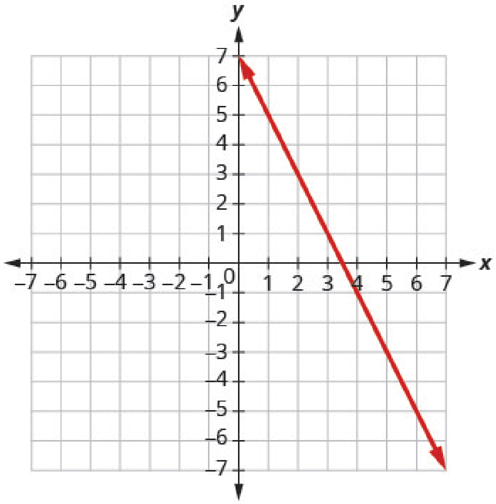
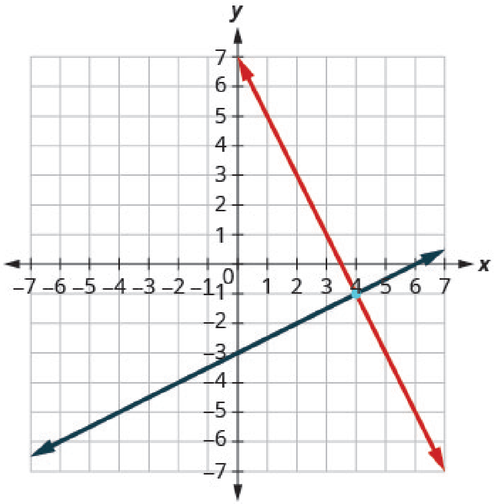
For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we'll use a different method in each section. After seeing the third method, you'll decide which method was the most convenient way to solve this system.

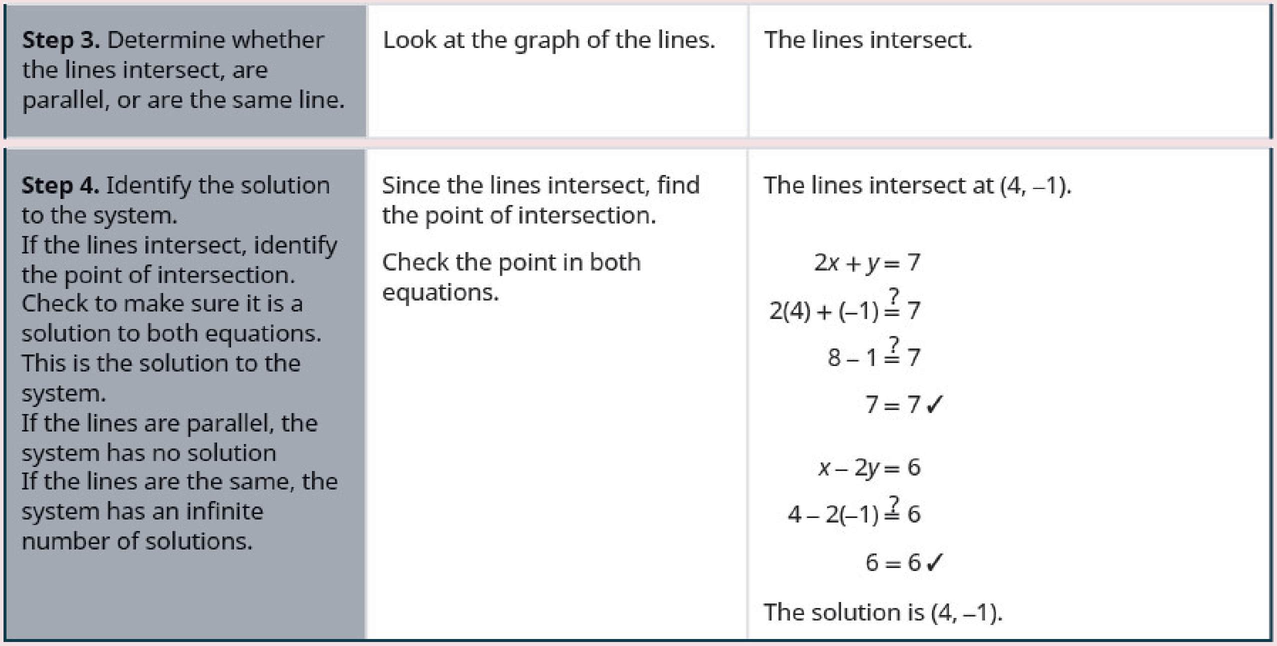
EXAMPLE 2

How to Solve a System of Linear Equations by Graphing

Solve the system by graphing: $\left\{\begin{array}{l} 2x + y = 7 \\ x - 2y = 6 \end{array}\right.$.

Solution

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x+y=7$ $y=-2x+7$ $m=-2$ $b=7$	$\begin{cases}2x+y=7\\x-2y=6\end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x-2y=6$ $(0,-3)$ $(6,0)$	



Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, –1). $2x + y = 7$ $2(4) + (-1) \overset{?}{=} 7$ $8 - 1 \overset{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \overset{?}{=} 6$ $6 = 6$ ✓ The solution is (4, –1).

TRY IT 2

Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$.

Show answer
$(3, 2)$

The steps to use to solve a system of linear equations by graphing are shown below

To solve a system of linear equations by graphing.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
    - If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
    - If the lines are parallel, the system has no solution.

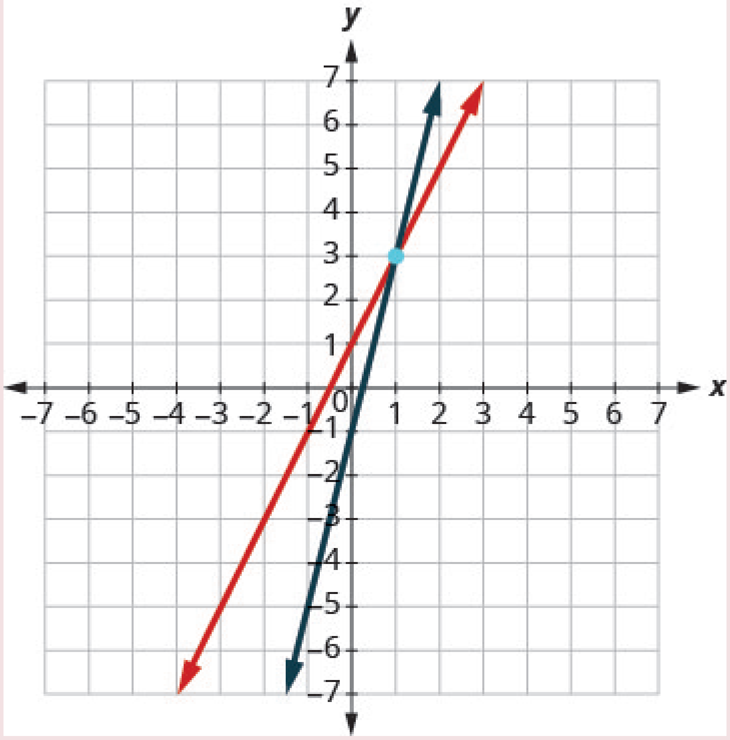
- If the lines are the same, the system has an infinite number of solutions.

EXAMPLE 3

Solve the system by graphing: $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$.

Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and *y*-intercepts to graph them. $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Find the slope and *y*-intercept of the first equation.	$y = 2x + 1$ $m = 2$ $b = 1$
Find the slope and *y*-intercept of the first equation.	$y = 4x - 1$ $m = 4$ $b = -1$
Graph the two lines.	
Determine the point of intersection.	The lines intersect at (1, 3).
	
Check the solution in both equations.	$y = 2x + 1$ $\quad$ $y = 4x - 1$ $3 \overset{?}{=} 21 + 1$ $\quad$ $3 \overset{?}{=} 41 - 1$ $3 = 3$ $\quad$ $3 = 3$
	The solution is (1, 3).

TRY IT 3

Solve each system by graphing: $\begin{cases} y = 2x + 2 \\ y = x - 4 \end{cases}$.

Show answer
$(-2, -2)$

Both equations in Example 3 were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we'll first re-write the equations into slope–intercept form.

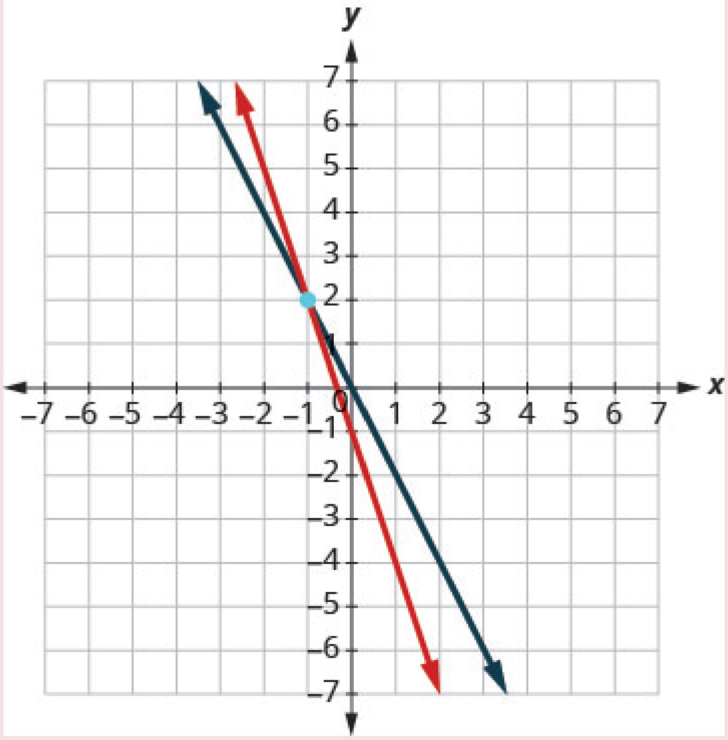
EXAMPLE 4

Solve the system by graphing: $\left\{\begin{array}{l} 3x + y = -1 \\ 2x + y = 0 \end{array}\right.$.

Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and *y*-intercepts.

$$\left\{\begin{array}{l} 3x + y = -1 \\ 2x + y = 0 \end{array}\right.$$

Solve the first equation for *y*. Find the slope and *y*-intercept. Solve the second equation for *y*. Find the slope and *y*-intercept.	$3x + y = -1$ $y = -3x - 1$ $m = -3$ $b = -1$ $2x + y = 0$ $y = -2x$ $m = -2$ $b = 0$
Graph the lines.	
Determine the point of intersection.	The lines intersect at (−1, 2).
Check the solution in both equations.	$3x + y = -1$ $3(-1) + 2 \overset{?}{=} -1$ $-1 = -1$ $2x + y = 0$ $2(-1) + 2 \overset{?}{=} 0$ $0 = 0$
	The solution is (−1, 2).

TRY IT 4

Solve each system by graphing: $\left\{\begin{array}{l} -x+y=1 \\ 2x+y=10 \end{array}\right.$.

Show answer
$(3,4)$

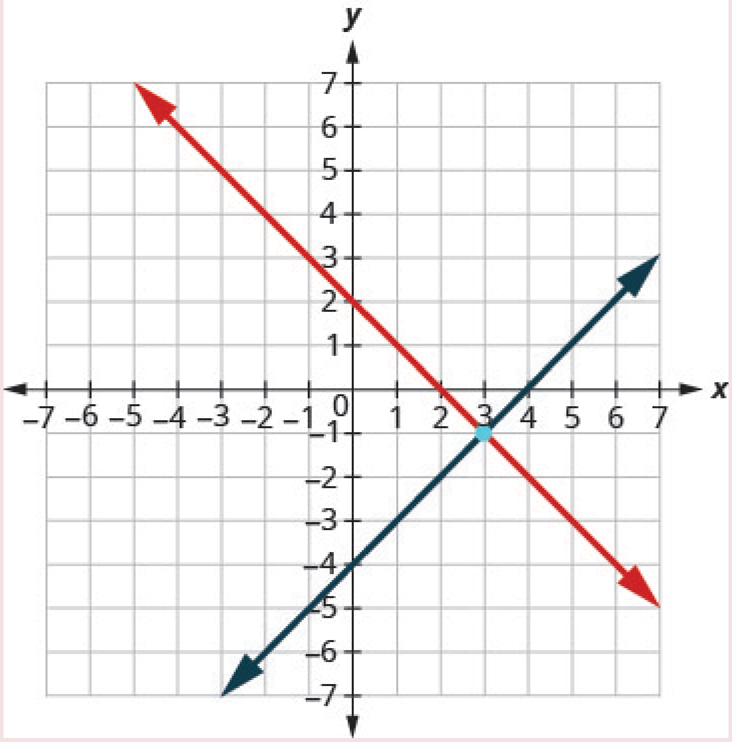
Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in the next example.

EXAMPLE 5

Solve the system by graphing: $\left\{\begin{array}{l} x+y=2 \\ x-y=4 \end{array}\right.$.

Solution

We will find the *x*– and *y*-intercepts of both equations and use them to graph the lines.

	$x+y=2$	
To find the intercepts, let *x* = 0 and solve for *y*, then let *y* = 0 and solve for *x*.	$\begin{aligned} x+y &= 2 \\ 0+y &= 2 \\ y &= 2 \end{aligned} \qquad \begin{aligned} x+y &= 2 \\ x+0 &= 2 \\ x &= 2 \end{aligned}$	$\begin{array}{\|c\|c\|}\hline x & y \\ \hline 0 & 2 \\ \hline 2 & 0 \\ \hline \end{array}$
	$x-y=4$	
To find the intercepts, let *x* = 0 then let *y* = 0.	$\begin{aligned} x-y &= 4 \\ 0-y &= 4 \\ -y &= 4 \\ y &= -4 \end{aligned} \qquad \begin{aligned} x-y &= 4 \\ x-0 &= 4 \\ x &= 4 \end{aligned}$	$\begin{array}{\|c\|c\|}\hline x & y \\ \hline 0 & -4 \\ \hline 4 & 0 \\ \hline \end{array}$
Graph the line.		
Determine the point of intersection.	The lines intersect at (3, −1).	
Check the solution in both equations.	$\begin{aligned} x+y &= 2 \\ 3+(-1) &\overset{?}{=} 2 \\ 2 &= 2 \end{aligned} \qquad \begin{aligned} x-y &= 4 \\ 3-(-1) &\overset{?}{=} 4 \\ 4 &= 4 \end{aligned}$ The solution is (3, −1).	

TRY IT 5

Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$.

Show answer

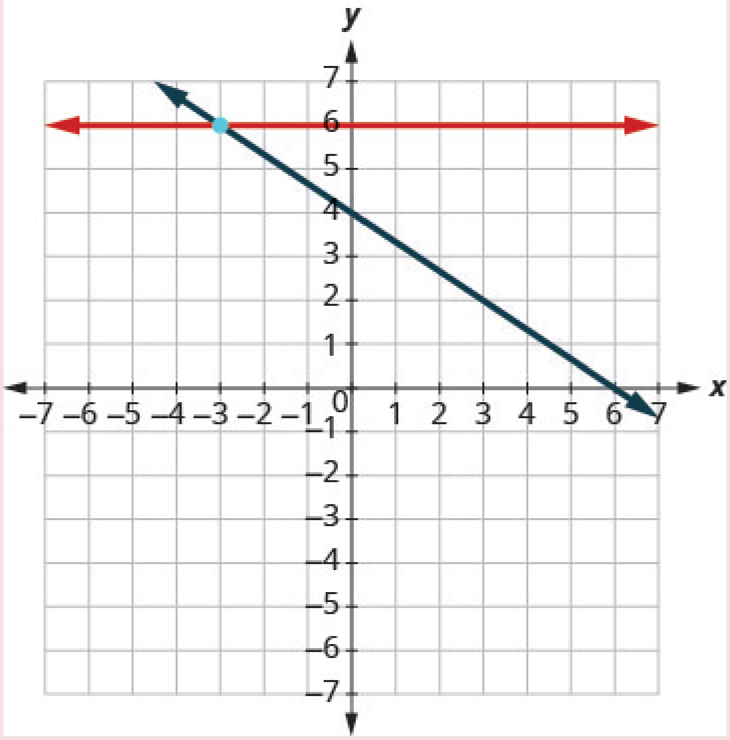
$(4,2)$

Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

EXAMPLE 6

Solve the system by graphing: $\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$.

Solution

<table>
<tr><td></td><td>$$\begin{cases} y = 6 \\ 2x + 3y = 12 \end{cases}$$</td></tr>
<tr><td>We know the first equation represents a horizontal line whose <i>y</i>-intercept is 6.</td><td>$y = 6$</td></tr>
<tr><td>The second equation is most conveniently graphed using intercepts.</td><td>$2x + 3y = 12$</td></tr>
<tr><td>To find the intercepts, let <i>x</i> = 0 and then <i>y</i> = 0.</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>4</td></tr><tr><td>6</td><td>0</td></tr></table></td></tr>
<tr><td>Graph the lines.</td><td>
</td></tr>
<tr><td>Determine the point of intersection.</td><td>The lines intersect at (−3, 6).</td></tr>
<tr><td>Check the solution to both equations.</td><td>$$\begin{aligned} y &= 6 \\ 6 &\overset{?}{=} 6 \\ 2 &= 2 \end{aligned} \qquad \begin{aligned} 2x + 3y &= 12 \\ 2(-3) + 3(6) &\overset{?}{=} 12 \\ -6 + 18 &\overset{?}{=} 12 \\ 12 &= 12 \end{aligned}$$</td></tr>
<tr><td></td><td>The solution is (−3, 6).</td></tr>
</table>

TRY IT 6

Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$.

Show answer

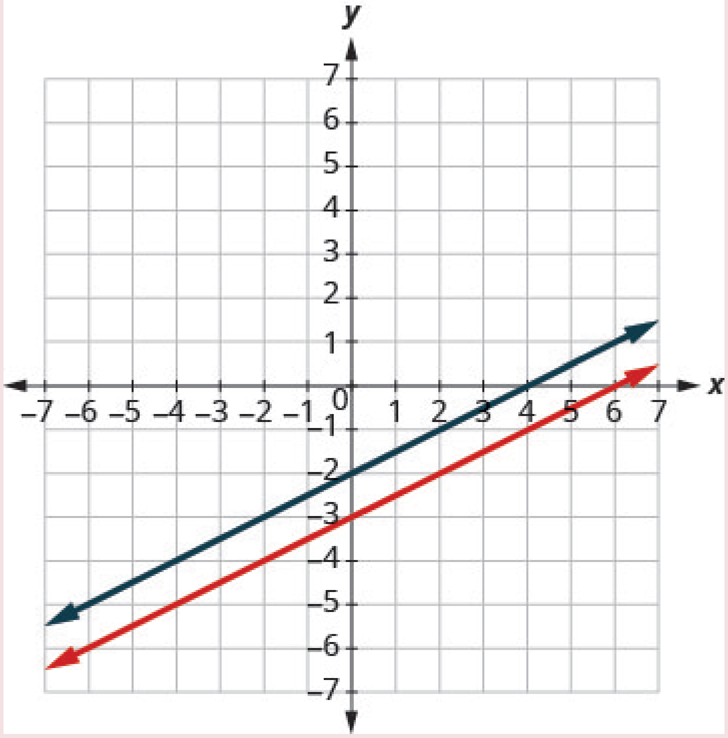
$(9, -1)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

EXAMPLE 7

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$.

Solution

<table>
<tr><td></td><td>$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$</td></tr>
<tr><td>To graph the first equation, we will use its slope and *y*-intercept.</td><td>$y = \frac{1}{2}x - 3$</td></tr>
<tr><td></td><td>$m = \frac{1}{2}$</td></tr>
<tr><td></td><td>$b = -3$</td></tr>
<tr><td>To graph the second equation, we will use the intercepts.</td><td>$x - 2y = 4$</td></tr>
<tr><td></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>–2</td></tr><tr><td>4</td><td>0</td></tr></table></td></tr>
<tr><td>Graph the lines.</td><td> </td></tr>
<tr><td>Determine the point of intersection.</td><td>The lines are parallel.</td></tr>
<tr><td></td><td>Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.</td></tr>
</table>

TRY IT 7

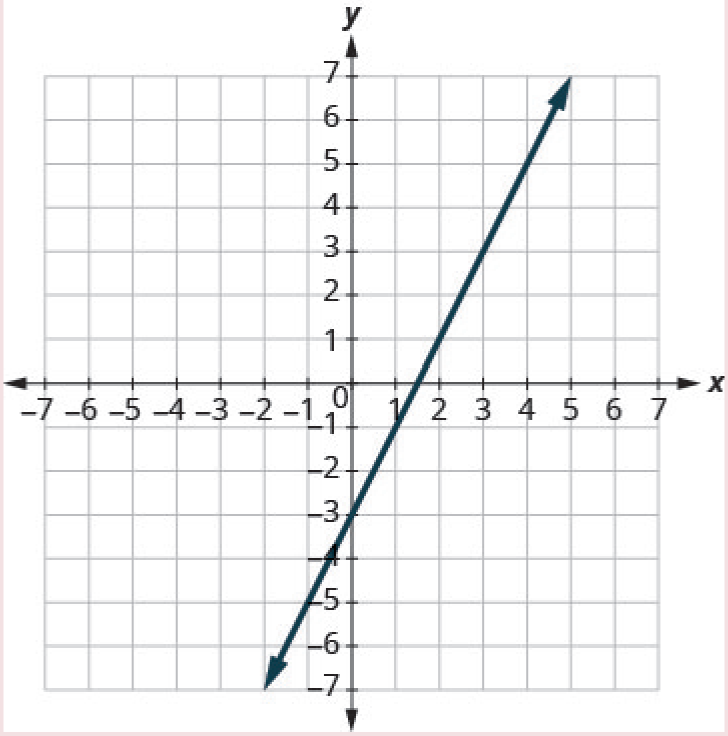
Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$.

Show answer
no solution

EXAMPLE 8

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$.

Solution

<table>
<tr><td></td><td>$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$</td></tr>
<tr><td>Find the slope and y-intercept of the first equation.</td><td>$y = 2x - 3$
$m = 2$
$b = -3$</td></tr>
<tr><td>Find the intercepts of the second equation.</td><td>$-6x + 3y = -9$</td></tr>
<tr><td></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>–3</td></tr><tr><td>$\frac{3}{2}$</td><td>0</td></tr></table></td></tr>
<tr><td>Graph the lines.</td><td>
</td></tr>
<tr><td>Determine the point of intersection.</td><td>The lines are the same!</td></tr>
<tr><td></td><td>Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.</td></tr>
<tr><td></td><td>There are infinitely many solutions to this system.</td></tr>
</table>

TRY IT 8

Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Show answer
infinitely many solutions

If you write the second equation in Example 8 in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. Coincident lines have the same slope and same *y*-intercept.

Coincident Lines

Coincident lines have the same slope and same *y*-intercept.

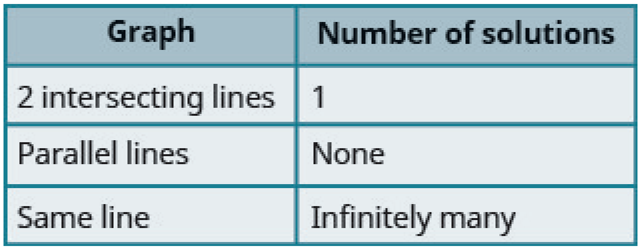
Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in Example 2 through Example 6 all had two intersecting lines. Each system had one solution.

A system with parallel lines, like Example 7, has no solution. What happened in Example 8? The equations have coincident lines, and so the system had infinitely many solutions.

We'll organize these results in (Table 1) below:



Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Table 1

Parallel lines have the same slope but different *y*-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in Example 7.

$$\begin{cases} y &= \frac{1}{2}x - 3 \\ x - 2y &= 4 \end{cases}$$

The first line is in slope-intercept form.

$$y = \frac{1}{2}x - 3$$

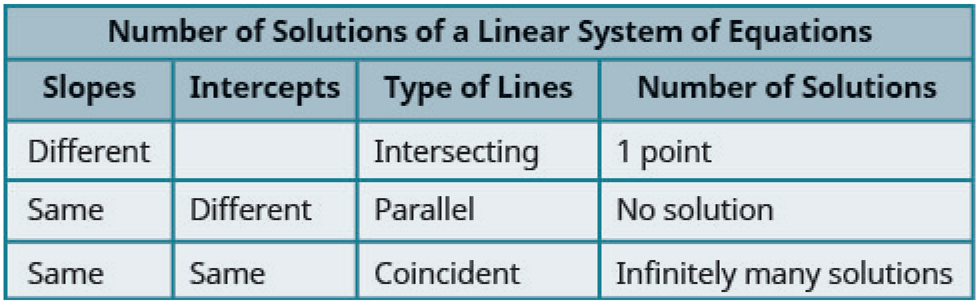
$$m = \frac{1}{2}, b = -3$$

If we solve the second equation for y, we get

$$\begin{aligned} x - 2y &= 4 \\ -2y &= x + 4 \\ y &= \frac{1}{2}x - 2 \\ m = \frac{1}{2}&, b = -2 \end{aligned}$$

The two lines have the same slope but different *y*-intercepts. They are parallel lines.

(Table 2) shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.



Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Table 2

Let's take one more look at our equations in (Example 7) that gave us parallel lines.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

When both lines were in slope-intercept form we had:

$$y = \frac{1}{2}x - 3 \qquad y = \frac{1}{2}x - 2$$

Do you recognize that it is impossible to have a single ordered pair (x, y) that is a solution to both of those equations?

We call a system of equations like this an inconsistent system. It has no solution.

A system of equations that has at least one solution is called a consistent system.

Consistent and Inconsistent Systems

A consistent system of equations is a system of equations with at least one solution.

An inconsistent system of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent equations, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

Independent and Dependent Equations

Two equations are independent if they have different solutions.

Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

Let’s sum this up by looking at the graphs of the three types of systems. See (Figure 3) and (Table 3).

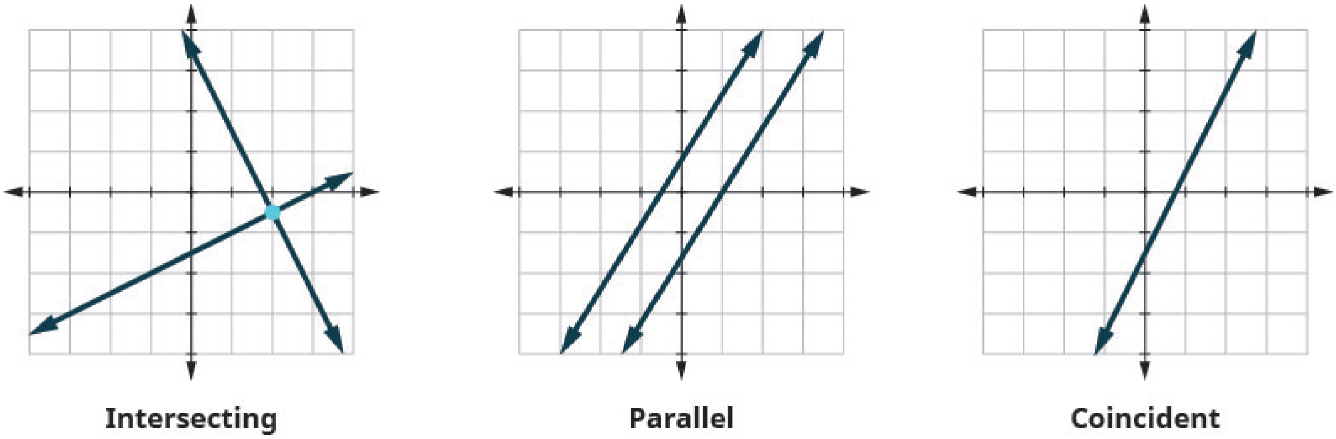


Figure 3

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

Table 3

EXAMPLE 9

Without graphing, determine the number of solutions and then classify the system of equations:

$$\left\{\begin{array}{l} y=3x-1 \\ 6x-2y=12 \end{array}\right. .$$

Solution

We will compare the slopes and intercepts of the two lines.	$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}.$
The first equation is already in slope-intercept form.	$y = 3x - 1$
Write the second equation in slope-intercept form.	$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x + 12 \\ \frac{-2y}{-2} &= \frac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$
Find the slope and intercept of each line.	$\begin{aligned} y &= 3x - 1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x - 6 \\ m &= 3 \\ b &= -6 \end{aligned}$
	Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

TRY IT 9

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$$

Show answer
no solution, inconsistent, independent

EXAMPLE 10

Without graphing, determine the number of solutions and then classify the system of equations:

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}.$$

Solution

We will compare the slope and intercepts of the two lines.	$\begin{cases} 2x+y=-3 \\ x-5y=5 \end{cases}$	
Write both equations in slope-intercept form.	$\begin{aligned} 2x+y &= -3 \\ y &= -2x-3 \end{aligned}$	$\begin{aligned} x-5y=5 &= 5 \\ -5y &= -x+5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x-1 \end{aligned}$
Find the slope and intercept of each line.	$\begin{aligned} y &= -2x-3 \\ m &= -2 \\ b &= -3 \end{aligned}$	$\begin{aligned} y &= \frac{1}{5}x-1 \\ m &= \frac{1}{5} \\ b &= -1 \end{aligned}$
	Since the slopes are different, the lines intersect.	

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

TRY IT 10

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x+2y=2 \\ 2x+y=1 \end{cases}$$

Show answer
one solution, consistent, independent

EXAMPLE 11

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x-2y=4 \\ y=\frac{3}{2}x-2 \end{cases}$$

Solution

We will compare the slopes and intercepts of the two lines.	$\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$
Write the first equation in slope-intercept form.	$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x - 2 \end{aligned}$
The second equation is already in slope-intercept form.	$y = \frac{3}{2}x - 2$
	Since the slopes are the same, they have the same slope and same y -intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

TRY IT 11

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x - 5y = 20 \\ y = \frac{4}{5}x - 4 \end{cases}$$

Show answer

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

We will modify the problem solving strategy slightly to set up and solve applications of systems of linear equations.

How to use a problem solving strategy for systems of linear equations.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.

5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

EXAMPLE 12

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution
Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3. Name what we are looking for. Choose variables to represent those quantities.

Let f = number of quarts of fruit juice.
c = number of quarts of club soda

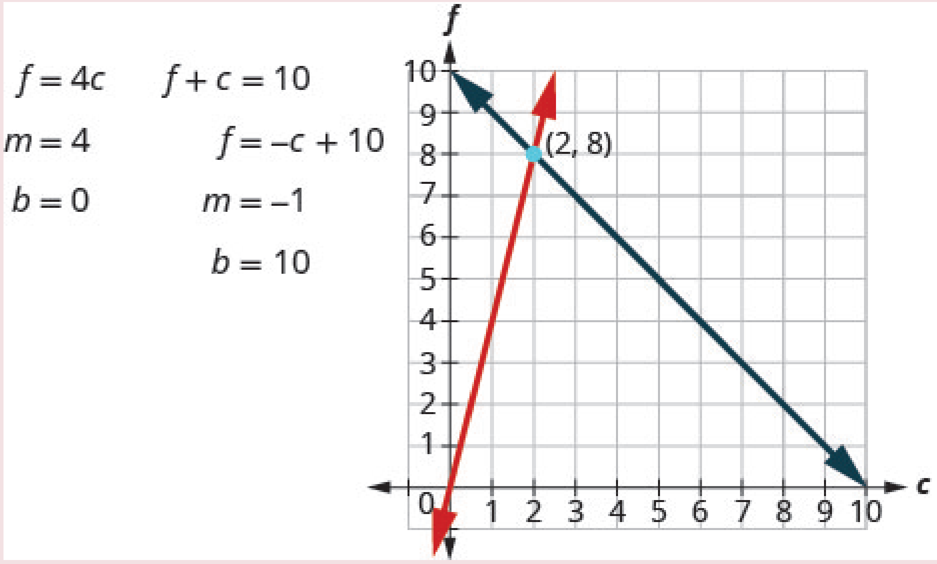
Step 4. Translate into a system of equations.

$$\text{The }\underbrace{\text{number of quarts of fruit juice}}_{f}\ \underbrace{\text{and}}_{+}\text{ the }\underbrace{\text{number of quarts of club soda}}_{c}\ \underbrace{\text{is 10}}_{=10}$$

$$\text{The }\underbrace{\text{number of quarts of fruit juice}}_{f}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four times the number of quarts of club soda}}_{4c}$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

Step 5. Solve the system of equations using good algebra techniques.



The point of intersection (2, 8) is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

Step 6. Check the answer in the problem and make sure it makes sense.

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

Step 7. Answer the question with a complete sentence.

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

TRY IT 12

Manu is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manu need?

Show answer
Manu needs 3 quarts juice concentrate and 9 quarts water.

Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

- **Problem Solving Strategy for Systems of Linear Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

coincident lines
Coincident lines are lines that have the same slope and same *y*-intercept.

consistent system
A consistent system of equations is a system of equations with at least one solution.

dependent equations
Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

inconsistent system
An inconsistent system of equations is a system of equations with no solution.

independent equations
Two equations are independent if they have different solutions.

solutions of a system of equations
Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (*x*, *y*).

system of linear equations
When two or more linear equations are grouped together, they form a system of linear equations.

4.1 Exercise Set

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$
   a. $(3,1)$
   b. $(-3,4)$

2. $\begin{cases} 2x+y=5 \\ x+y=1 \end{cases}$
   a. $(4,3)$
   b. $(2,0)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$
    a. $\left(\frac{8}{7},\frac{6}{7}\right)$
    b. $\left(1,\frac{3}{4}\right)$

4. $\begin{cases} x+5y=10 \\ y=\frac{3}{5}x+1 \end{cases}$
    a. $(-10,4)$
    b. (\left(\frac{5}{4},\frac{7}{4}\right)\)

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

6. $\begin{cases} -3x+y=-1 \\ 2x+y=4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

8. $\begin{cases} y=\frac{3}{2}x+1 \\ y=-\frac{1}{2}x+5 \end{cases}$

9. $\begin{cases} -x+y=-3 \\ 4x+4y=4 \end{cases}$

10. $\begin{cases} -3x+y=-1 \\ 2x+y=4 \end{cases}$

11. $\begin{cases} x+y=5 \\ 2x-y=4 \end{cases}$

12. $\begin{cases} x+y=2 \\ x-y=0 \end{cases}$

13. $\begin{cases} x+y=-5 \\ x-y=3 \end{cases}$

14. $\begin{cases} x+y=-4 \\ -x+2y=-2 \end{cases}$

15. $\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

16. $\begin{cases} 2x+3y=6 \\ y=-2 \end{cases}$

17. $\begin{cases} x-3y=-3 \\ y=2 \end{cases}$

18. $\begin{cases} 2x-y=-1 \\ x=1 \end{cases}$

19. $\begin{cases} x-3y=-6 \\ x=-3 \end{cases}$

20. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

21. $\begin{cases} -2x+4y=4 \\ y=\frac{1}{2}x \end{cases}$

22. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

24. $\begin{cases} x+3y=-6 \\ 4y=-\frac{4}{3}x-8 \end{cases}$

25. $\begin{cases} -3x+2y=-2 \\ y=x+4 \end{cases}$

Without graphing the following systems of equations, determine the number of solutions and then classify the system of equations.

26. $\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

27. $\begin{cases} y=-2x+1 \\ 4x+2y=8 \end{cases}$

28. missing

29. $\begin{cases} 4x+2y=10 \\ 4x-2y=-6 \end{cases}$

30. $\begin{cases} y=-\frac{1}{2}x+5 \\ x+2y=10 \end{cases}$

31. $\begin{cases} y=2x+3 \\ 2x-y=-3 \end{cases}$

In the following exercises, solve.

32. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?
33. Enrique is making a party mix that contains raisins and nuts. For each ounce of nuts, he uses twice the amount of raisins. How many ounces of nuts and how many ounces of raisins does he need to make 24 ounces of party mix?
34. Leo is planning his spring flower garden. He wants to plant tulip and daffodil bulbs. He will plant 6 times as many daffodil bulbs as tulip bulbs. If he wants to plant 350 bulbs, how many tulip bulbs and how many daffodil bulbs should he plant?

Answers

1. a) yes b) no
2. a) yes b) no
3. a) yes b) no
4. a) no b) yes
5. $(-2,3)$
6. $(1,2)$
7. $(0,2)$
8. $(2,4)$
9. $(2,-1)$
10. $(1,2)$
11. $(3,2)$
12. $(1,1)$
13. $(-1,-4)$
14. $(3,3)$
15. $(-5,6)$
16. $(6,-2)$
17. $(3,2)$
18. $(1,3)$
19. $(-3,1)$
20. no solution
21. no solution
22. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$
23. infinitely many solutions
24. infinitely many solutions
25. $(2,2)$
26. no solutions
27. no solutions
28. no solutions, inconsistent, independent
29. consistent, 1 solution
30. infinitely many solutions
31. infinitely many solutions
32. Molly needs 16 ounces of strawberry juice and 48 ounces of water.
33. Enrique needs 8 ounces of nuts and 16 ounces of water.
34. Leo should plant 50 tulips and 300 daffodils.

4.2 Solve Systems of Equations by Substitution

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with *x* and *y* both between −10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either *x* or *y*. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

EXAMPLE 1

How to Solve a System of Equations by Substitution

Solve the system by substitution. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for *y*.	$2x + y = 7$ $y = \boxed{7 - 2x}$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace *y* in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $\boxed{x = 4}$
Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace *x* with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into both equations and make sure they are both true.	$2x + y = 7$ $\quad$ $x - 2y = 6$ $2(4) + (-1) \overset{?}{=} 7$ $\quad$ $4 - 2(-1) \overset{?}{=} 6$ $7 = 7$ ✓ $\quad$ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

TRY IT 1

Solve the system by substitution. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

Show answer
$(6, 1)$

How to solve a system of equations by substitution.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done!

EXAMPLE 2

Solve the system by substitution.

$$\begin{cases} x + y = -1 \\ y = x + 5 \end{cases}$$

Solution

The second equation is already solved for *y*. We will substitute the expression in place of *y* in the first equation.

	$\begin{cases} x + y = -1 \\ y = x + 5 \end{cases}$
The second equation is already solved for *y*. We will substitute into the first equation.	
Replace the *y* with *x* + 5.	$y = x + 5$ $x + y = -1$
Solve the resulting equation for *x*.	$x + x + 5 = -1$
	$2x + 5 = -1$
	$2x = -6$
Substitute *x* = −3 into *y* = *x* + 5 to find *y*.	$x = -3$ $y = x + 5$
	$y = -3 + 5$
The ordered pair is (−3, 2).	$y = 2$
Check the ordered pair in both equations: $x + y = -1$ $\quad y = x + 5$ $-3 + 2 \overset{?}{=} -1$ $\quad 2 \overset{?}{=} -3 + 5$ $-1 = -1$ $\quad 2 = 2$	
	The solution is (−3, 2).

TRY IT 2

Solve the system by substitution. $\begin{cases} x + y = 6 \\ y = 3x - 2 \end{cases}$

Show answer

$(2, 4)$

If the equations are given in standard form, we’ll need to start by solving for one of the variables. In this next example, we’ll solve the first equation for *y*.

EXAMPLE 3

Solve the system by substitution. $\begin{cases} 3x + y = 5 \\ 2x + 4y = -10 \end{cases}$

Solution

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for *y*. Substitute into the other equation.	$3x + y = 5$ $y = -3x + 5$ $2x + 4y = -10$
Replace the *y* with −3*x* + 5.	$2x + 4(-3x + 5) = -10$
Solve the resulting equation for *x*.	$2x - 12x + 20 = -10$
	$-10x + 20 = -10$ $-10x = -30$
Substitute *x* = 3 into 3*x* + *y* = 5 to find *y*.	$x = -3$ $3x + y = 5$
	$3(3) + y = 5$ $9 + y = 5$
The ordered pair is (3, −4).	$y = -4$
Check the ordered pair in both equations: $\begin{aligned} 3x + y &= 5 \\ 3 \cdot 3 + (-4) &\overset{?}{=} 5 \\ 9 - 4 &\overset{?}{=} 5 \\ 5 &= 5 \end{aligned}$ $\begin{aligned} 2x + 4y &= -10 \\ 2 \cdot 3 + 4(-4) &= -10 \\ 6 - 16 &\overset{?}{=} -10 \\ -10 &= -10 \end{aligned}$	
	The solution is (3, −4).

TRY IT 3

Solve the system by substitution. $\begin{cases} 4x + y = 2 \\ 3x + 2y = -1 \end{cases}$

Show answer
$(1, -2)$

In example 3, it was easiest to solve for *y* in the first equation because it had a coefficient of 1. In the next example it will be easier to solve for *x*.

EXAMPLE 4

Solve the system by substitution. $\begin{cases} x - 2y = -2 \\ 3x + 2y = 34 \end{cases}$

Solution

We will solve the first equation for x and then substitute the expression into the second equation.

	$x - 2y = -2$
Solve for *x*. Substitute into the other equation.	$x = \boxed{2y - 2}$ $3x + 2y = 34$
Replace the *x* with 2*y* − 2.	$3(2y - 2) + 2y = 34$
Solve the resulting equation for *y*.	$6y - 6 + 2y = 34$
Substitute *y* = 5 into *x* − 2*y* = −2 to find *x*.	$8y - 6 = 34$ $8y = 40$ $\boxed{y = 5}$ $x - 2y = -2$ $x - 2 \cdot 5 = -2$ $x - 10 = -2$ $x = 8$
The ordered pair is (8, 5).	
Check the ordered pair in both equations: $\begin{array}{rcl} x - 2y & = & -2 \\ 8 - 2 \cdot 5 & \overset{?}{=} & -2 \\ 8 - 10 & \overset{?}{=} & -2 \\ -2 & = & -2 \end{array} \qquad \begin{array}{rcl} 3x + 2y & = & 34 \\ 3 \cdot 8 + 2 \cdot 5 & \overset{?}{=} & 34 \\ 24 + 10 & \overset{?}{=} & 34 \\ 34 & = & 34 \end{array}$	
	The solution is (8, 5).

TRY IT 4

Solve the system by substitution. $\begin{cases} x - 5y = 13 \\ 4x - 3y = 1 \end{cases}$

Show answer

$(-2, -3)$

When both equations are already solved for the same variable, it is easy to substitute!

EXAMPLE 5

Solve the system by substitution. $\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$

Solution

Since both equations are solved for y, we can substitute one into the other.

Substitute $\frac{1}{2}x$ for *y* in the first equation.	$y = \frac{1}{2}x$ $y = -2x + 5$
Replace the *y* with $\frac{1}{2}x$.	$\frac{1}{2}x = -2x + 5$
Solve the resulting equation. Start by clearing the fraction.	$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$
Solve for *x*.	$x = -4x + 10$
	$5x = 10$
Substitute *x* = 2 into *y* = $\frac{1}{2}x$ to find *y*.	$x = 2$ $y = \frac{1}{2}x$ $y = \frac{1}{2} \cdot 2$ $y = 1$
The ordered pair is (2,1).	
Check the ordered pair in both equations: $\begin{aligned} y &= \tfrac{1}{2}x \\ 1 &\overset{?}{=} \tfrac{1}{2} \cdot 2 \\ 1 &= 1 \end{aligned}$ $\begin{aligned} y &= -2x + 5 \\ 1 &\overset{?}{=} -2 \cdot 2 + 5 \\ 1 &= -4 + 5 \\ 1 &= 1 \end{aligned}$	
	The solution is (2,1).

TRY IT 5

Solve the system by substitution. $\left\{\begin{array}{l} y = 3x - 16 \\ y = \frac{1}{3}x \end{array}\right.$

Show answer

$(6, 2)$

Be very careful with the signs in the next example.

EXAMPLE 6

Solve the system by substitution. $\begin{cases} 4x + 2y = 4 \\ 6x - y = 8 \end{cases}$

Solution

We need to solve one equation for one variable. We will solve the first equation for *y*.

	$4x+2y=4$
Solve the first equation for *y*.	$2y=-4x+4$
Substitute −2*x* + 2 for *y* in the second equation.	$y=-2x+2$ $6x-y=8$
Replace the *y* with −2*x* + 2.	$6x-(-2x+2)=8$
Solve the equation for *x*.	$6x+2x-2=8$
	$8x-2=8$ $8x=10$
Substitute $x=\frac{5}{4}$ into 4*x* + 2*y* = 4 to find *y*.	$x=\frac{5}{4}$ $4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
The ordered pair is $\left(\frac{5}{4},-\frac{1}{2}\right)$.	
Check the ordered pair in both equations. $4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\overset{?}{=}4$ $5-1\overset{?}{=}4$ $4=4$ $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\overset{?}{=}8$ $\frac{15}{4}-\left(-\frac{1}{2}\right)\overset{?}{=}8$ $\frac{16}{2}\overset{?}{=}8$ $8=8$	
	The solution is $\left(\frac{5}{4},-\frac{1}{2}\right)$.

TRY IT 6

Solve the system by substitution. $\begin{cases} x - 4y = -4 \\ -3x + 4y = 0 \end{cases}$

Show answer

$\left(2, \frac{3}{2}\right)$

In the next example, it will take a little more work to solve one equation for *x* or *y*.

EXAMPLE 7

Solve the system by substitution. $\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$

Solution

We need to solve one equation for one variable. We will solve the first equation for *x*.

	$4x - 3y = 6$
Solve the first equation for *x*.	$4x = 3y + 6$
Substitute $\frac{3}{4}y + \frac{3}{2}$ for *x* in the second equation.	$x = \frac{3}{4}y + \frac{3}{2}$ $15y - 20x = -30$
Replace the *x* with $\frac{3}{4}y + \frac{3}{2}$.	$15y - 20\left(\frac{3}{4}y + \frac{3}{2}\right) = -30$
Solve for *y*.	$15y - 15y - 30 = -30$
	$0 - 30 = -30$
	$0 = 0$

Since 0 = 0 is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

TRY IT 7

Solve the system by substitution. $\begin{cases} 2x - 3y = 12 \\ -12y + 8x = 48 \end{cases}$

Show answer
infinitely many solutions

Look back at the equations in example 6. Is there any way to recognize that they are the same line?

Let’s see what happens in the next example.

EXAMPLE 8

Solve the system by substitution. $\begin{cases} 5x - 2y = -10 \\ y = \frac{5}{2}x \end{cases}$

Solution

The second equation is already solved for *y*, so we can substitute for *y* in the first equation.

Substitute *x* for *y* in the first equation.	$y = \frac{5}{2}x$ $5x - 2y = -10$
Replace the *y* with $\frac{5}{2}x$.	$5x - 2\left(\frac{5}{2}x\right) = -10$
Solve for *x*.	$5x - 5x = -10$
	$0 \neq -10$

Since 0 = −10 is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

TRY IT 8

Solve the system by substitution. $\begin{cases} 3x + 2y = 9 \\ y = -\frac{3}{2}x + 1 \end{cases}$

Show answer
no solution

Solve Applications of Systems of Equations by Substitution

We'll copy here the problem solving strategy we used in the last sub chapter for solving systems of equations. Now that we know how to solve systems by substitution, that's what we'll do in Step 5.

How to use a problem solving strategy for systems of linear equations.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

EXAMPLE 9

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number Let $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$
	One number is nine less than the other.
	$n = m - 9$
The system is:	$\begin{cases} n + m = 0 \\ n = m - 9 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for *n*.	
Substitute $m - 9$ for n in the first equation.	$n = \boxed{m - 9}$ $n + m = 0$
Solve for *m*.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for *n*.	$m = \boxed{\frac{9}{2}}$ $n = m - 9$
	$m = \frac{9}{2} - 9$
	$m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$

Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

TRY IT 9

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Show answer

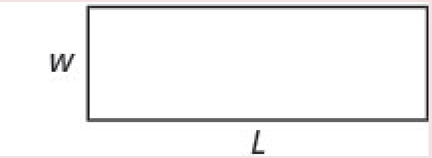
The numbers are 3 and 7.

In the next example, we'll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

EXAMPLE 10

The perimeter of a rectangle is 88. The length is five more than twice the width. Find the length and the width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L = the length W = the width
Step 4. Translate into a system of equations.	The perimeter of a rectangle is 88.
	$2L + 2W = P$ $2L + 2W = 88$
	The length is five more than twice the width.
	$L = 2W + 5$
The system is:	$\begin{cases} 2L + 2W = 88 \\ L = 2W + 5 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for L. Substitute $2W + 5$ for L in the first equation.	$L = 2W + 5$ $2L + 2W = 88$
Solve for W.	$2(2W + 5) + 2W = 88$
	$4W + 10 + 2W = 88$
	$6W + 10 = 88$
	$6W = 78$
Substitute $W = 13$ into the second equation and then solve for L.	$W = 13$ $L = 2W + 5$
	$L = 2 \cdot 13 + 5$
	$L = 31$
Step 6. Check the answer in the problem.	Does a rectangle with length 31 and width 13 have perimeter 88? Yes.

Step 7. Answer the equation.	The length is 31 and the width is 13.

TRY IT 10

The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Show answer
The length is 12 and the width is 8.

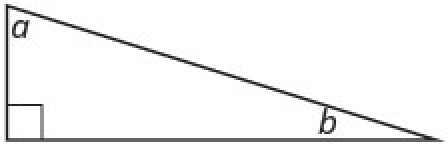
For the next example we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

EXAMPLE 11

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a = the measure of the 1st angle b = the measure of the 2nd angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is:	$\begin{cases} a = 3b + 10 \\ a + b + 90 = 180 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for *a*.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute 3*b* + 10 for *a* in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for *b*.	$4b + 100 = 180$
	$4b = 80$
	$b = 20$ $a = 3b + 10$
Substitute *b* = 20 into the first equation and then solve for *a*.	$a = 3 \cdot 20 + 10$ $a = 70$

Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70.

TRY IT 11

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Show answer

The measure of the angles are 22 degrees and 68 degrees.

EXAMPLE 12

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 plus $15 for each training session. Option B would pay her $10,000 + $40 for each training session. How many training sessions would make the salary options equal?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let $s =$ Heather's salary. $n =$ the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her \$25,000 plus \$15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her \$10,000 + \$40 for each training session
	$s = 10{,}000 + 40n$
The system is:	$\begin{cases} s = 25{,}000 + 15n \\ s = 10{,}000 + 40n \end{cases}$
Step 5. Solve the system of equations. We will use substitution.	$s = \boxed{25{,}000 + 15n}$ $s = 10{,}000 + 40n$
Substitute 25,000 + 15*n* for *s* in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for *n*.	$25{,}000 = 10{,}000 + 25n$
	$15{,}000 = 25n$
	$600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when $n = 600$?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

TRY IT 12

Geraldine has been offered positions by two insurance companies. The first company pays a salary of \$12,000 plus a commission of \$100 for each policy sold. The second pays a salary of \$20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Show answer
There would need to be 160 policies sold to make the total pay the same.

Access these online resources for additional instruction and practice with solving systems of equations by substitution.

- Instructional Video-Solve Linear Systems by Substitution
- Instructional Video-Solve by Substitution

Key Concepts

- **Solve a system of equations by substitution**
    1. Solve one of the equations for either variable.
    2. Substitute the expression from Step 1 into the other equation.
    3. Solve the resulting equation.
    4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
    5. Write the solution as an ordered pair.
    6. Check that the ordered pair is a solution to both original equations.

4.2 Exercise Set

In the following exercises, solve the systems of equations by substitution.

1. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

2. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

3. $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

4. $\begin{cases} 2x+3y=3 \\ y=x+3 \end{cases}$

5. $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

6. $\begin{cases} 3x-2y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

7. $\begin{cases} 2x+y=10 \\ -x+y=-5 \end{cases}$

8. $\begin{cases} 3x+y=1 \\ -4x+y=15 \end{cases}$

9. $\begin{cases} x+3y=1 \\ 3x+5y=-5 \end{cases}$

10. $\begin{cases} 2x+y=5 \\ x-2y=-15 \end{cases}$

11. $\begin{cases} y = -2x - 1 \\ y = -\frac{1}{3}x + 4 \end{cases}$

12. $\begin{cases} y = 2x - 8 \\ y = \frac{3}{5}x + 6 \end{cases}$

13. $\begin{cases} 4x + 2y = 8 \\ 8x - y = 1 \end{cases}$

14. $\begin{cases} 15x + 2y = 6 \\ -5x + 2y = -4 \end{cases}$

15. $\begin{cases} y = 3x \\ 6x - 2y = 0 \end{cases}$

16. $\begin{cases} 2x + 16y = 8 \\ -x - 8y = -4 \end{cases}$

17. $\begin{cases} y = -4x \\ 4x + y = 1 \end{cases}$

18. $\begin{cases} y = \frac{7}{8}x + 4 \\ -7x + 8y = 6 \end{cases}$

In the following exercises, translate to a system of equations and solve.

19. The sum of two numbers is 15. One number is 3 less than the other. Find the numbers
20. The sum of two numbers is −26. One number is 12 less than the other. Find the numbers.
21. The perimeter of a rectangle is 60. The length is 10 more than the width. Find the length and width.
22. The perimeter of a rectangle is 84. The length is 10 more than three times the width. Find the length and width.
23. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.
24. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.
25. Jackie has been offered positions by two cable companies. The first company pays a salary of $ 14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?
26. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?
27. Stephanie left Riverside, California, driving her motorhome north on Interstate 15 towards Salt Lake City at a speed of 56 miles per hour. Half an hour later, Tina left Riverside in her car on the same route as Stephanie, driving 70 miles per hour. Solve the system $\begin{cases} 56s = 70t \\ s = t + \frac{1}{2} \end{cases}$.
    a. for t to find out how long it will take Tina to catch up to Stephanie.
    b. what is the value of s, the number of hours Stephanie will have driven before Tina catches up to her?

Answers:

1. $(-2, 0)$

2. $(7, 6)$

3. $(0, 3)$
4. $(6, -3)$
5. $(3, -1)$
6. $(6, 6)$
7. $(5, 0)$
8. $(-2, 7)$
9. $(-5, 2)$
10. $(-1, 7)$
11. $(-3, 5)$
12. (10, 12)
13. $\left(\frac{1}{2}, 3\right)$
14. $\left(\frac{1}{2}, -\frac{3}{4}\right)$
15. Infinitely many solutions
16. Infinitely many solutions
17. No solution
18. No solution
19. The numbers are 6 and 9.
20. The numbers are −7 and −19.
21. The length is 20 and the width is 10.
22. The length is 34 and the width is 8.
23. The measures are 16° and 74°.
24. The measures are 45° and 45°.
25. 80 cable packages would need to be sold.
26. Mitchell would need to sell 120 stoves.
27.
    a. $t = 2$ hours
    b. $s = 2\frac{1}{2}$ hours

4.3 Solve Systems of Equations by Elimination

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*,

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{aligned} 3x + y &= 5 \\ 2x - y &= 0 \\ 5x \quad &= 5 \end{aligned}$$

The *y*’s add to zero and we have one equation with one variable.

Let’s try another one:

$$\begin{cases} x + 4y = 2 \\ 2x + 5y = -2 \end{cases}$$

This time we don’t see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by −2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by −2.

$$\begin{cases} -2(x + 4y) = -2(2) \\ 2x + 5y = -2 \end{cases}$$

$$\begin{cases} -2x - 8y = -4 \\ 2x + 5y = -2 \end{cases}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

Add the equations yourself—the result should be −3*y* = −6. And that looks easy to solve, doesn’t it? Here is what it would look like.

$$\left\{\begin{array}{r} -2x - 8y = -4 \\ 2x + 5y = -2 \\ \hline -3y = -6 \end{array}\right.$$

We’ll do one more:

$$\begin{cases} 4x - 3y = 10 \\ 3x + 5y = -7 \end{cases}$$

It doesn’t appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we’ll have to multiply both equations by a constant.

We can make the coefficients of *x* be opposites if we multiply the first equation by 3 and the second by −4, so we get 12*x* and −12*x*.

$$3(4x - 3y) = 3(10)$$

$$-4(3x + 5y) = -4(-7)$$

This gives us these two new equations:

$$\begin{cases} 12x - 9y = 30 \\ -12x - 20y = 28 \end{cases}$$

When we add these equations,

$$\begin{cases} 12x - 9y = 30 \\ -12x - 20y = 28 \end{cases}$$
$$-29y = 58$$

the x's are eliminated and we just have $-29y = 58$.

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

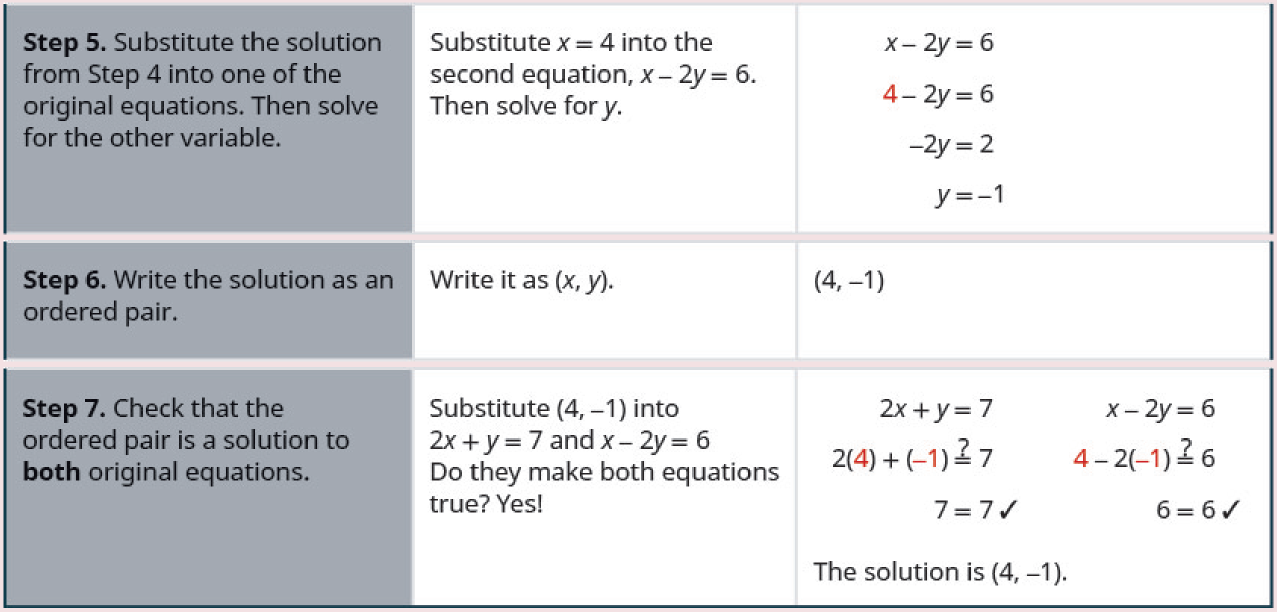
EXAMPLE 1

How to Solve a System of Equations by Elimination

Solve the system by elimination. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ x - 2y = 6 \end{cases}$ $5x \quad = 20$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$



Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$ $\quad$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $\quad$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $\quad$ $6 = 6$ ✓ The solution is $(4, -1)$.

TRY IT 1

Solve the system by elimination. $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Show answer
$(2, -1)$

The steps are listed below for easy reference.

How to solve a system of equations by elimination.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
    - Decide which variable you will eliminate.
    - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

First we'll do an example where we can eliminate one variable right away.

EXAMPLE 2

Solve the system by elimination. $\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$

Solution

	$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$
Both equations are in standard form.	
The coefficients of *y* are already opposites.	
Add the two equations to eliminate *y*. The resulting equation has only 1 variable, *x*.	$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$ $\overline{2x \qquad = 22}$
Solve for *x*, the remaining variable. Substitute *x* = 11 into one of the original equations.	$x = \textcircled{11}$ $x + y = 10$
	$11 + y = 10$
Solve for the other variable, *y*.	$y = -1$
Write the solution as an ordered pair.	The ordered pair is (11, −1).
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} x + y &= 10 \\ 11 + (-1) &\overset{?}{=} 10 \\ 10 &= 10 \end{aligned}$ $\qquad$ $\begin{aligned} x - y &= 12 \\ 11 - (-1) &\overset{?}{=} 12 \\ 12 &= 12 \end{aligned}$	
	The solution is (11, −1).

TRY IT 2

Solve the system by elimination. $\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$

Show answer

$(3, -1)$

In the next example, we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

EXAMPLE 3

Solve the system by elimination. $\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$

Solution

	$\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$
Both equations are in standard form.	
None of the coefficients are opposites.	
We can make the coefficients of *y* opposites by multiplying the first equation by −3.	$\begin{cases} -3(3x - 2y) = -3(-2) \\ 5x - 6y = 10 \end{cases}$
Simplify.	$\begin{cases} -9x + 6y = 6 \\ 5x - 6y = 10 \end{cases}$
Add the two equations to eliminate *y*.	$\begin{array}{l} \begin{cases} -9x + 6y = 6 \\ 5x - 6y = 10 \end{cases} \\ \hline -4x \quad = 16 \end{array}$
Solve for the remaining variable, *x*. Substitute *x* = −4 into one of the original equations.	$x = -4$ $3x - 2y = -2$
	$3(-4) - 2y = -2$
Solve for *y*.	$-12 - 2y = -2$ $-2y = 10$ $y = -5$
Write the solution as an ordered pair.	The ordered pair is (−4, −5).
Check that the ordered pair is a solution to both original equations. $\begin{aligned} 3x - 2y &= -2 \\ 3(-4) - 2(-5) &\stackrel{?}{=} -2 \\ -12 + 10 &\stackrel{?}{=} -2 \\ -2y &= -2 \end{aligned}$ $\quad$ $\begin{aligned} 5x - 6y &= 10 \\ 3(-4) - 6(-5) &\stackrel{?}{=} 10 \\ -20 + 30 &\stackrel{?}{=} 10 \\ 10 &= 10 \end{aligned}$	
	The solution is (−4, −5).

TRY IT 3

Solve the system by elimination. $\begin{cases} 4x - 3y = 1 \\ 5x - 9y = -4 \end{cases}$

Show answer
$(1, 1)$

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

EXAMPLE 4

Solve the system by elimination. $\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$

Solution
In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a constant to get the opposites.

	$\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$
Both equations are in standard form. To get opposite coefficients of *y*, we will multiply the first equation by 2 and the second equation by 3.	$\begin{cases} 2(4x - 3y) = 2(9) \\ 3(7x + 2y) = 3(-6) \end{cases}$
Simplify.	$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$
Add the two equations to eliminate *y*.	$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$ $39x = 0$
Solve for *x*. Substitute *x* = 0 into one of the original equations.	$x = 0$ $7x + 2y = -6$
	$7 \cdot 0 + 2y = -6$
Solve for *y*.	$2y = -6$
	$y = -3$
Write the solution as an ordered pair.	The ordered pair is (0, −3).
Check that the ordered pair is a solution to **both** original equations. $4x - 3y = 9$ $\quad$ $7x + 2y = -6$ $4(0) - 3(-3) \overset{?}{=} 9$ $\quad$ $7(0) + 2(-3) \overset{?}{=} -6$ $9 = 9$ $\quad$ $-6 = -6$	
	The solution is (0, −3).

What other constants could we have chosen to eliminate one of the variables? Would the solution be the same?

TRY IT 4

Solve the system by elimination. $\begin{cases} 3x - 4y = -9 \\ 5x + 3y = 14 \end{cases}$

Show answer

$(1, 3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

EXAMPLE 5

Solve the system by elimination. $\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

	$\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$
To clear the fractions, multiply each equation by its LCD.	$\begin{cases} 2\left(x + \frac{1}{2}y\right) = 2(6) \\ 6\left(\frac{3}{2}x + \frac{2}{3}y\right) = 6\left(\frac{17}{2}\right) \end{cases}$
Simplify.	$\begin{cases} 2x + y = 12 \\ 9x + 4y = 51 \end{cases}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate *y* multiplying the top equation by −4.	$\begin{cases} -4(2x + y) = -4(12) \\ 9x + 4y = 51 \end{cases}$
Simplify and add. Substitute *x* = 3 into one of the original equations.	$\begin{cases} -8x - 4y = -48 \\ 9x + 4y = 51 \end{cases}$ $x = \textcircled{3}$ $x + \frac{1}{2}y = 6$
Solve for *y*.	$3 + \frac{1}{2}y = 6$
	$\frac{1}{2}y = 3$
	$y = 6$
Write the solution as an ordered pair.	The ordered pair is (3, 6).

Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} x+\frac{1}{2}y &= 6 \\ 3+\frac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3+6 &\stackrel{?}{=} 6 \\ 6 &= 6 \end{aligned}$ $\begin{aligned} \frac{3}{2}x+\frac{2}{3}y &= \frac{17}{2} \\ \frac{3}{2}(3)+\frac{2}{3}(6) &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2}+4 &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2}+\frac{8}{2} &\stackrel{?}{=} \frac{17}{2} \\ \frac{17}{2} &= \frac{17}{2} \end{aligned}$	
	The solution is (3, 6).

TRY IT 5

Solve the system by elimination. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Show answer

$(6, 2)$

When we were solving systems of linear equations by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

EXAMPLE 6

Solve the system by elimination:

a) $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$

b) $\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$

c) $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

d) $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Solution

a)	$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$
Write the second equation in standard form.	$\begin{cases} 3x+4y & = & 12 \\ \frac{3}{4}x+y & = & 3 \end{cases}$
Clear the fractions by multiplying the second equation by 4.	$\begin{cases} 3x+4y & = & 12 \\ 4\left(\frac{3}{4}x+y\right) & = & 4\left(3\right) \end{cases}$
Simplify.	$\begin{cases} 3x+4y & = & 12 \\ 3x+4y & = & 12 \end{cases}$
To eliminate a variable, we multiply the second equation by -1. Simplify and add.	$\begin{cases} 3x+4y & = & 12 \\ -3x-4y & = & -12 \end{cases}$ $0=0$
This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.	
After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.	

b)	$\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$
infinitely many solutions	

c)	$\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$
infinitely many solutions	

d)	$\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$
The equations are in standard form.	$\begin{cases} -6x+15y & = & 10 \\ 2x-5y & = & -5 \end{cases}$
Multiply the second equation by 3 to eliminate a variable.	$\begin{cases} -6x+15y & = & 10 \\ 3\left(2x-5y\right) & = & 3\left(-5\right) \end{cases}$
Simplify and add.	$\begin{cases} -6x+15y & = & 10 \\ 6x-15y & = & -15 \end{cases}$ $0 \neq -5$
This statement is false. The equations are inconsistent and so their graphs would be parallel lines.	
The system does not have a solution.	

TRY IT 6

Solve the system by elimination. $\begin{cases} -3x+2y=8 \\ 9x-6y=13 \end{cases}$

Show answer
no solution

Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

EXAMPLE 7

The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $n =$ the first number. $m =$ the second number.
Step 4. Translate into a system of equations. The system is:	The sum of two numbers is 39. $n + m = 39$ Their difference is 9. $n - m = 9$ $\begin{cases} n + m = 39 \\ n - m = 9 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form and the coefficients of m are opposites. Add. Solve for n. Substitute $n = 24$ into one of the original equations and solve for m.	$\begin{cases} n + m = 39 \\ n - m = 9 \end{cases}$ $2n \quad = 48$ $n = 24$ $n + m = 39$ $24 + m = 39$ $m = 15$
Step 6. Check the answer.	Since $24 + 15 = 39$ and $24 - 15 = 9$, the answers check.
Step 7. Answer the question.	The numbers are 24 and 15.

TRY IT 7

The sum of two numbers is 42. Their difference is 8. Find the numbers.

Show answer

The numbers are 25 and 17.

EXAMPLE 8

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories in one order of medium fries and in one small soda.
Step 3. Name what we are looking for.	Let *f* = the number of calories in 1 order of medium fries. *s* = the number of calories in 1 small soda.
Step 4. Translate into a system of equations:	one medium fries and two small sodas had a total of 620 calories
	$f+2s=620$
	two medium fries and one small soda had a total of 820 calories.
	$2f+s=820$
Our system is:	$\begin{cases} f+2s=620 \\ 2f+s=820 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of *f*, multiply the top equation by −2.	$\begin{cases} -2(f+2s)=-2(620) \\ 2f+\quad s\ =820 \end{cases}$
Simplify and add.	$\begin{cases} -2f-4s=-1240 \\ 2f+\quad s=820 \end{cases}$ $\underline{\qquad\qquad\qquad}$ $-3s=-420$
Solve for *s*.	$s=140$
Substitute *s* = 140 into one of the original equations and then solve for *f*.	$f+2s=620$
	$f+2\cdot 140=620$
	$f+280=620$
	$f=340$

Step 6. Check the answer.	Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!
Step 7. Answer the question.	The small soda has 140 calories and the fries have 340 calories.

TRY IT 8

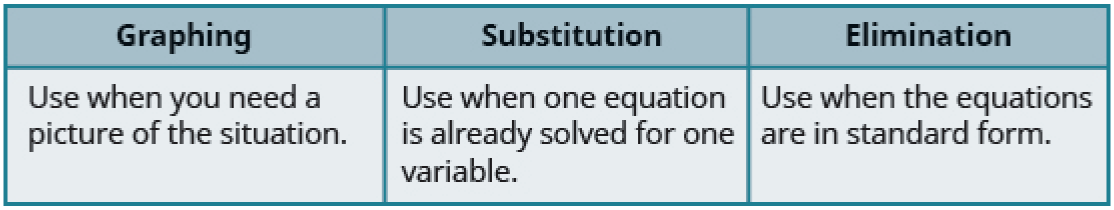
Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $37. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $87. How much does a bag of diapers cost? How much is one can of formula?

Show answer

The bag of diapers costs ?11 and the can of formula costs ?13.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.



Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

EXAMPLE 9

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a) $\begin{cases} 3x + 8y = 40 \\ 7x - 4y = -32 \end{cases}$

b) $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Solution

a) $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$

Since both equations are in standard form, using elimination will be most convenient.

b) $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Since one equation is already solved for y, using substitution will be most convenient.

TRY IT 9

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a) $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$

b) $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Show answer

a) Since both equations are in standard form, using elimination will be most convenient.

b) Since one equation is already solved for x, using substitution will be most convenient.

Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**
    1. Write both equations in standard form. If any coefficients are fractions, clear them.
    2. Make the coefficients of one variable opposites.
        - Decide which variable you will eliminate.

    - Multiply one or both equations so that the coefficients of that variable are opposites.

3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

4.3 Exercise Set

In the following exercises, solve the systems of equations by elimination.

1. $\begin{cases} -3x+y=-9 \\ x-2y=-12 \end{cases}$
2. $\begin{cases} 3x-y=-7 \\ 4x+2y=-6 \end{cases}$
3. $\begin{cases} x+y=-8 \\ x-y=-6 \end{cases}$
4. $\begin{cases} -7x+6y=-10 \\ x-6y=22 \end{cases}$
5. $\begin{cases} 5x+2y=1 \\ -5x-4y=-7 \end{cases}$
6. $\begin{cases} 3x-4y=-11 \\ x-2y=-5 \end{cases}$
7. $\begin{cases} 6x-5y=-75 \\ -x-2y=-13 \end{cases}$
8. $\begin{cases} 2x-5y=7 \\ 3x-y=17 \end{cases}$
9. $\begin{cases} 7x+y=-4 \\ 13x+3y=4 \end{cases}$
10. $\begin{cases} 3x-5y=-9 \\ 5x+2y=16 \end{cases}$
11. $\begin{cases} 4x+7y=14 \\ -2x+3y=32 \end{cases}$
12. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$
13. $\begin{cases} 3x+8y=67 \\ 5x+3y=60 \end{cases}$
14. $\begin{cases} \frac{1}{3}x-y=-3 \\ x+\frac{5}{2}y=2 \end{cases}$
15. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{2}x-\frac{1}{3}y=-2 \end{cases}$
16. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$
17. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$
18. $\begin{cases} 3x+2y=6 \\ -6x-4y=-12 \end{cases}$
19. $\begin{cases} -11x+12y=60 \\ -22x+24y=90 \end{cases}$
20. $\begin{cases} 5x-3y=15 \\ y=\frac{5}{3}x-2 \end{cases}$

In the following exercises, translate to a system of equations and solve.

21. The sum of two numbers is 65. Their difference is 25. Find the numbers.
22. The sum of two numbers is −27. Their difference is −59. Find the numbers.

23. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for \$114 or she is able to buy 2 shirts and 4 sweaters for \$164. How much does a shirt cost? How much does a sweater cost?
24. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

25\.
   a. $\begin{cases} 8x - 15y = -32 \\ 6x + 3y = -5 \end{cases}$
   b. $\begin{cases} x = 4y - 3 \\ 4x - 2y = -6 \end{cases}$

26\.
   a. $\begin{cases} y = 4x + 9 \\ 5x - 2y = -21 \end{cases}$
   b. $\begin{cases} 9x - 4y = 24 \\ 3x + 5y = -14 \end{cases}$

27. Norris can row 3 miles upstream against the current in the same amount of time it takes him to row 5 miles downstream, with the current. Solve the system. $\begin{cases} r - c = 3 \\ r + c = 5 \end{cases}$
   a. for r, his rowing speed in still water.
   b. Then solve for c, the speed of the river current.

Answers:

1. (6, 9)
2. $(-2, 1)$
3. $(-7, -1)$
4. $(-2, -4)$
5. $(-1, 3)$
6. $(-1, 2)$
7. $(-5, 9)$
8. (6, 1)
9. $(-2, 10)$
10. (2, 3)
11. $(-7, 6)$
12. $(-9, 3)$
13. (9, 5)
14. $(-3, 2)$
15. $(-2, 3)$
16. infinitely many solutions
17. infinitely many solutions
18. infinitely many solutions
19. inconsistent, no solution
20. inconsistent, no solution
21. The numbers are 20 and 45.
22. The numbers are 16 and −43.
23. A shirt costs \$16 and a sweater costs \$33.
24. There are 860 mg in a hot dog. There are 1,000 mg in a cup of cottage cheese.
25\.
   a. elimination
   b. substitution
26\.
   a. substitution
   b. elimination

27. a. $r = 4$ b. $c = 1$

4.4. Solve Applications with Systems of Equations

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Translate to a system of equations
- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Previously in this chapter we solved several applications with systems of linear equations. In this section, we'll look at some specific types of applications that relate two quantities. We'll translate the words into linear equations, decide which is the most convenient method to use, and then solve them.

We will use our Problem Solving Strategy for Systems of Linear Equations.

Use a problem solving strategy for systems of linear equations.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Translate to a System of Equations

Many of the problems we solved in earlier applications related two quantities.

Let's see how we can translate these problems into a system of equations with two variables. We'll focus on Steps 1 through 4 of our Problem Solving Strategy.

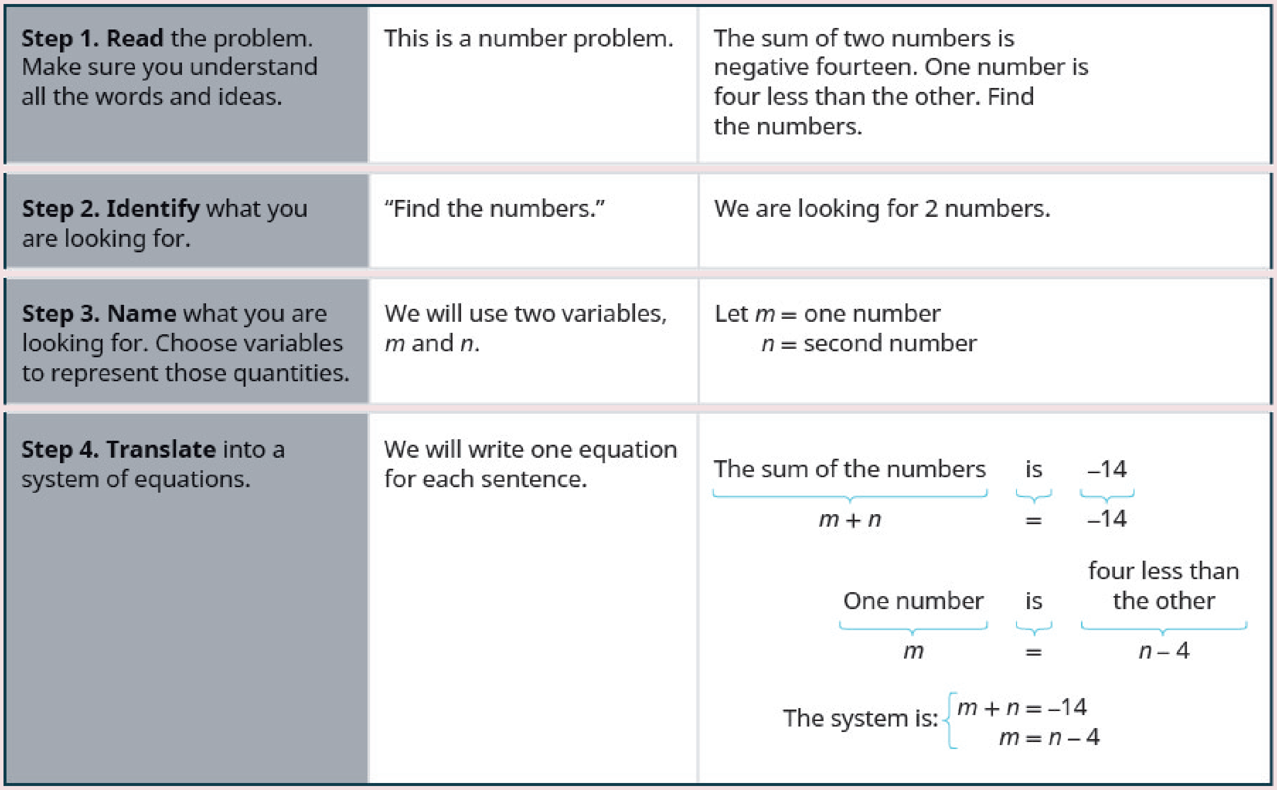
EXAMPLE 1

How to Translate to a System of Equations

Translate to a system of equations:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution



Step 1. Read the problem. Make sure you understand all the words and ideas.	This is a number problem.	The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.
Step 2. Identify what you are looking for.	“Find the numbers.”	We are looking for 2 numbers.
Step 3. Name what you are looking for. Choose variables to represent those quantities.	We will use two variables, m and n.	Let m = one number n = second number
Step 4. Translate into a system of equations.	We will write one equation for each sentence.	$\underbrace{\text{The sum of the numbers}}_{m+n}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$ $\underbrace{\text{One number}}_{m}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four less than the other}}_{n-4}$ The system is: $\begin{cases} m+n=-14 \\ m=n-4 \end{cases}$

TRY IT 1

Translate to a system of equations:

The sum of two numbers is negative twenty-three. One number is 7 less than the other. Find the numbers.

Show answer

$$\begin{cases} m+n=-23 \\ m=n-7 \end{cases}$$

We’ll do another example where we stop after we write the system of equations.

EXAMPLE 2

Translate to a system of equations:

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

We are looking for the amount that the husband and wife each earn.	Let $h =$ the amount the husband earns. $w =$ the amount the wife earns.
Translate.	A married couple together earns $110,000.
	$w + h = 110,000$
	The wife earns $16,000 less than twice what husband earns.
	$w = 2h - 16,000$
The system of equations is:	$\begin{cases} w + h = 110,000 \\ w = 2h - 16,000 \end{cases}$

TRY IT 2

Translate to a system of equations:

A couple has a total household income of $84,000. The husband earns $18,000 less than twice what the wife earns. How much does the wife earn?

Show answer

$$\begin{cases} w + h = 84,000 \\ h = 2w - 18,000 \end{cases}$$

Solve Direct Translation Applications

We set up, but did not solve, the systems of equations in examples 1 and 2. Now we'll translate a situation to a system of equations and then solve it.

EXAMPLE 3

Translate to a system of equations and then solve:

Devon is 26 years older than his son Cooper. The sum of their ages is 50. Find their ages.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the ages of Devon and Cooper.
Step 3. Name what we are looking for.	Let $d =$ Devon's age. $c =$ Cooper's age
Step 4. Translate into a system of equations.	Devon is 26 years older than Cooper.
	$d = c + 26$
	The sum of their ages is 50.
	$d + c = 50$
The system is:	$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$
Step 5. Solve the system of equations. Solve by substitution.	$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$ $d + c = 50$
Substitute *c* + 26 into the second equation.	$c + 26 + c = 50$
Solve for *c*.	$2c + 26 = 50$
	$2c = 24$
	$c = 12$ $d = c + 26$
Substitute *c* = 12 into the first equation and then solve for *d*.	$d = 12 + 26$
	$d = 38$
Step 6. Check the answer in the problem.	Is Devon's age 26 more than Cooper's? Yes, 38 is 26 more than 12. Is the sum of their ages 50? Yes, 38 plus 12 is 50.
Step 7. Answer the question.	Devon is 38 and Cooper is 12 years old.

TRY IT 3

Translate to a system of equations and then solve:

Ali is 12 years older than his youngest sister, Jameela. The sum of their ages is 40. Find their ages.

Show answer
Ali is 28 and Jameela is 16.

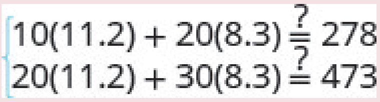
EXAMPLE 4

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories does she burn for each minute of circuit training?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let $e =$ number of calories burned per minute on the elliptical trainer. $c =$ number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of *e*.	$\begin{cases} -2(10e + 20c) = -2(278) \\ 20e + 30c = 473 \end{cases}$
Simplify and add the equations. Solve for *c*.	$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$ $-10c = -83$ $c = 8.3$
Substitute $c = 8.3$ into one of the original equations to solve for *e*.	$10e + 20c = 278$
	$10e + 20(8.3) = 278$
	$10e + 166 = 278$
	$10e = 112$
	$e = 11.2$

Step 6. Check the answer in the problem.	Check the math on your own.
 $\begin{cases}10(11.2)+20(8.3)\overset{?}{=}278\\20(11.2)+30(8.3)\overset{?}{=}473\end{cases}$	
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

TRY IT 4

Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Show answer
Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Solve Geometry Applications

We solved geometry applications using properties of triangles and rectangles. Now we'll add to our list some properties of angles.

The measures of two complementary angles add to 90 degrees. The measures of two supplementary angles add to 180 degrees.

Complementary and Supplementary Angles

Two angles are **complementary** if the sum of the measures of their angles is 90 degrees.

Two angles are **supplementary** if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

EXAMPLE 5

Translate to a system of equations and then solve:

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle $x =$. $m =$ the measure of the second angle.
Step 4. Translate into a system of equations.	The angles are complementary. $x + y = 90$
	The difference of the two angles is 26 degrees. $x - y = 26$
The system is	$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$
Step 5. Solve the system of equations by elimination.	$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$ $2x \qquad = 116$
Substitute $x = 58$ into the first equation.	$\begin{aligned} x + y &= 90 \\ 58 + y &= 90 \\ y &= 32 \end{aligned}$
Step 6. Check the answer in the problem. $58 + 32 = 90$ $58 - 32 = 26$	
Step 7. Answer the question.	The angle measures are 58 degrees and 42 degrees.

TRY IT 5

Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Show answer
The angle measures are 55 degrees and 35 degrees.

EXAMPLE 6

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle. $y =$ the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x + y = 180$
	The larger angle is twelve less than five times the smaller angle
	$y = 5x - 12$
The system is: **Step 5. Solve** the system of equations substitution.	$\begin{cases} x + y = 180 \\ y = 5x - 12 \end{cases}$ $x + y = 180$
Substitute 5x − 12 for y in the first equation.	$x + 5x - 12 = 180$
Solve for x.	$6x - 12 = 180$
	$6x = 192$
	$x = 32$ $y = 5x - 12$
Substitute 32 for in the second equation, then solve for y.	$y = 5 \cdot 32 - 12$
	$y = 160 - 12$
	$y = 148$
Step 6. Check the answer in the problem. $\begin{aligned} 32 + 158 &= 180 \\ 5 \cdot 32 - 12 &= 147 \end{aligned}$	
Step 7. Answer the question.	The angle measures are 148 and 32.

TRY IT 6

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

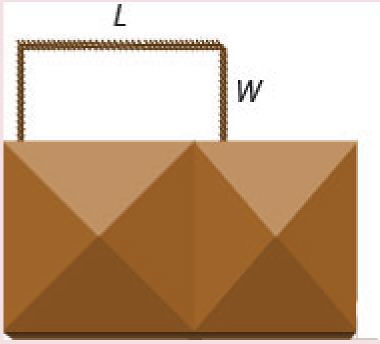
Show answer
The angle measures are 42 degrees and 138 degrees.

EXAMPLE 7

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the rectangular part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
	
Step 3. Name what we are looking for.	Let L = the length of the fenced yard. W = the width of the fenced yard
Step 4. Translate into a system of equations.	One length and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is: **Step 5. Solve** the system of equations by substitution.	$\begin{cases} L + 2W = 125 \\ L = \boxed{4W + 5} \end{cases}$ $L + 2W = 125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$
	$6W + 5 = 125$
	$6W = 120$
Substitute 20 for W in the second equation, then solve for L.	$W = \boxed{20}$ $L = 4W + 5$
	$L = 4 \cdot 20 + 5$
	$L = 80 + 5$
	$L = 85$

Step 6. Check the answer in the problem. $$\begin{aligned}20+28+20 &= 125\\ 85 &= 4\cdot 20+5\end{aligned}$$	
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

TRY IT 7

Translate to a system of equations and then solve:

Mario wants to put a rectangular fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Show answer
The length is 60 feet and the width is 35 feet.

Solve Uniform Motion Applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance travelled, r is the rate, and t is the time.

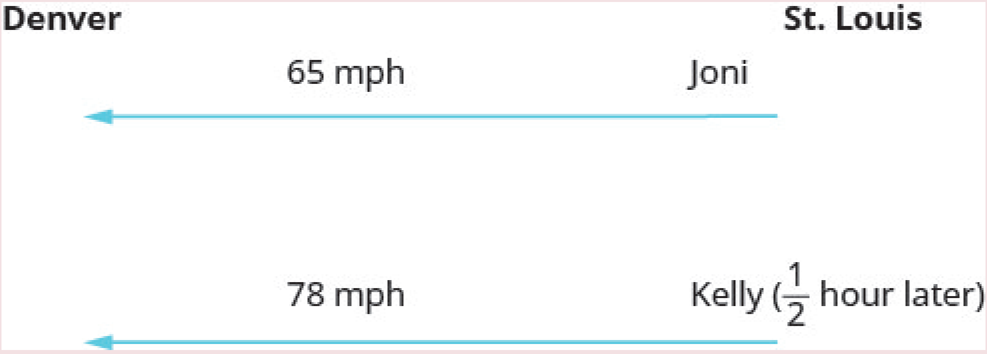
Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

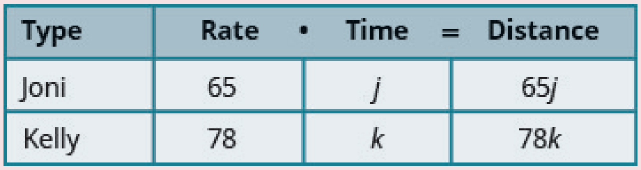
EXAMPLE 8

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Solution
A diagram is useful in helping us visualize the situation.


Denver
St. Louis
65 mph
Joni
78 mph
Kelly ($\frac{1}{2}$ hour later)

<table>
<tr><td>Identify and name what we are looking for.
A chart will help us organize the data.
We know the rates of both Joni and Kelly, and so
we enter them in the chart.</td><td></td></tr>
<tr><td>We are looking for the length of time Kelly, <i>k</i>, and Joni, <i>j</i>, will each drive.
Since $D = r \cdot t$ we can fill in the Distance column.</td><td>
<table>
<tr><th>Type</th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>Joni</td><td>65</td><td>j</td><td>$65j$</td></tr>
<tr><td>Kelly</td><td>78</td><td>k</td><td>$78k$</td></tr>
</table></td></tr>
<tr><td>Translate into a system of equations.
To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So, $65j = 78k$.

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time.

So, $k = j - \frac{1}{2}$.</td><td></td></tr>
<tr><td>Now we have the system.</td><td>$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$</td></tr>
<tr><td>Solve the system of equations by substitution.</td><td>$65j = 78k$</td></tr>
<tr><td>Substitute $k = j - \frac{1}{2}$ into the second equation, then solve for <i>j</i>.</td><td>$65j = 78\left(j - \frac{1}{2}\right)$</td></tr>
<tr><td></td><td>$65j = 78j - 39$</td></tr>
<tr><td></td><td>$-13j = -39$</td></tr>
<tr><td></td><td>$j = 3$</td></tr>
<tr><td>To find Kelly's time, substitute <i>j</i> = 3 into the first equation, then solve for <i>k</i>.</td><td>$k = j - \frac{1}{2}$</td></tr>
<tr><td></td><td>$k = 3 - \frac{1}{2}$</td></tr>
<tr><td></td><td>$k = \frac{5}{2}$ or $k = 2\frac{1}{2}$</td></tr>
</table>

Check the answer in the problem. Joni 3 hours (65 mph) = 195 miles. Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles. Yes, they will have traveled the same distance when they meet.	
Answer the question.	Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

TRY IT 8

Translate to a system of equations and then solve: Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Show answer

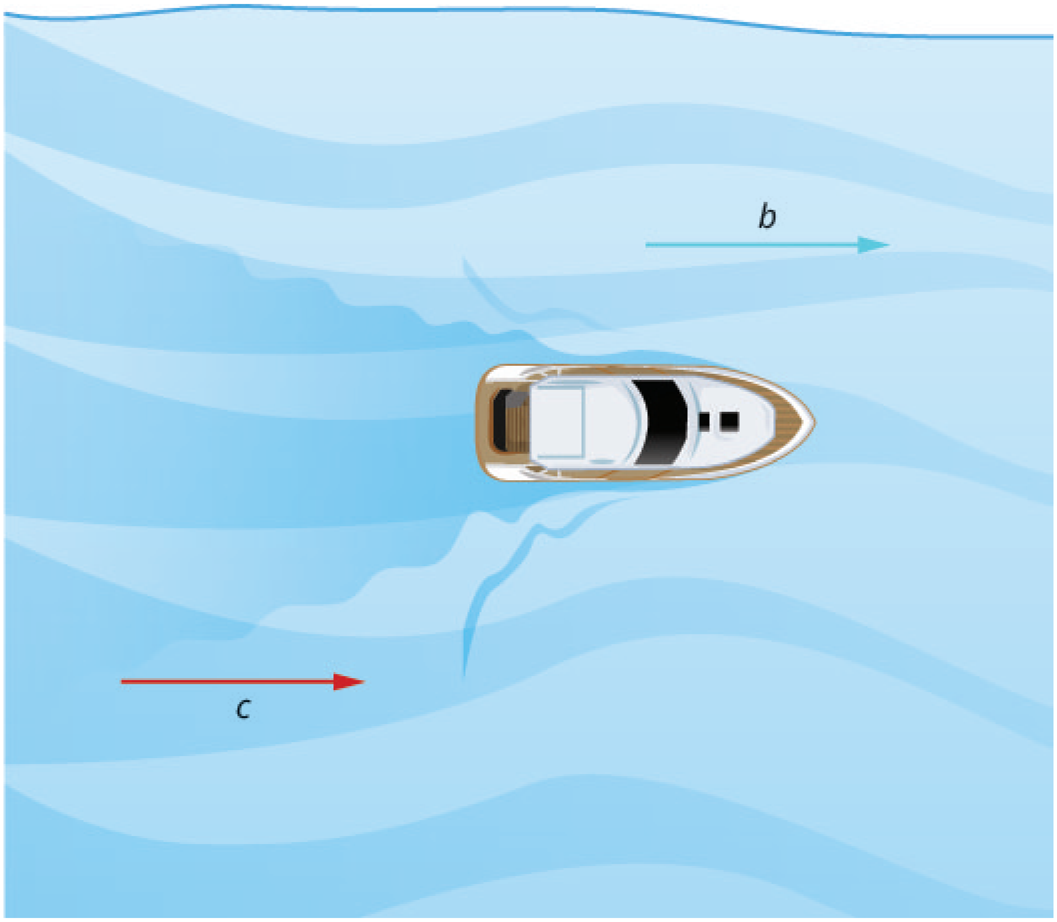
It will take Clark 4 hours to catch Mitchell.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights generally take longer going west than going east because of the prevailing wind currents.

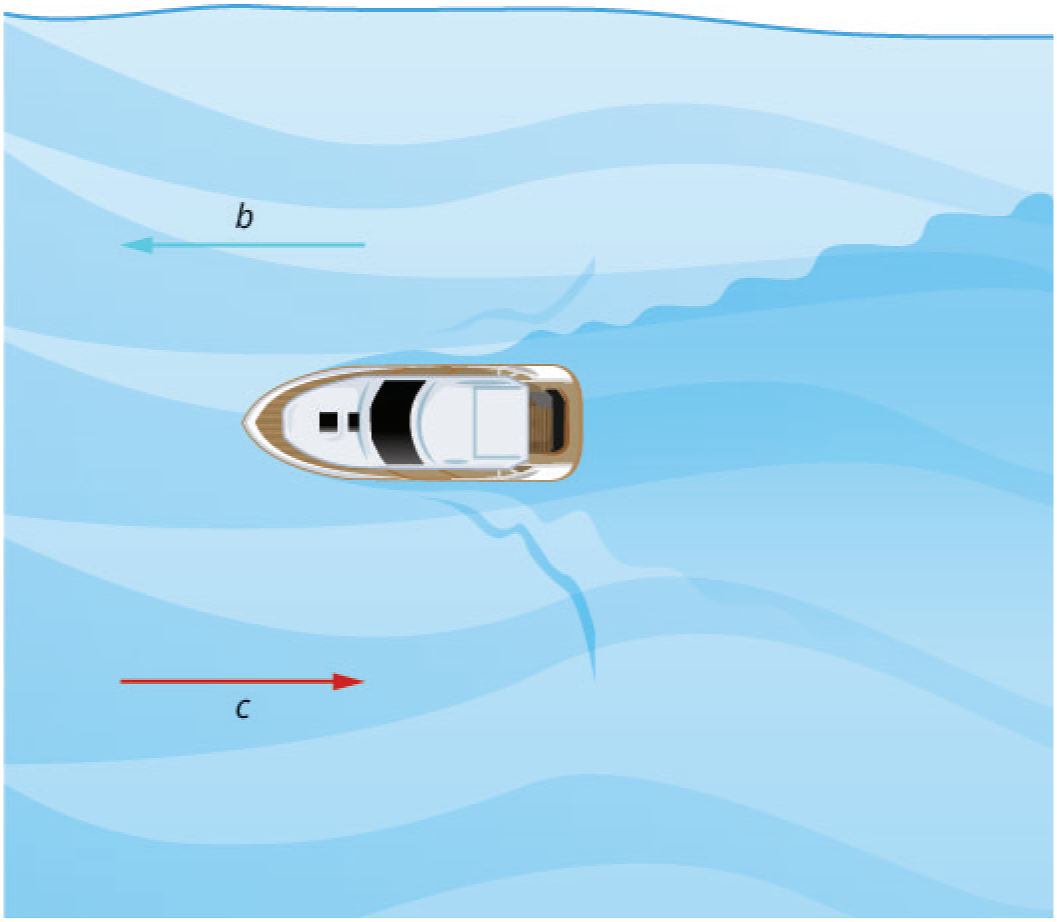
Let’s take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually travelling. We’ll call the speed of the boat in still water b and the speed of the river current c.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat’s actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b + c$.



The boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b - c$.



We’ll put some numbers to this situation in the next example.

EXAMPLE 9

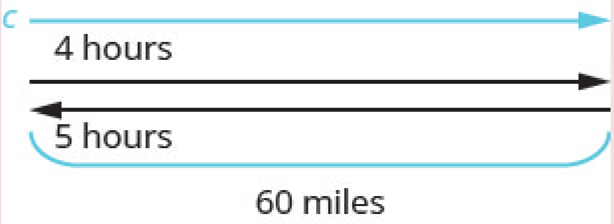
Translate to a system of equations and then solve:

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize the situation.



<table>
<tr><td>Identify what we are looking for.</td><td>We are looking for the speed of the ship in still water and the speed of the current.</td></tr>
<tr><td>Name what we are looking for.</td><td>Let s = the rate of the ship in still water.
c = the rate of the current</td></tr>
<tr><td>A chart will help us organize the information.
The ship goes downstream and then upstream.
Going downstream, the current helps the ship; therefore, the ship's actual rate is $s + c$.
Going upstream, the current slows the ship;
therefore, the actual rate is $s - c$.</td><td><table>
<tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>downstream</td><td>$s + c$</td><td>4</td><td>60</td></tr>
<tr><td>upstream</td><td>$s - c$</td><td>5</td><td>60</td></tr>
</table></td></tr>
<tr><td>Downstream it takes 4 hours.
Upstream it takes 5 hours.
Each way the distance is 60 miles.</td><td></td></tr>
<tr><td>Translate into a system of equations.
Since rate times time is distance, we can write the system of equations.</td><td>$\begin{cases} 4(s+c) = 60 \\ 5(s-c) = 60 \end{cases}$</td></tr>
<tr><td>Solve the system of equations.
Distribute to put both equations in standard
form, then solve by elimination.</td><td>$\begin{cases} 4s + 4c = 60 \\ 5s - 5c = 60 \end{cases}$</td></tr>
<tr><td>Multiply the top equation by 5 and the bottom equation by 4.
Add the equations, then solve for s.</td><td>$\begin{array}{l} \begin{cases} 20s + 20c = 300 \\ 20s - 20c = 240 \end{cases} \\ \hline 40s \qquad = 540 \end{array}$</td></tr>
<tr><td>Substitute $s = 13.5$ into one of the original equations.</td><td>$s \qquad = \boxed{13.5}$
$4(s + c) = 60$</td></tr>
<tr><td></td><td>$4(13.5 + c) = 60$</td></tr>
<tr><td></td><td>$54 + 4c = 60$</td></tr>
<tr><td></td><td>$4c = 6$</td></tr>
<tr><td></td><td>$c = 1.5$</td></tr>
</table>

Check the answer in the problem. The downstream rate would be 13.5 + 1.5 = 15 mph. In 4 hours the ship would travel 15 \cdot 4 = 60 miles. The upstream rate would be 13.5 − 1.5 = 12 mph. In 5 hours the ship would travel 12 \cdot 5 = 60 miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

TRY IT 9

Translate to a system of equations and then solve: Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Show answer
The speed of the canoe is 7 mph and the speed of the current is 1 mph.

Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

EXAMPLE 10

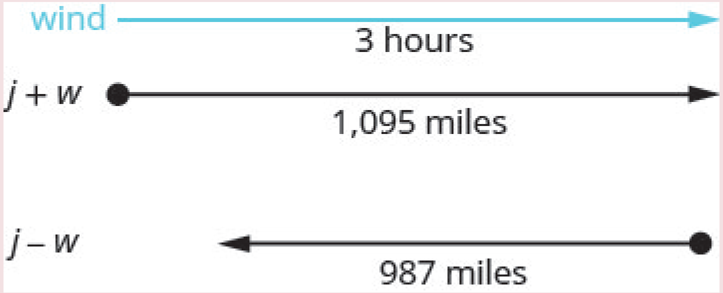
Translate to a system of equations and then solve:

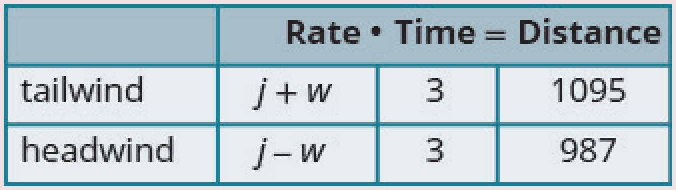
A private jet can fly 1095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize.


wind
3 hours
j + w
1,095 miles
j – w
987 miles

<table>
<tr><td>Identify what we are looking for.</td><td>We are looking for the speed of the jet in still air and the speed of the wind.</td></tr>
<tr><td>Name what we are looking for.</td><td>Let j = the speed of the jet in still air.
w = the speed of the wind</td></tr>
<tr><td>A chart will help us organize the information.
The jet makes two trips-one in a tailwind and one in a headwind.
In a tailwind, the wind helps the jet and so the rate is $j + w$.
In a headwind, the wind slows the jet and so the rate is $j - w$.</td><td>
<table>
<tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>tailwind</td><td>$j+w$</td><td>3</td><td>1095</td></tr>
<tr><td>headwind</td><td>$j-w$</td><td>3</td><td>987</td></tr>
</table></td></tr>
<tr><td>Each trip takes 3 hours.
In a tailwind the jet flies 1095 miles.
In a headwind the jet flies 987 miles.</td><td></td></tr>
<tr><td>Translate into a system of equations.
Since rate times time is distance, we get the system of equations.</td><td>$\begin{cases} 3(j+w)=1095 \\ 3(j-w)=987 \end{cases}$</td></tr>
<tr><td>Solve the system of equations.
Distribute, then solve by elimination.</td><td>$\begin{array}{l} \begin{cases} 3j+3w=1095 \\ 3j-3w=987 \end{cases} \\ \hline 6j \qquad = 2082 \end{array}$</td></tr>
<tr><td>Add, and solve for j.

Substitute $j = 347$ into one of the original equations, then solve for w.</td><td>$j \qquad = \boxed{347}$
$3(j+w)=1095$</td></tr>
<tr><td></td><td>$3(347+w)=1095$</td></tr>
<tr><td></td><td>$1041+3w=1095$</td></tr>
<tr><td></td><td>$3w=54$</td></tr>
<tr><td></td><td>$w=18$</td></tr>
</table>

Check the answer in the problem. With the tailwind, the actual rate of the jet would be 347 + 18 = 365 mph. In 3 hours the jet would travel 365 \cdot 3 = 1095 miles. Going into the headwind, the jet's actual rate would be 347 − 18 = 329 mph. In 3 hours the jet would travel 329 \cdot 3 = 987 miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

TRY IT 10

Translate to a system of equations and then solve: A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1025 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Show answer
The speed of the jet is 235 mph and the speed of the wind is 30 mph.

Glossary

complementary angles
Two angles are complementary if the sum of the measures of their angles is 90 degrees.
supplementary angles
Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

4.4 Exercise Set

In the following exercises, translate to a system of equations and solve the system.

1. The sum of two numbers is fifteen. One number is three less than the other. Find the

numbers.

2. The sum of two numbers is negative thirty. One number is five times the other. Find the numbers.
3. Twice a number plus three times a second number is twenty-two. Three times the first number plus four times the second is thirty-one. Find the numbers.
4. Three times a number plus three times a second number is fifteen. Four times the first plus twice the second number is fourteen. Find the numbers.
5. A married couple together earn ?75,000. The husband earns ?15,000 more than five times what his wife earns. What does the wife earn?
6. Daniela invested a total of ?50,000, some in a certificate of deposit (CD) and the remainder in bonds. The amount invested in bonds was ?5000 more than twice the amount she put into the CD. How much did she invest in each account?
7. In her last two years in college, Marlene received? 42,000 in loans. The first year she received a loan that was ?6,000 less than three times the amount of the second year's loan. What was the amount of her loan for each year?

In the following exercises, translate to a system of equations and solve.

8. Alyssa is twelve years older than her sister, Bethany. The sum of their ages is forty-four. Find their ages.
9. The age of Noelle's dad is six less than three times Noelle's age. The sum of their ages is seventy-four. Find their ages.
10. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?
11. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?
12. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

In the following exercises, translate to a system of equations and solve.

13. The difference of two complementary angles is 30 degrees. Find the measures of the angles.
14. The difference of two supplementary angles is 70 degrees. Find the measures of the angles.
15. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.
16. The difference of two complementary angles is 55 degrees. Find the measures of the angles.
17. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

18. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.
19. Wayne is hanging a string of lights 45 feet long around the three sides of his rectangular patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.
20. A frame around a rectangular family portrait has a perimeter of 60 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

In the following exercises, translate to a system of equations and solve.

21. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?
22. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?
23. The Jones family took a 12 mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.
24. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.).
25. A small jet can fly 1,072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.
26. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.
27. At a school concert, 425 tickets were sold. Student tickets cost $5 each and adult tickets cost $8 each. The total receipts for the concert were $2,851. Solve the system $\begin{cases} s+a=425 \\ 5s+8a=2,851 \end{cases}$ to find s, the number of student tickets and a, the number of adult tickets.

Answers:

1. The numbers are 6 and 9.
2. The numbers are −5 and −25.
3. The numbers are 5 and 4.
4. The numbers are 2 and 3.
5. $10,000
6. She put $15,000 into a CD and $35,000 in bonds.

7. The amount of the first year’s loan was \$30,000 and the amount of the second year’s loan was \$12,000.
8. Bethany is 16 years old and Alyssa is 28 years old.
9. Noelle is 20 years old and her dad is 54 years old.
10. The small container holds 20 gallons and the large container holds 30 gallons.
11. There were 10 calories burned jogging and 10 calories burned cycling.
12. Notebooks are \$4 and thumb drives are \$20.
13. The measures are 60 degrees and 30 degrees.
14. The measures are 125 degrees and 55 degrees.
15. 94 degrees and 86 degrees
16. 72.5 degrees and 17.5 degrees
17. The measures are 44 degrees and 136 degrees.
18. The measures are 34 degrees and 56 degrees.
19. The width is 10 feet and the length is 25 feet.
20. The width is 15 feet and the length is 15 feet.
21. It took Sarah’s sister 12 hours.
22. It took Lucy’s friend 2 hours.
23. The canoe rate is 5 mph and the current rate is 1 mph.
24. The boat rate is 6.5 mph and the current rate is 2.5 mph.
25. The jet rate is 240 mph and the wind speed is 28 mph.
26. The jet rate is 415 mph and the wind speed is 19 mph.
27. $s = 183, a = 242$

4.5 Graphing Systems of Linear Inequalities

Lynn Marecek and MaryAnne Anthony-Smith

Learning Objectives

By the end of this section it is expected that you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities
- Solve a system of linear inequalities by graphing
- Solve applications of systems of inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

The definition of a system of linear inequalities is very similar to the definition of a system of linear equations.

System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown below.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of a System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the *x-y* coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

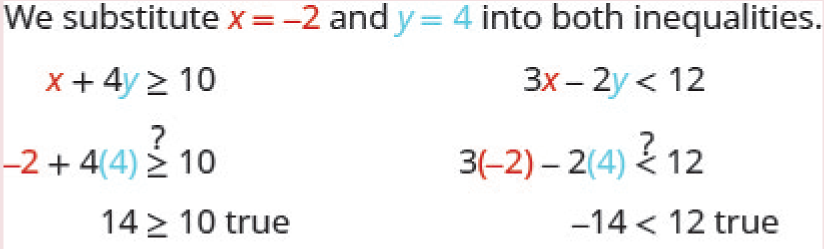
EXAMPLE 1

Determine whether the ordered pair is a solution to the system. $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a) (−2, 4) b) (3,1)

Solution

a) Is the ordered pair (−2, 4) a solution?

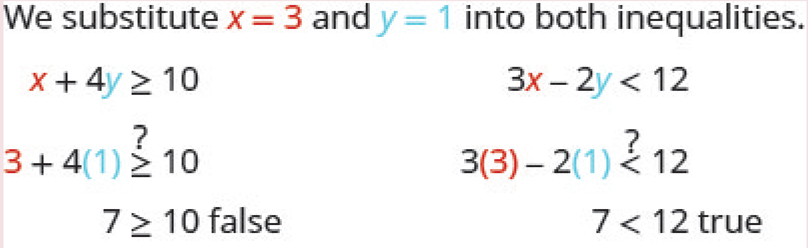


We substitute $x = -2$ and $y = 4$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$-2 + 4(4) \overset{?}{\geq} 10$	$3(-2) - 2(4) \overset{?}{<} 12$
$14 \geq 10$ true	$-14 < 12$ true

The ordered pair (−2, 4) made both inequalities true. Therefore (−2, 4) is a solution to this system.

b) Is the ordered pair (3,1) a solution?



We substitute $x = 3$ and $y = 1$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$3 + 4(1) \overset{?}{\geq} 10$	$3(3) - 2(1) \overset{?}{<} 12$
$7 \geq 10$ false	$7 < 12$ true

The ordered pair (3,1) made one inequality true, but the other one false. Therefore (3,1) is not a solution to this system.

TRY IT 1

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$$

a) $(3, -1)$ b) $(6, -3)$

Show answer
a) no b) yes

Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

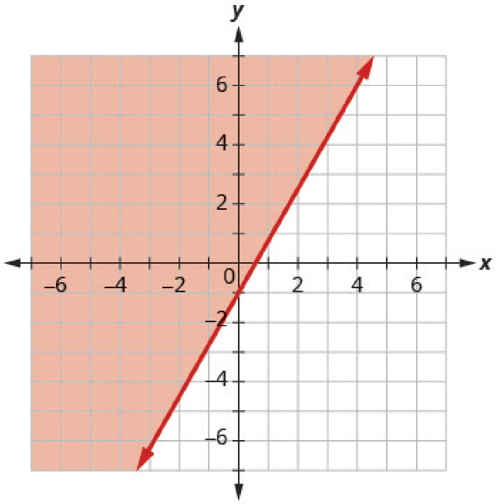
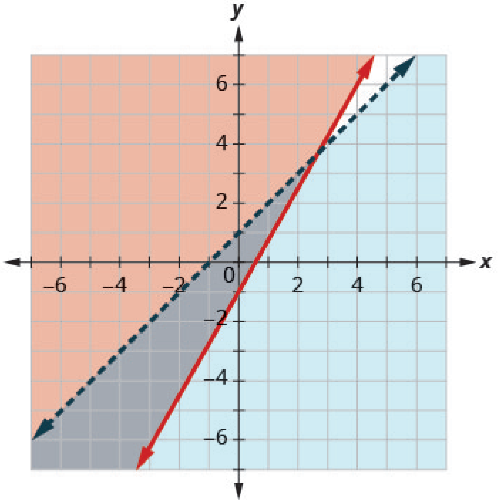
EXAMPLE 2

How to Solve a System of Linear inequalities

Solve the system by graphing.

$$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$$

Solution

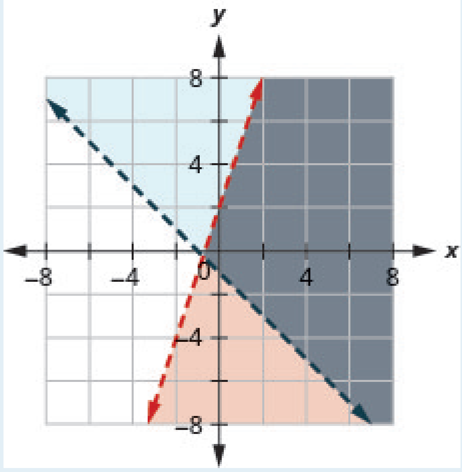
Step 1. Graph the first inequality.	We will graph $y \geq 2x - 1$.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$
Graph the boundary line.	We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$.	
Shade in the side of the boundary line where the inequality is true.	We choose (0,0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in the left side of the boundary line.	
Step 2. On the same grid, graph the second inequality.	We will graph $y < x + 1$ on the same grid.	
Graph the boundary line.	We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$.	
Shade in the side of that boundary line where the inequality is true.	Again, we use (0,0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	

Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the darker shaded region.
Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \ge 2x - 1$? $-1 \overset{?}{\ge} 2(-1) - 1$ $-1 \ge -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \overset{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

TRY IT 2

Solve the system by graphing. $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$

Show answer



Solve a system of linear inequalities by graphing.

1. Graph the first inequality.
    - Graph the boundary line.
    - Shade in the side of the boundary line where the inequality is true.
2. On the same grid, graph the second inequality.
    - Graph the boundary line.
    - Shade in the side of that boundary line where the inequality is true.
3. The solution is the region where the shading overlaps.
4. Check by choosing a test point.

EXAMPLE 3

Solve the system by graphing.

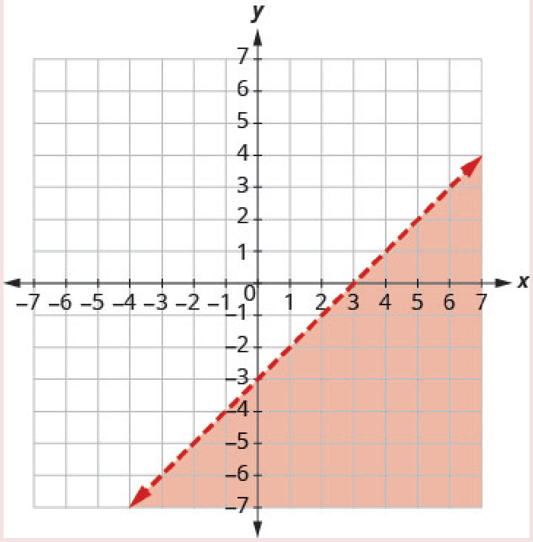
$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

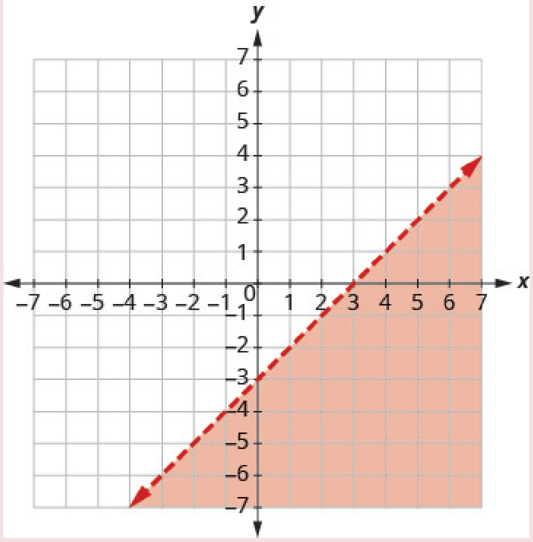
Solution

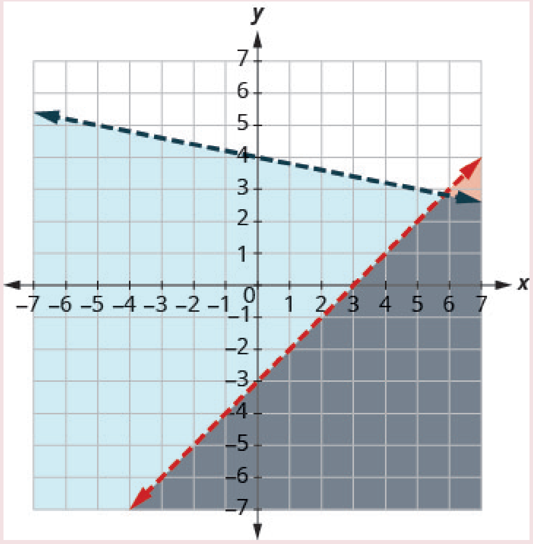
Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

Test (0, 0). It makes the inequality false. So, shade the side that does not contain (0, 0) red.





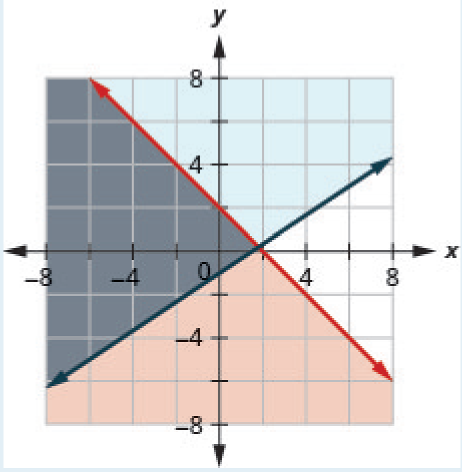
Graph $y < -\frac{1}{5}x + 4$ by graphing $y = -\frac{1}{5}x + 4$ using the slope $m = -\frac{1}{5}$ and *y*−intercept *b* = 4. The boundary line will be dashed. Test (0, 0). It makes the inequality true, so shade the side that contains (0, 0) blue. Choose a test point in the solution and verify that it is a solution to both inequalities.	

The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice which is the darker-shaded region.

TRY IT 3

Solve the system by graphing. $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Show answer

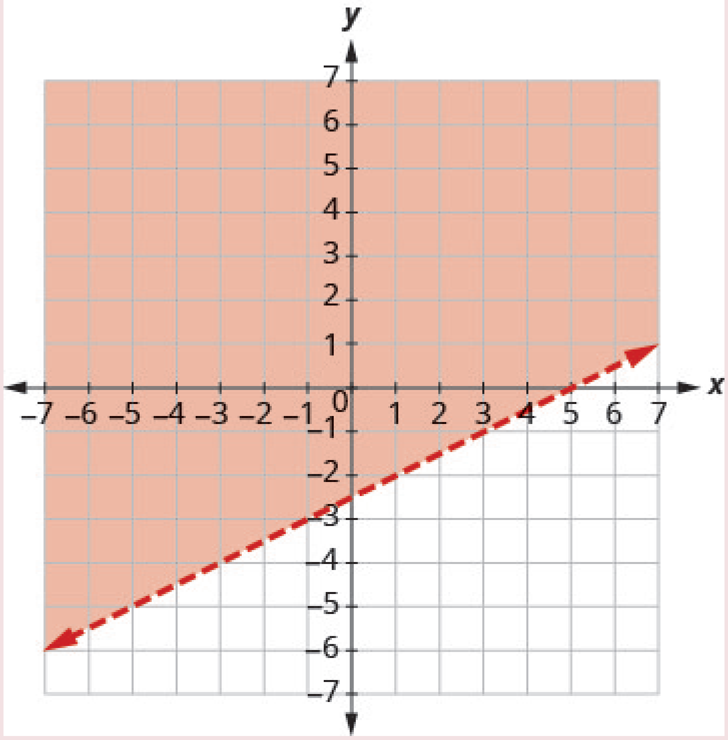
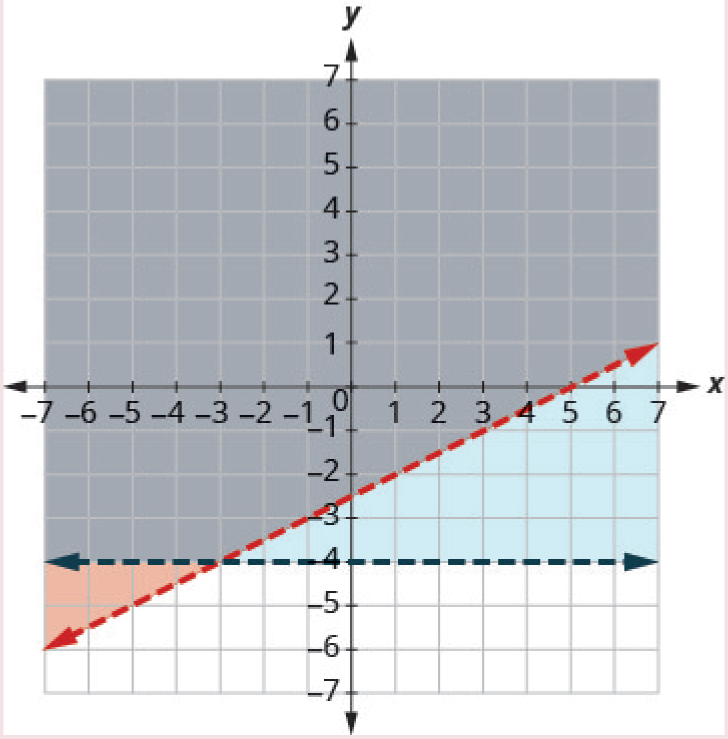


EXAMPLE 4

Solve the system by graphing.

$$\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$$

Solution

Graph $x - 2y < 5$, by graphing $x - 2y = 5$ and testing a point. The intercepts are $x = 5$ and $y = -2.5$ and the boundary line will be dashed. Test (0, 0). It makes the inequality true. So, shade the side that contains (0, 0) red.	
Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed. Test (0, 0). It makes the inequality true. So, shade (blue) the side that contains (0, 0) blue.	

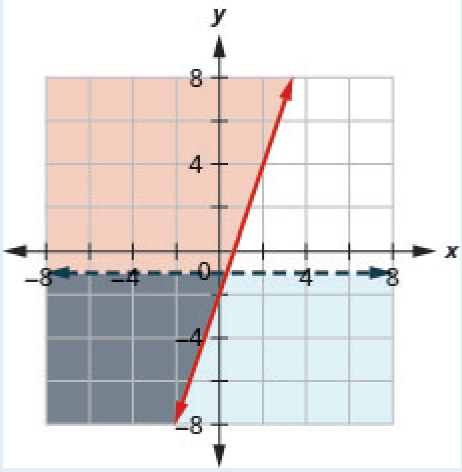
The point (0, 0) is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice which is the darker-shaded region.

TRY IT 4

Solve the system by graphing. $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

Show answer



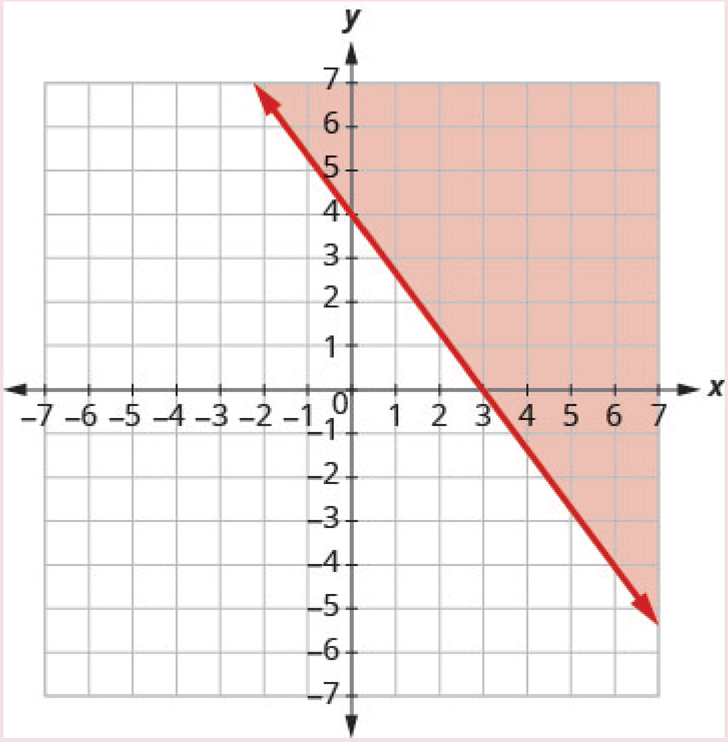
Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

EXAMPLE 5

Solve the system by graphing. $\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$

Solution

<table>
<tr>
<td>Graph $4x + 3y \geq 12$, by graphing $4x + 3y = 12$ and testing a point. The intercepts are $x = 3$ and $y = 4$ and the boundary line will be solid.

Test (0, 0). It makes the inequality false. So, shade the side that does not contain (0, 0) red.</td>
<td>
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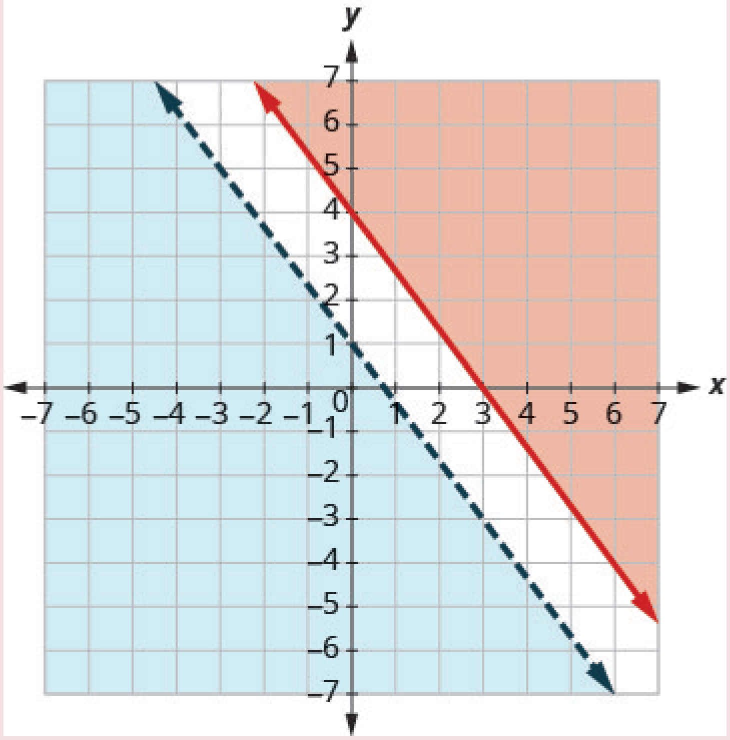
<tr>
<td>Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = \frac{4}{3}$ and the y-intercept $b = 1$. The boundary line will be dashed.

Test (0, 0). It makes the inequality true. So, shade the side that contains (0, 0) blue.</td>
<td>
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</table>

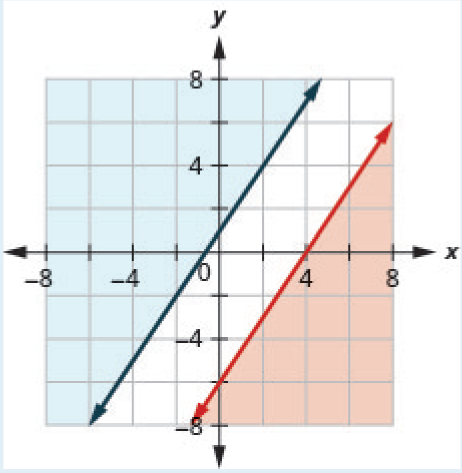
There is no point in both shaded regions, so the system has no solution. This system has no solution.

TRY IT 5

Solve the system by graphing. $\begin{cases} 3x - 2y \leq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$

Show answer

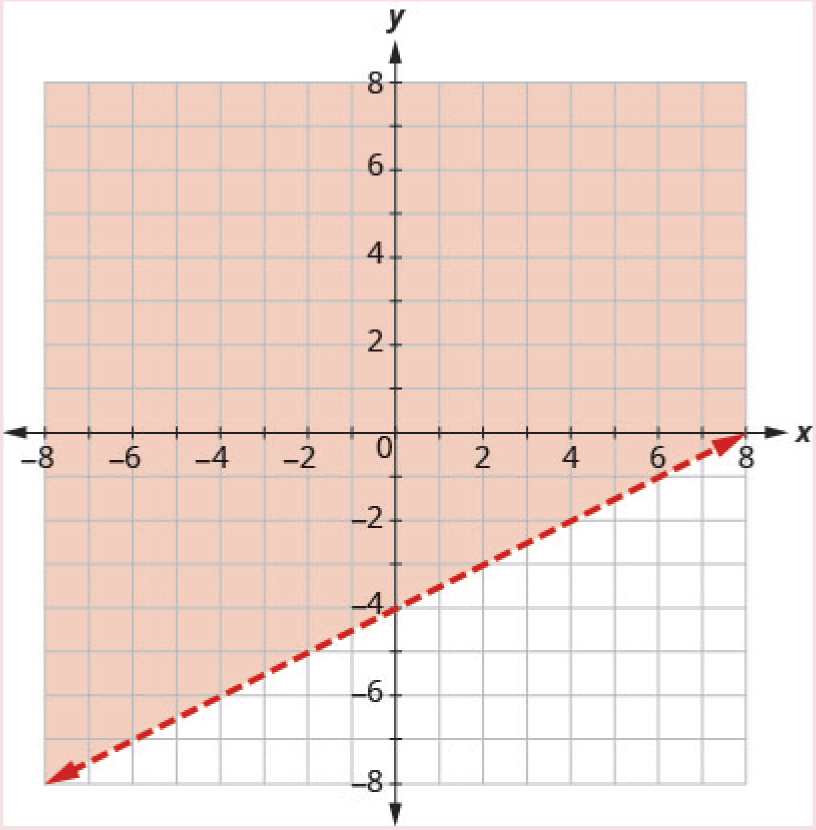
no solution

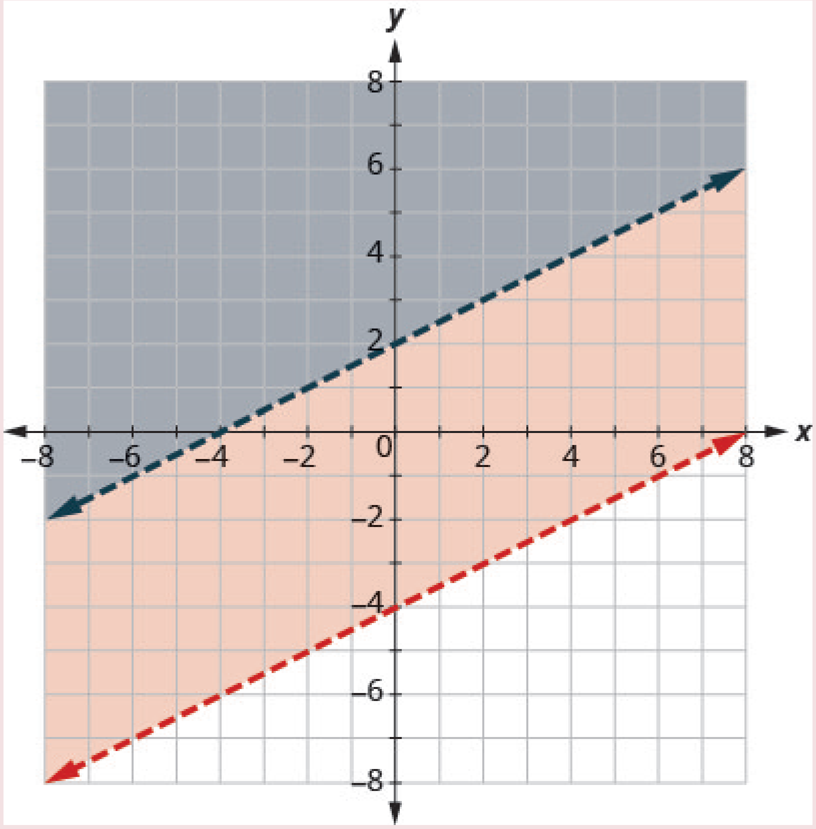


EXAMPLE 6

Solve the system by graphing. $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Solution

Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept *b* = −4. The boundary line will be dashed. Test (0, 0). It makes the inequality true. So, shade the side that contains (0, 0) red.	

Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are *x* = −4 and *y* = 2 and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalities.	

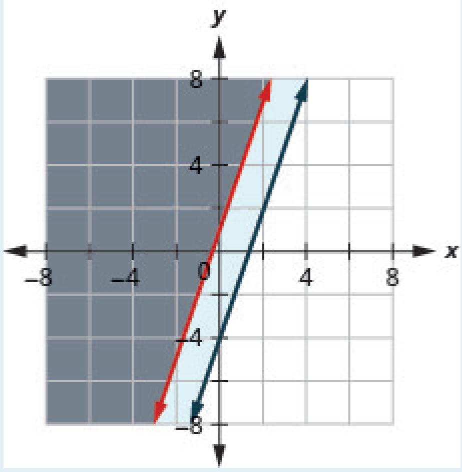
No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice, which is also the solution to $x - 2y < -4$.

TRY IT 6

Solve the system by graphing. $\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$

Show answer
$y \geq 3x + 1$



Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system as we did above to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so their graphs will only show Quadrant I.

EXAMPLE 7

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her ?4 and each large photo costs her ?10. She doesn't want to spend more than ?200 on photos to display.

a) Write a system of inequalities to model this situation.

b) Graph the system.

c) Could she display 15 small and 5 large photos?

d) Could she display 3 large and 22 small photos?

Solution

a) Let $x =$ the number of small photos.
$y =$ the number of large photos
To find the system of inequalities, translate the information.

She wants to have at least 25 photos.

The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

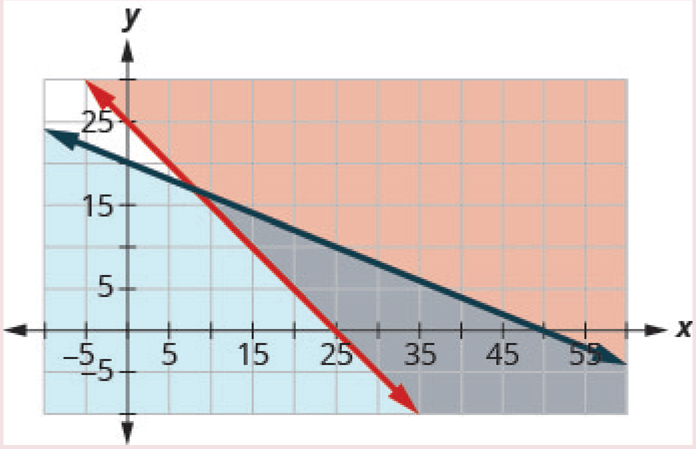
$$4x + 10y \leq 200$$

We have our system of inequalities. $\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \end{cases}$

b)

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line.
Choose (0, 0) as a test point. Since it does not make the inequality
true, shade the side that does not include the point (0, 0) red.

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line.
Choose (0, 0) as a test point. Since it does not make the inequality
true, shade the side that includes the point (0, 0) blue.



The solution of the system is the region of the graph that is double shaded and so is shaded darker.

c) To determine if 10 small and 20 large photos would work, we see if the point (10, 20) is in the solution region. It is not. Christy would not display 10 small and 20 large photos.

d) To determine if 20 small and 10 large photos would work, we see if the point (20, 10) is in the solution region. It is. Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

TRY IT 7

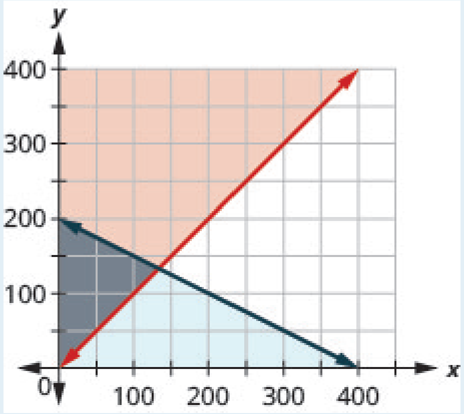
Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost \$2 and the answer sheets cost \$1. Mary's budget for these supplies allows for a maximum cost of \$400.

a) Write a system of inequalities to model this situation.
b) Graph the system.
c) Could Mary purchase 100 pencils and 100 answer sheets?
d) Could Mary purchase 150 pencils and 150 answer sheets?

Show answer

a) $\begin{cases} a \geq p + 5 \\ a + 2p \leq 400 \end{cases}$

b)



c) no

d) no

EXAMPLE 8

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

a) Write a system of inequalities to model this situation.
b) Graph the system.
c) Could he eat 3 hamburgers and 1 cookie?
d) Could he eat 2 hamburgers and 4 cookies?

Solution
a) Let $h =$ the number of hamburgers.
$c =$ the number of cookies
To find the system of inequalities, translate the information.
The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$240h + 160c \geq 800$

The amount spent on hamburgers at $1.40 each, plus the amount spent on cookies at $0.50 each must be no more than $5.00.

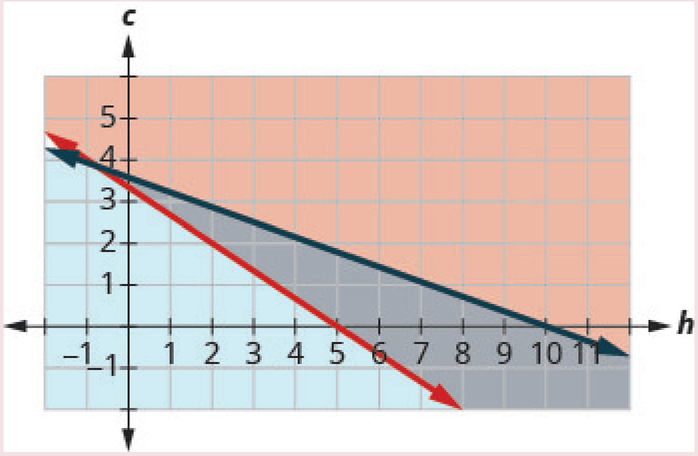
$1.40h + 0.50c \leq 5$

We have our system of inequalities. $\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \end{cases}$

b)

To graph $240h + 160c \geq 800$ graph $240h + 160c = 800$ as a solid line.
Choose (0, 0) as a test point. it does not make the inequality true.
So, shade (red) the side that does not include the point (0, 0).

To graph $1.40h + 0.50c \leq 5$, graph $1.40h + 0.50c = 5$ as a solid line.
Choose (0,0) as a test point. It makes the inequality true. So, shade (blue) the side that includes the point.



The solution of the system is the region of the graph that is double shaded and so is shaded darker.

c) To determine if 3 hamburgers and 2 cookies would meet Omar’s criteria, we see if the point (3, 1) is in the solution region. It is. He might choose to eat 3 hamburgers and 2 cookies.

d) To determine if 2 hamburgers and 4 cookies would meet Omar’s criteria, we see if the point (2, 4) is in the solution region. It is. He might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

TRY IT 8

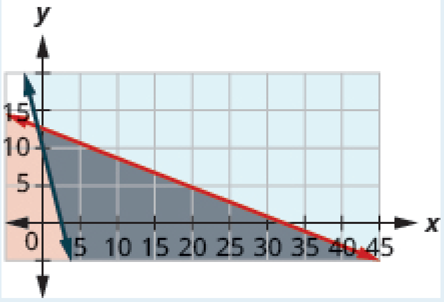
Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

a) Write a system of inequalities that models this situation.
b) Graph the system.
c) Can he buy 8 donuts and 4 energy drinks?
d) Can he buy 1 donut and 3 energy drinks?

Show answer

a) $\begin{cases} 0.75d + 2e \leq 25 \\ 360d + 110e \geq 1000 \end{cases}$

b)



c) yes

d) no

Access these online resources for additional instruction and practice with graphing systems of linear inequalities.

- Graphical System of Inequalities
- Systems of Inequalities
- Solving Systems of Linear Inequalities by Graphing

Key Concepts

- **To Solve a System of Linear Inequalities by Graphing**
    1. Graph the first inequality.
        - Graph the boundary line.
        - Shade in the side of the boundary line where the inequality is true.
    2. On the same grid, graph the second inequality.
        - Graph the boundary line.
        - Shade in the side of that boundary line where the inequality is true.
    3. The solution is the region where the shading overlaps.
    4. Check by choosing a test point.

4.5 Exercise Set

In the following exercises, determine whether each ordered pair is a solution to the system.

1. $$\begin{cases} 3x + y > 5 \\ 2x - y \le 10 \end{cases}$$

a. $(3, -3)$

b. $(7, 1)$

2. $\begin{cases} y > \frac{2}{3}x - 5 \\ x + \frac{1}{2}y \leq 4 \end{cases}$
   a. $(6, -4)$
   b. $(3, 0)$

3. $\begin{cases} 7x + 2y > 14 \\ 5x - y \leq 8 \end{cases}$
   a. $(2, 3)$
   b. $(7, -1)$

4. $\begin{cases} 2x + 3y \geq 2 \\ 4x - 6y < -1 \end{cases}$
   a. $\left(\frac{3}{2}, \frac{4}{3}\right)$
   b. $\left(\frac{1}{4}, \frac{7}{6}\right)$

In the following exercises, solve each system by graphing.

5. $\begin{cases} y \leq 3x + 2 \\ y > x - 1 \end{cases}$

6. $\begin{cases} y < 2x - 1 \\ y \leq -\frac{1}{2}x + 4 \end{cases}$

7. $\begin{cases} x - y > 1 \\ y < -\frac{1}{4}x + 3 \end{cases}$

8. $\begin{cases} 3x - y \leq 6 \\ y \geq -\frac{1}{2}x \end{cases}$

9. $\begin{cases} 2x - 5y < 10 \\ 3x + 4y \geq 12 \end{cases}$

10. $\begin{cases} 2x + 2y > -4 \\ -x + 3y \geq 9 \end{cases}$

11. $\begin{cases} x - 2y < 3 \\ y \leq 1 \end{cases}$

12. $\begin{cases} y \geq -\frac{1}{2}x - 3 \\ x \leq 2 \end{cases}$

13. $\begin{cases} y \geq \frac{3}{4}x - 2 \\ y < 2 \end{cases}$

14. $\begin{cases} 3x - 4y < 8 \\ x < 1 \end{cases}$

15. $\begin{cases} x \geq 3 \\ y \leq 2 \end{cases}$

16. $\begin{cases} 2x + 4y > 4 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$

17. $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$

18. $\begin{cases} y \geq -3x + 2 \\ 3x + y > 5 \end{cases}$

19. $\begin{cases} y \leq -\frac{1}{4}x - 2 \\ x + 4y < 6 \end{cases}$

20. $\begin{cases} 3y > x + 2 \\ -2x + 6y > 8 \end{cases}$

In the following exercises, translate to a system of inequalities and solve.

21. Caitlyn sells her drawings at the county fair. She wants to sell at least 60 drawings and has portraits and landscapes. She sells the portraits for $15 and the landscapes for $10. She needs to sell at least $800 worth of drawings in order to earn a profit.
    a. Write a system of inequalities to model this situation.
    b. Graph the system.
    c. Will she make a profit if she sells 20 portraits and 35 landscapes?
    d. Will she make a profit if she sells 50 portraits and 20 landscapes?

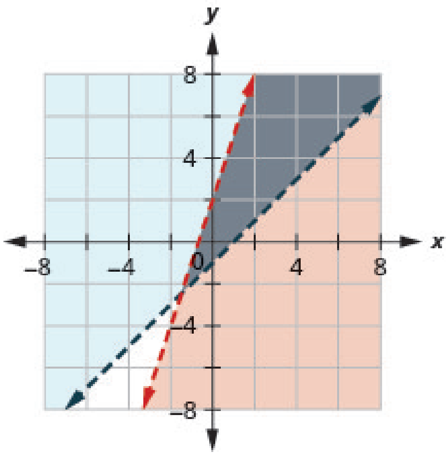
22. Reiko needs to mail her Christmas cards and packages and wants to keep her mailing costs to no more than $500. The number of cards is at least 4 more than twice the number of packages. The cost of mailing a card (with pictures enclosed) is $3 and for a package the cost is $7.
    a. Write a system of inequalities to model this situation.
    b. Graph the system.
    c. Can she mail 60 cards and 26 packages?
    d. Can she mail 90 cards and 40 packages?
23. Jocelyn is pregnant and needs to eat at least 500 more calories a day than usual. When buying groceries one day with a budget of $15 for the extra food, she buys bananas that have 90 calories each and chocolate granola bars that have 150 calories each. The bananas cost $0.35 each and the granola bars cost $2.50 each.
    a. Write a system of inequalities to model this situation.
    b. Graph the system.
    c. Could she buy 5 bananas and 6 granola bars?
    d. Could she buy 3 bananas and 4 granola bars?
24. Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.
    a. Write a system of inequalities to model this situation.
    b. Graph the system.
    c. Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?
    d. Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?
25. Tickets for an American Baseball League game for 3 adults and 3 children cost less than ?75, while tickets for 2 adults and 4 children cost less than $62.
    a. Write a system of inequalities to model this problem.
    b. Graph the system.
    c. Could the tickets cost $20 for adults and $8 for children?
    d. Could the tickets cost $15 for adults and $5 for children?

Answers:

a. true
b. false
a. false
b. true

a. true
b. false
a. true
b. true

5.



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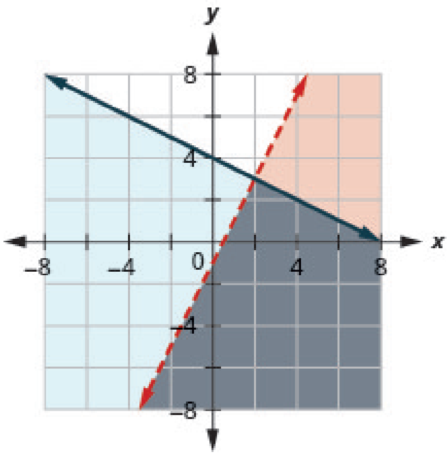

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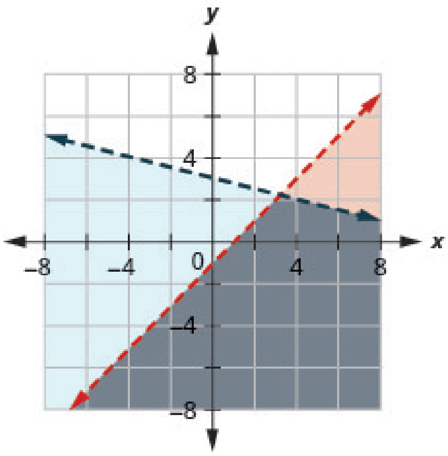

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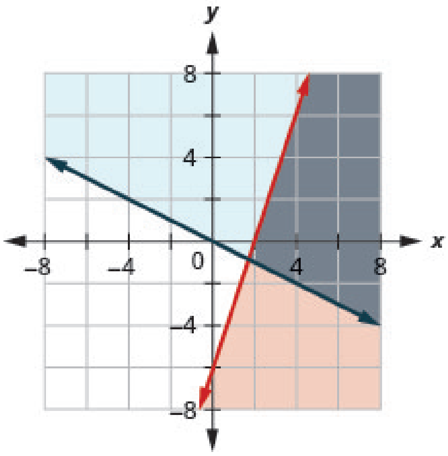

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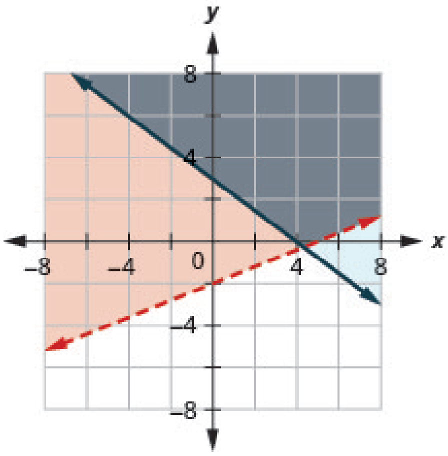

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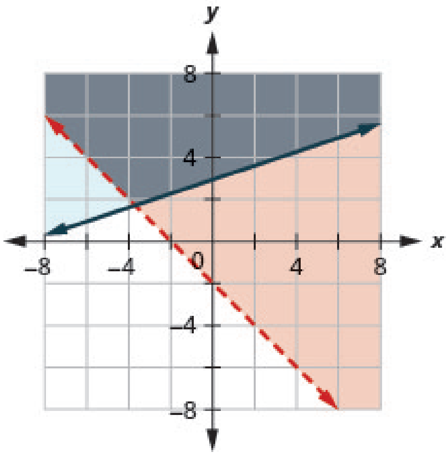

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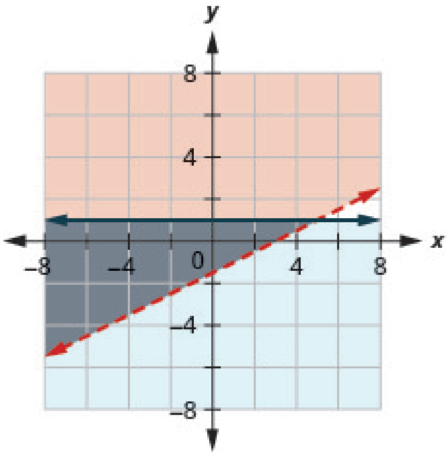

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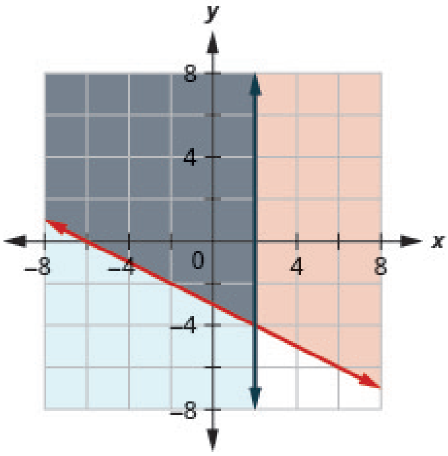

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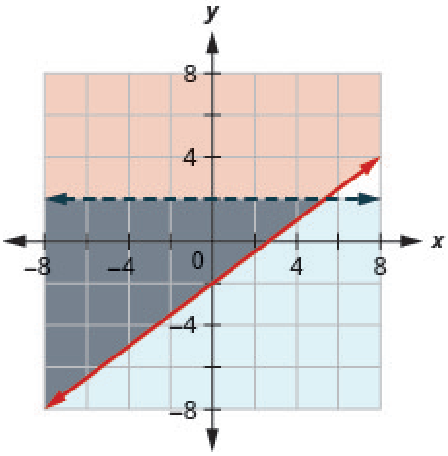

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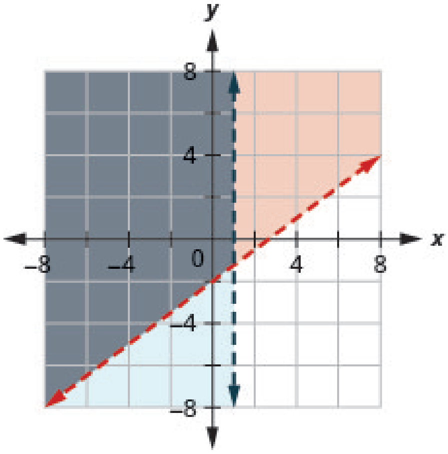

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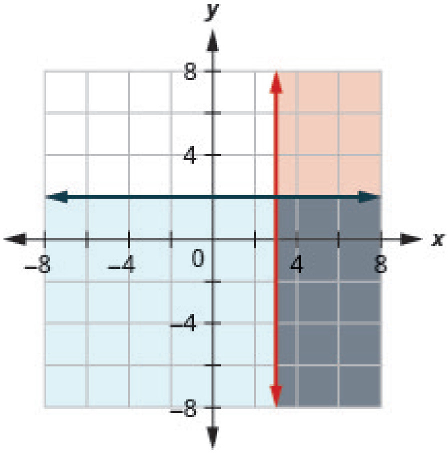
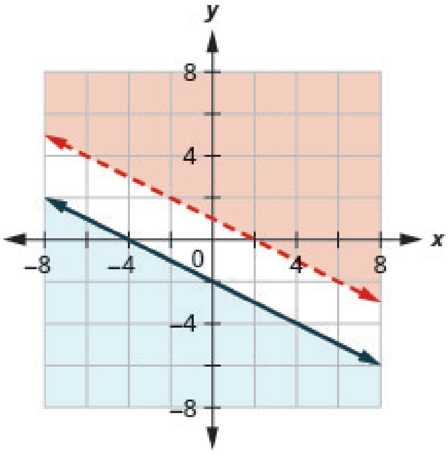
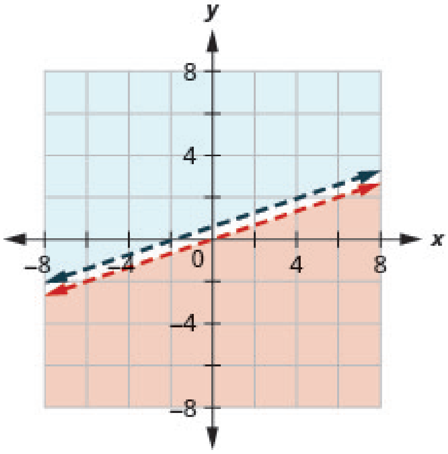


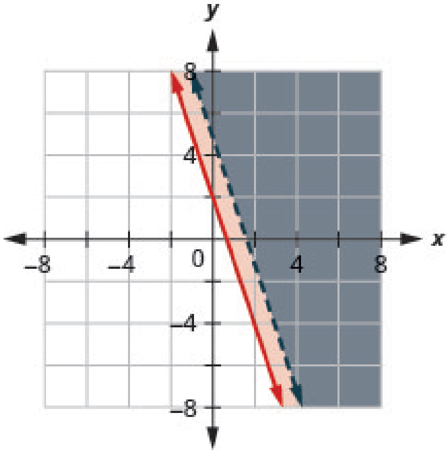
16. No solution



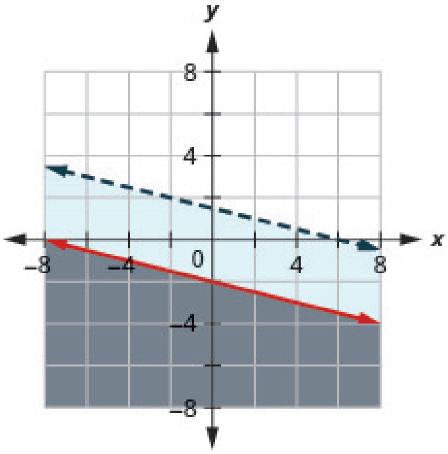
17. No solution



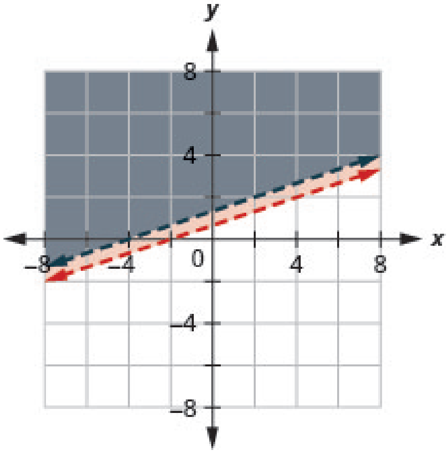
18.



19. $x + 4y < 6$

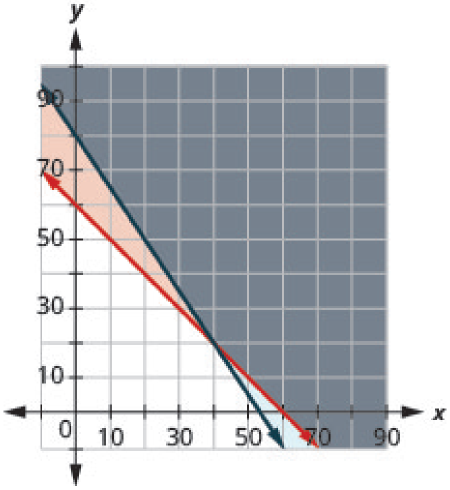


20. $-2x + 6y > 8$



21. a) $\begin{cases} p + l \geq 60 \\ 15p + 10l \geq 800 \end{cases}$

b)

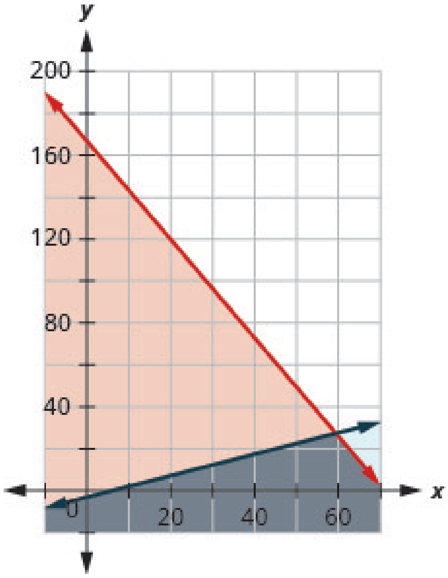


c) No

d) Yes

22. a) $\begin{cases} 7p + 3c \leq 500 \\ p \geq 2c + 4 \end{cases}$

b)

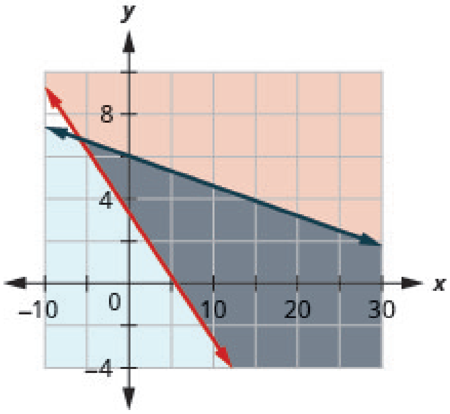


c) Yes

d) No

23. a) $\begin{cases} 90b + 150g \geq 500 \\ 0.35b + 2.50g \leq 15 \end{cases}$

b)

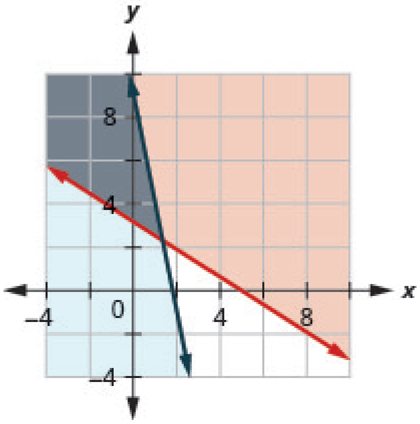


c) No

d) Yes

24. a) $\begin{cases} 7c + 11p \geq 35 \\ 110c + 22p \leq 200 \end{cases}$

b)

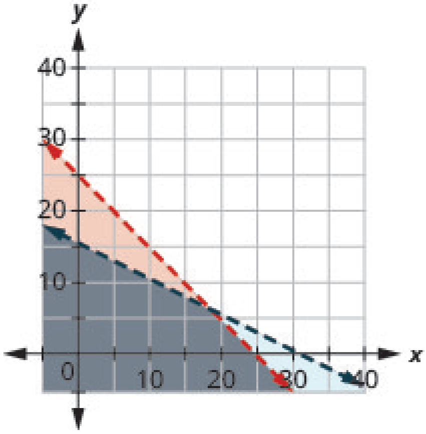


c) Yes

d) No

25. a) $\begin{cases} 3a + 3c < 75 \\ 2a + 4c < 62 \end{cases}$

b)



c) No

d) Yes

5. Trigonometry

Trigonometry is a part of geometry that takes its origin in the ancient study of the relationship of the sides and angles of a right triangle. “Trigon” from Greek means triangle and “metron” means measure.

Applications of trigonometry are essential to many disciplines like carpentry, engineering, surveying, and astronomy, just to name a few.

How tall is the Riverpole? Do we have to climb the pole to find out? Fortunately, with the knowledge of trigonometry, we can find out the measurements of tall objects without too much hassle.

In this chapter we will explore the basic properties of angles and triangles, and the applications of the Pythagorean Theorem and trigonometric ratios.

5.1 Use Properties of Angles, Triangles, and the Pythagorean Theorem

Learning Objectives

By the end of this section it is expected that you will be able to

- Use the properties of angles
- Use the properties of triangles
- Use the Pythagorean Theorem

Use the Properties of Angles

Are you familiar with the phrase 'do a 180'? It means to make a full turn so that you face the opposite direction. It comes from the fact that the measure of an angle that makes a straight line is 180 degrees. See (Figure 1).

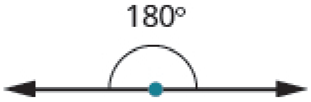


Figure 1

An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle and the common endpoint is called the vertex. An angle is named by its vertex. In (Figure 2), $\angle A$ is the angle with vertex at point A. The measure of $\angle A$ is written $m\angle A$.
$\angle A$ is the angle with vertex at $\text{point } A$.

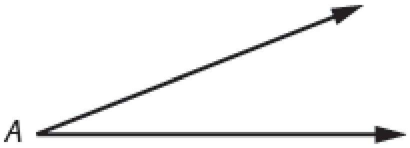


Figure 2

We measure angles in degrees, and use the symbol ° to represent degrees. We use the abbreviation m to for the *measure* of an angle. So if $\angle A$ is 27°, we would write $m\angle A = 27$.

If the sum of the measures of two angles is $180°$, then they are called supplementary angles. In (Figure 3), each pair of angles is supplementary because their measures add to $180°$. Each angle is the *supplement* of the other.
The sum of the measures of supplementary angles is $180°$.

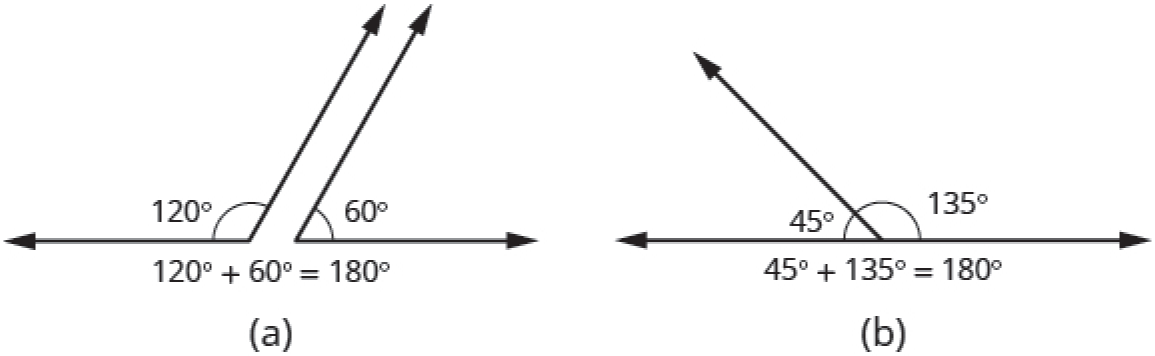


Figure 3

If the sum of the measures of two angles is $90°$, then the angles are complementary angles. In (Figure 4), each pair of angles is complementary, because their measures add to $90°$. Each angle is the *complement* of the other.
The sum of the measures of complementary angles is $90°$.

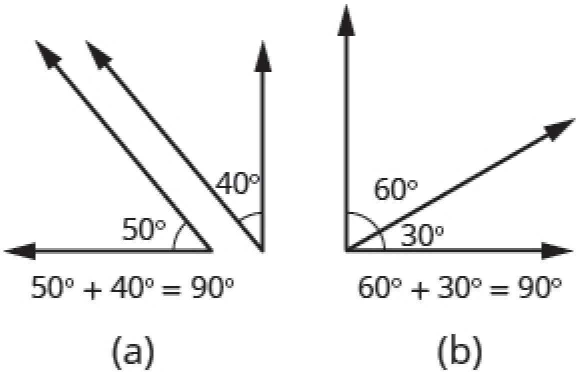


Figure 4

Supplementary and Complementary Angles

If the sum of the measures of two angles is $180°$, then the angles are supplementary.

If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180°$.

If the sum of the measures of two angles is $90°$, then the angles are complementary.

If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90°$.

In this section and the next, you will be introduced to some common geometry formulas. We will adapt our Problem Solving Strategy for Geometry Applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve geometric shapes, it will be helpful to draw a figure and then label it with the information from the problem. We will include this step in the Problem Solving Strategy for Geometry Applications.

HOW TO: Use a Problem Solving Strategy for Geometry Applications

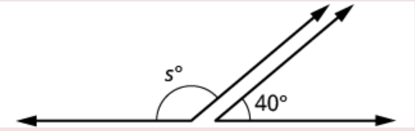
1. **Read** the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for and choose a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

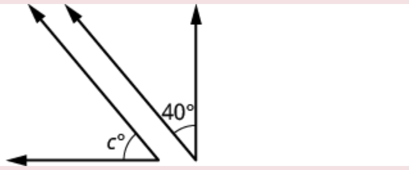
The next example will show how you can use the Problem Solving Strategy for Geometry Applications to answer questions about supplementary and complementary angles.

EXAMPLE 1

An angle measures $40°$. Find a) its supplement, and b) its complement.

Solution

a)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the supplement of a 40° angle.
Step 3. **Name.** Choose a variable to represent it.	let s = the measure of the supplement
Step 4. **Translate.** Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 180$ $s + 40 = 180$
Step 5. **Solve** the equation.	$s = 140$
Step 6. **Check:** $140 + 40 \overset{?}{=} 180$ $180 = 180$ ✓	
Step 7. **Answer** the question.	The supplement of the 40° angle is 140°.

b)	
Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the complement of a 40° angle.
Step 3. **Name.** Choose a variable to represent it.	let c = the measure of the complement
Step 4. **Translate.** Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 90$
Step 5. **Solve** the equation.	$c + 40 = 90$ $c = 50$
Step 6. **Check:** $50 + 40 \overset{?}{=} 90$ $90 = 90 \checkmark$	
Step 7. **Answer** the question.	The complement of the 40° angle is 50°.

TRY IT 1

An angle measures $25°$. Find its: a) supplement b) complement.

Show answer

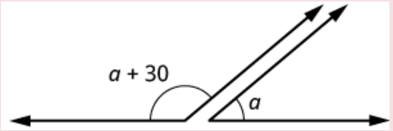
a. 155°
b. 65°

Did you notice that the words complementary and supplementary are in alphabetical order just like 90 and 180 are in numerical order?

EXAMPLE 2

Two angles are supplementary. The larger angle is $30°$ more than the smaller angle. Find the measure of both angles.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measures of both angles
Step 3. **Name.** Choose a variable to represent it. The larger angle is 30° more than the smaller angle.	let a = measure of smaller angle $a+30$ = measure of larger angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B = 180$
Step 5. **Solve** the equation.	$(a+30)+a=180$ $2a+30=180$ $2a=150$ $a=75$ measure of smaller angle $a+30$ measure of larger angle $75+30$ 105
Step 6. **Check:** $m\angle A + m\angle B = 180$ $75+105 \overset{?}{=} 180$ $180=180\checkmark$	
Step 7. **Answer** the question.	The measures of the angles are 75° and 105°.

TRY IT 2

Two angles are supplementary. The larger angle is $100°$ more than the smaller angle. Find the measures of both angles.

Show answer

40°, 140°

Use the Properties of Triangles

What do you already know about triangles? Triangle have three sides and three angles. Triangles are named by their vertices. The triangle in (Figure 5) is called ΔABC, read ‘triangle ABC’. We label each side with a lower case letter to match the upper case letter of the opposite vertex.

ΔABC has vertices $A, B,$ and C and sides $a, b,$ and $c.$

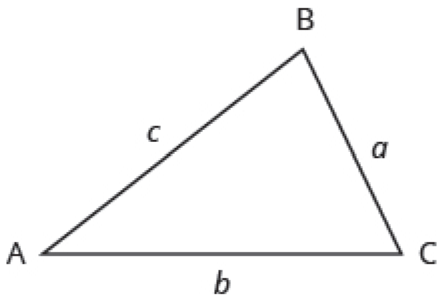


Figure 5

The three angles of a triangle are related in a special way. The sum of their measures is $180°$.
$m\angle A + m\angle B + m\angle C = 180°$

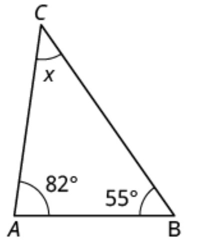
Sum of the Measures of the Angles of a Triangle

For any ΔABC, the sum of the measures of the angles is $180°$.
$m\angle A + m\angle B + m\angle C = 180°$

EXAMPLE 3

The measures of two angles of a triangle are $55°$ and $82°$. Find the measure of the third angle.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of the third angle in a triangle
Step 3. **Name.** Choose a variable to represent it.	let x = the measure of the angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. **Solve** the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x = 43$
Step 6. **Check:** $55 + 82 + 43 \overset{?}{=} 180$ $180 = 180$ ✓	
Step 7. **Answer** the question.	The measure of the third angle is 43 degrees.

TRY IT 3

The measures of two angles of a triangle are $31°$ and $128°$. Find the measure of the third angle.

Show answer
21°

Right Triangles

Some triangles have special names. We will look first at the right triangle. A right triangle has one $90°$ angle, which is often marked with the symbol shown in (Figure 6).

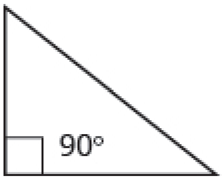


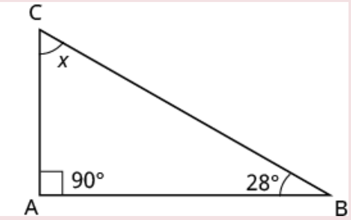
Figure 6

If we know that a triangle is a right triangle, we know that one angle measures $90°$ so we only need the measure of one of the other angles in order to determine the measure of the third angle.

EXAMPLE 4

One angle of a right triangle measures $28°$. What is the measure of the third angle?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of an angle
Step 3. **Name.** Choose a variable to represent it.	let x = the measure of the angle
Step 4. **Translate.** Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. **Solve** the equation.	$x + 90 + 28 = 180$ $x + 118 = 180$ $x = 62$
Step 6. **Check:** $180 \overset{?}{=} 90 + 28 + 62$ $180 = 180 \checkmark$	
Step 7. **Answer** the question.	The measure of the third angle is 62°.

TRY IT 4

One angle of a right triangle measures $56°$. What is the measure of the other angle?

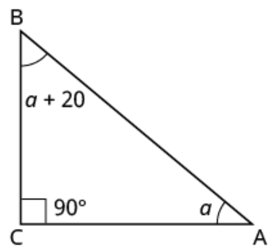
Show answer
34°

In the examples so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. So we will wait to draw the figure until we write expressions for all the angles we are looking for.

EXAMPLE 5

The measure of one angle of a right triangle is $20°$ more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the measures of all three angles
Step 3. **Name.** Choose a variable to represent it. Now draw the figure and label it with the given information.	Let a = 1st angle $a + 20$ = 2nd angle 90 = 3rd angle (the right angle) 
Step 4. **Translate.** Write the appropriate formula and substitute into the formula.	$m\angle A + m\angle B + m\angle C = 180$ $a + (a + 20) + 90 = 180$
Step 5. **Solve** the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35$ first angle $a + 20$ second angle $35 + 20$ 55 90 third angle
Step 6. **Check:** $35 + 55 + 90 \overset{?}{=} 180$ $180 = 180 ✓$	
Step 7. **Answer** the question.	The three angles measure 35°, 55°, and 90°.

TRY IT 5

The measure of one angle of a right triangle is $50°$ more than the measure of the smallest angle. Find the measures of all three angles.

Show answer

20°, 70°, 90°

Similar Triangles

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with similar figures. In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles are have the same measures.

The two triangles in (Figure 7) are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ and their corresponding angles have equal measures.

ΔABC and ΔXYZ are similar triangles. Their corresponding sides have the same ratio and the corresponding angles have the same measure.

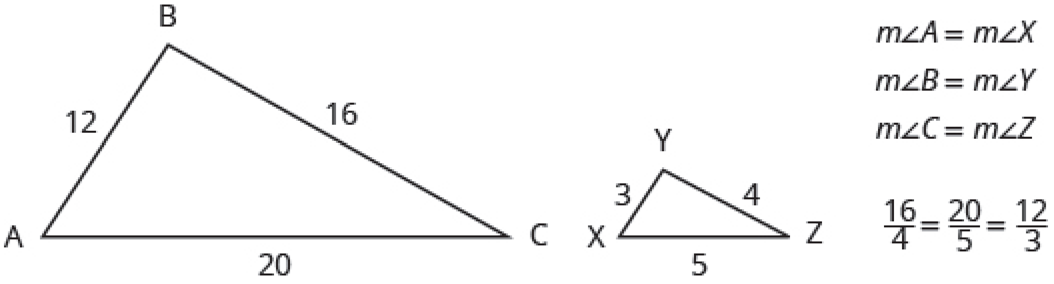
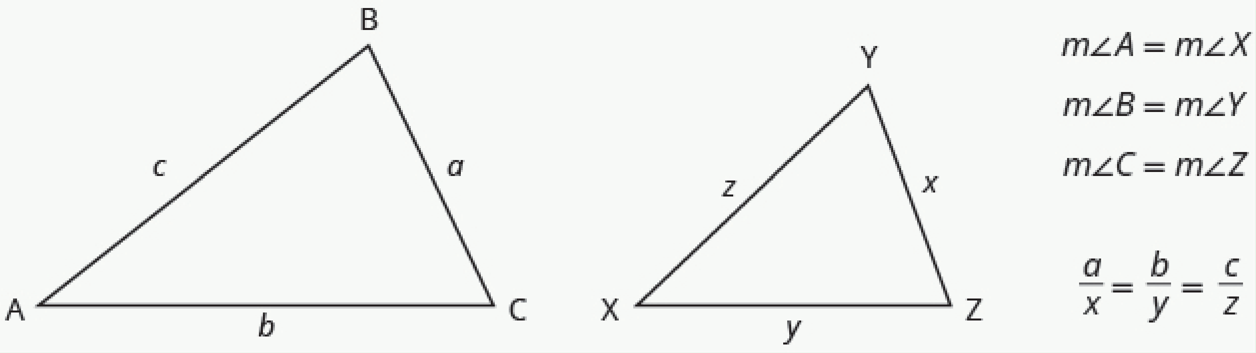


Figure 7

Properties of Similar Triangles

If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.



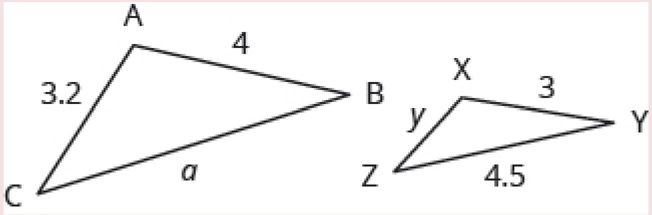
The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in ΔABC:

$$\text{the length } a \text{ can also be written } BC$$
$$\text{the length } b \text{ can also be written } AC$$
$$\text{the length } c \text{ can also be written } AB$$

We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

EXAMPLE 6

ΔABC and ΔXYZ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.

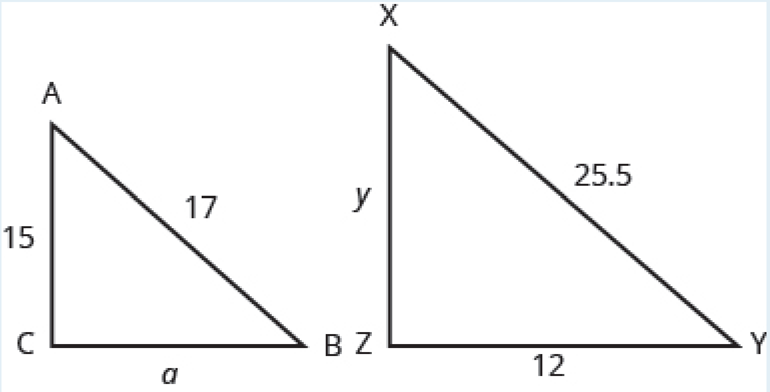


Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	The figure is provided.
Step 2. **Identify** what you are looking for.	The length of the sides of similar triangles
Step 3. **Name.** Choose a variable to represent it.	Let *a* = length of the third side of ΔABC *y* = length of the third side ΔXYZ
Step 4. **Translate.**	The triangles are similar, so the corresponding sides are in the same ratio. So $\frac{AB}{XY}=\frac{BC}{YZ}=\frac{AC}{XZ}$ Since the side $AB=4$ corresponds to the side $XY=3$, we will use the ratio $\frac{\text{AB}}{\text{XY}}=\frac{4}{3}$ to find the other sides. Be careful to match up corresponding sides correctly. To find *a*: To find *y*: sides of large triangle → $\frac{AB}{XY}=\frac{BC}{YZ}$ $\frac{AB}{XY}=\frac{AC}{XZ}$ sides of small triangle → $\frac{4}{3}=\frac{a}{4.5}$ $\frac{4}{3}=\frac{3.2}{y}$
Step 5. **Solve** the equation.	$3a=4(4.5)$ $4y=3(3.2)$ $3a=18$ $4y=9.6$ $a=6$ $y=2.4$
Step 6. **Check.**	$\frac{4}{3}\overset{?}{=}\frac{6}{4.5}$ $\frac{4}{3}\overset{?}{=}\frac{3.2}{2.4}$ $4(4.5)\overset{?}{=}6(3)$ $4(2.4)\overset{?}{=}3.2(3)$ $18=18$ ✓ $9.6=9.6$ ✓
Step 7. **Answer** the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

TRY IT 6

ΔABC is similar to ΔXYZ. Find a.



Show answer
8

Use the Pythagorean Theorem

The Pythagorean Theorem is a special property of right triangles that has been used since ancient times. It is named after the Greek philosopher and mathematician Pythagoras who lived around 500 BCE.

Remember that a right triangle has a $90°$ angle, which we usually mark with a small square in the corner. The side of the triangle opposite the $90°$ angle is called the hypotenuse, and the other two sides are called the legs. See (Figure 8).

In a right triangle, the side opposite the $90°$ angle is called the hypotenuse and each of the other sides is called a leg.

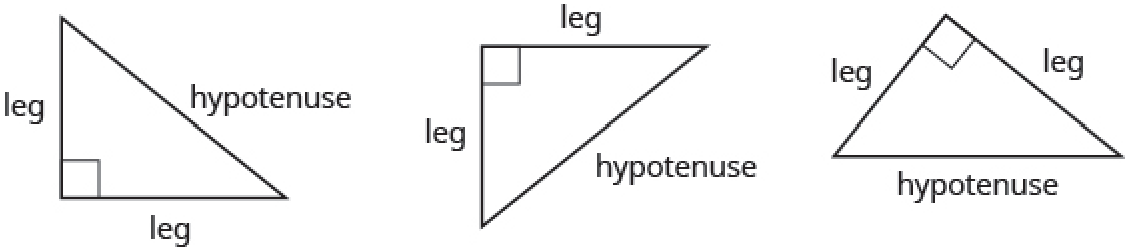


Figure 8

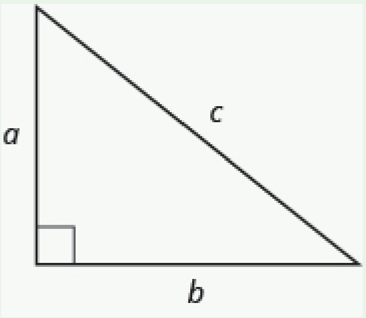
The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

The Pythagorean Theorem

In any right triangle $\Delta ABC,$

$a^2 + b^2 = c^2$

where c is the length of the hypotenuse a and b are the lengths of the legs.



To solve problems that use the Pythagorean Theorem, we will need to find square roots. We defined the notation $\sqrt{m}$ in this way:

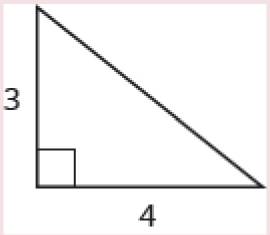
If $m = n^2$, then $\sqrt{m} = n$ for $n \geq 0$

For example, we found that $\sqrt{25}$ is 5 because $5^2 = 25$.

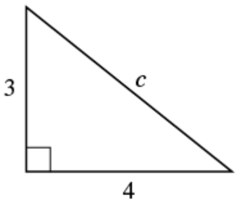
We will use this definition of square roots to solve for the length of a side in a right triangle.

EXAMPLE 7

Use the Pythagorean Theorem to find the length of the hypotenuse.

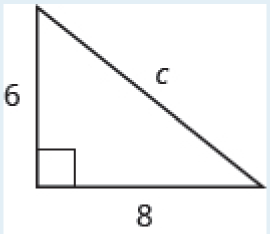


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length of the hypotenuse of the triangle
Step 3. **Name.** Choose a variable to represent it.	Let $c =$ the length of the hypotenuse 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $3^2 + 4^2 = c^2$
Step 5. **Solve** the equation.	$9 + 16 = c^2$ $25 = c^2$ $\sqrt{25} = c^2$ $5 = c$
Step 6. **Check:** $3^2 + 4^2 = 5^2$ $9 + 16 \overset{?}{=} 25$ $25 = 25$ ✓	
Step 7. **Answer** the question.	The length of the hypotenuse is 5.

TRY IT 7

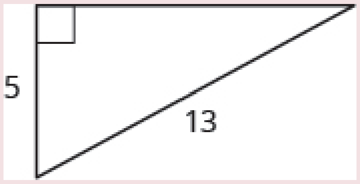
Use the Pythagorean Theorem to find the length of the hypotenuse.



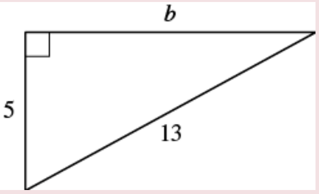
Show answer
10

EXAMPLE 8

Use the Pythagorean Theorem to find the length of the longer leg.

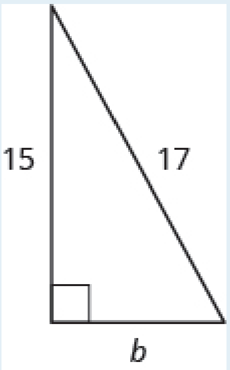


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	The length of the leg of the triangle
Step 3. **Name.** Choose a variable to represent it.	Let $b =$ the leg of the triangle Label side *b* 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $5^2 + b^2 = 13^2$
Step 5. **Solve** the equation. Isolate the variable term. Use the definition of the square root. Simplify.	$25 + b^2 = 169$ $b^2 = 144$ $b^2 = \sqrt{144}$ $b = 12$
Step 6. **Check:**	$5^2 + 12^2 \overset{?}{=} 13^2$ $25 + 144 \overset{?}{=} 169$ $169 = 169$ ✓
Step 7. **Answer** the question.	The length of the leg is 12.

TRY IT 8

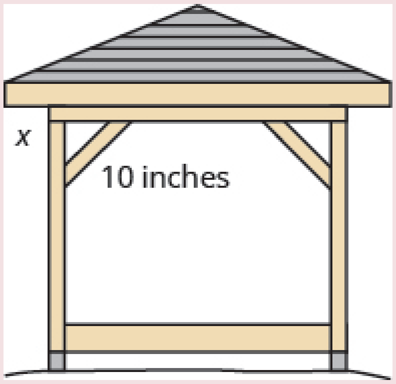
Use the Pythagorean Theorem to find the length of the leg.



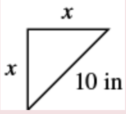
Show answer
8

EXAMPLE 9

Kelvin is building a gazebo and wants to brace each corner by placing a 10-inch wooden bracket diagonally as shown. How far below the corner should he fasten the bracket if he wants the distances from the corner to each end of the bracket to be equal? Approximate to the nearest tenth of an inch.



Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the distance from the corner that the bracket should be attached
Step 3. **Name.** Choose a variable to represent it.	Let x = the distance from the corner 
Step 4. **Translate.** Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $x^2 + x^2 = 10^2$
Step 5. **Solve** the equation. Isolate the variable. Use the definition of the square root. Simplify. Approximate to the nearest tenth.	$2x^2 = 100$ $x^2 = 50$ $x = \sqrt{50}$ $b \approx 7.1$
Step 6. **Check:** $a^2 + b^2 = c^2$ $(7.1)^2 + (7.1)^2 \overset{?}{\approx} 10^2$ Yes.	
Step 7. **Answer** the question.	Kelvin should fasten each piece of wood approximately 7.1″ from the corner.

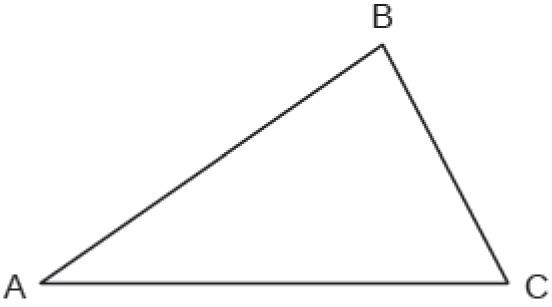
TRY IT 9

John puts the base of a 13-ft ladder 5 feet from the wall of his house. How far up the wall does the ladder reach?

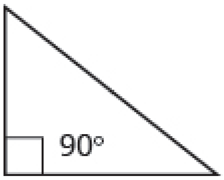
Show answer
12 feet

Key Concepts

- **Supplementary and Complementary Angles**
    - If the sum of the measures of two angles is 180°, then the angles are supplementary.
    - If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180$.
    - If the sum of the measures of two angles is 90°, then the angles are complementary.
    - If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90$.
- **Solve Geometry Applications**
    1. Read the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
    2. Identify what you are looking for.
    3. Name what you are looking for and choose a variable to represent it.
    4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
    5. Solve the equation using good algebra techniques.
    6. Check the answer in the problem and make sure it makes sense.
    7. Answer the question with a complete sentence.
- **Sum of the Measures of the Angles of a Triangle**
    - For any ΔABC, the sum of the measures is 180°
    - $m\angle A + m\angle B = 180$



- **Right Triangle**
    - A right triangle is a triangle that has one 90° angle, which is often marked with a ? symbol.



- **Properties of Similar Triangles**
    - If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths have the same ratio.

Glossary

angle

An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle.

complementary angles

If the sum of the measures of two angles is $90°$, then they are called complementary angles.

hypotenuse

The side of the triangle opposite the 90° angle is called the hypotenuse.

legs of a right triangle

The sides of a right triangle adjacent to the right angle are called the legs.

right triangle

A right triangle is a triangle that has one $90°$ angle.

similar figures

In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures.

supplementary angles

If the sum of the measures of two angles is $180°$, then they are called supplementary angles.

triangle

A triangle is a geometric figure with three sides and three angles.

vertex of an angle

When two rays meet to form an angle, the common endpoint is called the vertex of the angle.

5.1 Exercise Set

In the following exercises, find a) the supplement and b) the complement of the given angle.

1. 53°
2. 29°

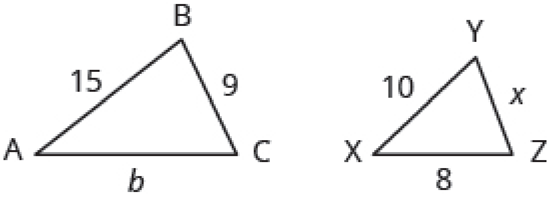
In the following exercises, use the properties of angles to solve.

3. Find the supplement of a 135° angle.
4. Find the complement of a 27.5° angle.
5. Two angles are supplementary. The larger angle is 56° more than the smaller angle. Find the measures of both angles.
6. Two angles are complementary. The smaller angle is 34° less than the larger angle. Find the measures of both angles.

In the following exercises, solve using properties of triangles.

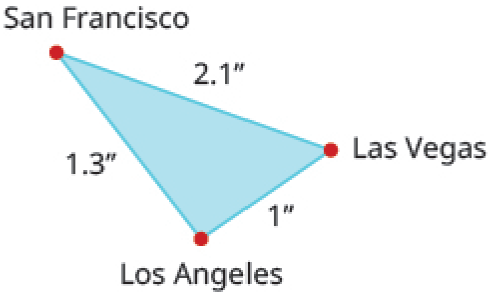
7. The measures of two angles of a triangle are 26° and 98°. Find the measure of the third angle.
8. The measures of two angles of a triangle are 105° and 31°. Find the measure of the third angle.
9. One angle of a right triangle measures 33°. What is the measure of the other angle?
10. One angle of a right triangle measures 22.5°. What is the measure of the other angle?
11. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.
12. The angles in a triangle are such that the measure of one angle is twice the measure of the smallest angle, while the measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



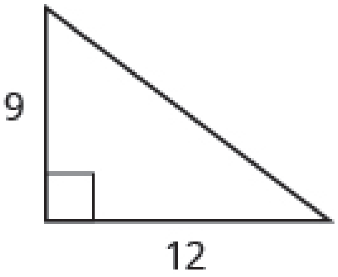
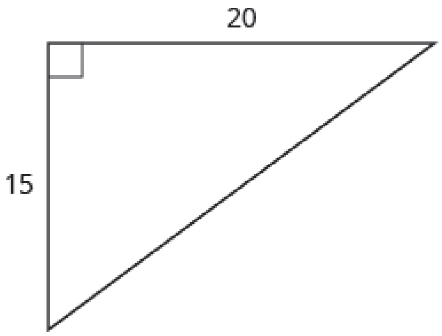
13. side b

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. The actual distance from Los Angeles to Las Vegas is 270 miles.



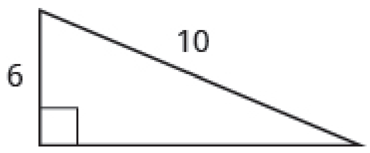
14. Find the distance from Los Angeles to San Francisco.

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

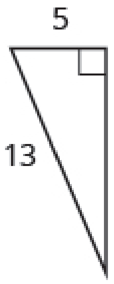
15. 	16. 

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

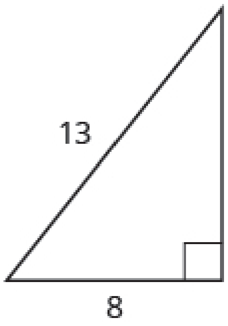
17.



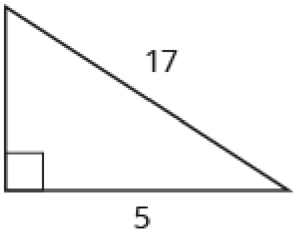
18.



19.

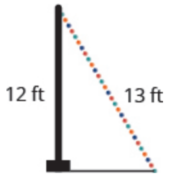


20.

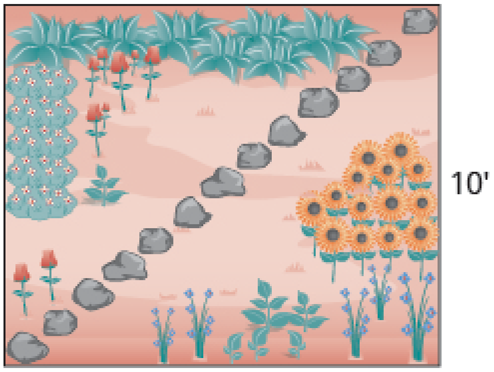


In the following exercises, solve. Approximate to the nearest tenth, if necessary.

21. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display. How far from the base of the pole should the end of the string of lights be anchored?



22. Chi is planning to put a path of paving stones through her flower garden. The flower garden is a square with sides of 10 feet. What will the length of the path be?



23. Building a scale model Joe wants to build a doll house for his daughter. He wants the doll house to look just like his house. His house is 30 feet wide and 35 feet tall at the highest point of the roof. If the dollhouse will be 2.5 feet wide, how tall will its highest point be?

Answers:

1.
    a. 127°
    b. 37°
2.
    a. 151°
    b. 61°
3. 45°
4. 62.5°
5. 62°, 118°
6. 62°, 28°
7. 56°
8. 44°
9. 57°
10. 67.5°
11. 45°, 45°, 90°
12. 30°, 60°, 90°
13. 12
14. 351 miles
15. 15
16. 25
17. 8
18. 12
19. 10.2
20. 8
21. 21.5 feet
22. 14.1 feet
23. 2.9 feet

Attributions:

This chapter has been adapted from "Use Properties of Angles, Triangles, and the Pythagorean Theorem" in *Prealgebra* (OpenStax) by Lynn Marecek, MaryAnne Anthony-Smith, and Andrea Honeycutt Mathis, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Adaptation Statement for more information.

5.2 Solve Applications: Sine, Cosine and Tangent Ratios.

Learning Objectives

By the end of this section it is expected that you will be able to

- Find missing side of a right triangle using sine, cosine, or tangent ratios
- Find missing angle of a right triangle using sine, cosine, or tangent ratios
- Solve applications using right angle trigonometry

Now, that we know the fundamentals of algebra and geometry associated with a right triangle, we can start exploring trigonometry. Many real life problems can be represented and solved using right angle trigonometry.

Sine, Cosine, and Tangent Ratios

We know that any right triangle has three sides and a right angle. The side opposite to the right angle is called the hypotenuse. The other two angles in a right triangle are acute angles (with a measure less than 90 degrees). One of those angles we call reference angle and we use θ (theta) to represent it.

The hypotenuse is always the longest side of a right triangle. The other two sides are called opposite side and adjacent side. The names of those sides depends on which of the two acute angles is being used as a reference angle.

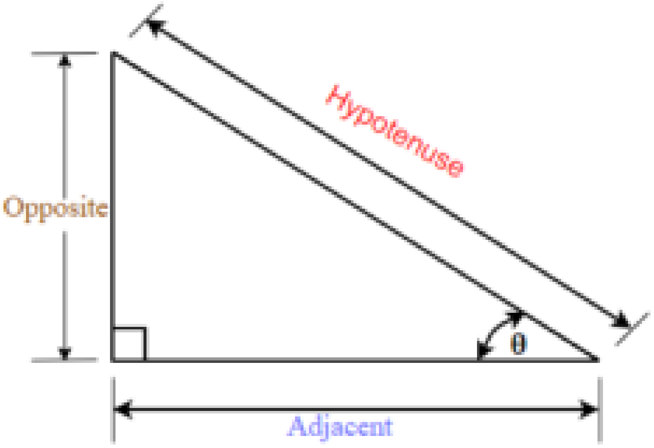
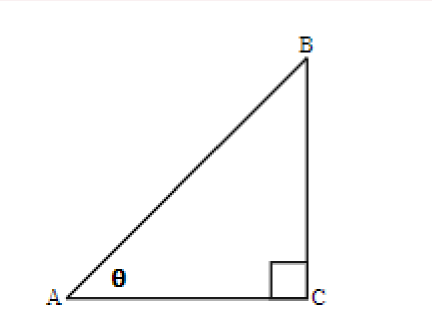


Figure 1.

In the right triangle each side is labeled with a lowercase letter to match the uppercase letter of the opposite vertex.

EXAMPLE 1

Label the sides of the triangle and find the hypotenuse, opposite, and adjacent.



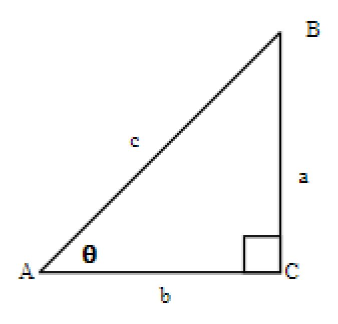
Solution

We labeled the sides with a lowercase letter to match the uppercase letter of the opposite vertex.

c is hypotenuse

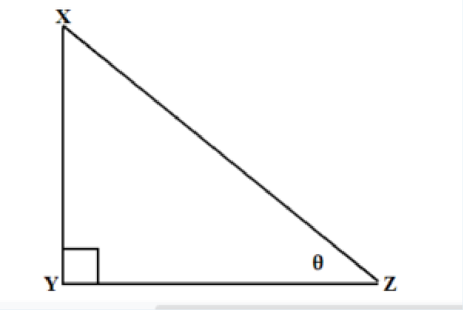
a is opposite

b is adjacent



TRY IT 1

Label the sides of the triangle and find the hypotenuse, opposite and adjacent.



Show answer
y is hypotenuse

z is opposite

x is adjacent

Trigonometric Ratios

Trigonometric ratios are the ratios of the sides in the right triangle. For any right triangle we can define three basic trigonometric ratios: sine, cosine, and tangent.

Let us refer to Figure 1 and define the three basic trigonometric ratios as:

Three Basic Trigonometric Ratios

- sine θ = $\frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}$
- cosine θ = $\frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}$
- tangent θ = $\frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}$

Where θ is the measure of a reference angle measured in degrees.

Very often we use the abbreviations for sine, cosine, and tangent ratios.

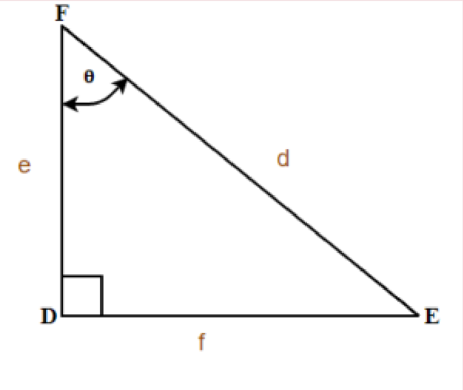
- sin θ = $\frac{\text{opp}}{\text{hyp}}$
- cos θ = $\frac{\text{adj}}{\text{hyp}}$
- tan θ = $\frac{\text{opp}}{\text{adj}}$

Some people remember the definition of the trigonometric ratios as SOH CAH TOA.

Let's use the ΔDEF from Example 1 to find the three ratios.

EXAMPLE 2

For the given triangle find the sine, cosine and tangent ratio.



Solution

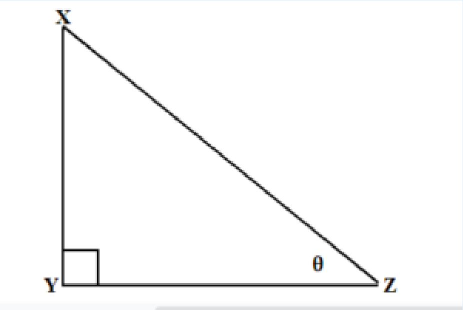
$\sin \theta = \frac{f}{d}$

$\cos \theta = \frac{e}{d}$

$\tan \theta = \frac{f}{e}$

TRY IT 2

For the given triangle find the sine cosine and tangent ratio.



Show answer

$\sin \theta = \frac{z}{y}$

$\cos \theta = \frac{x}{y}$

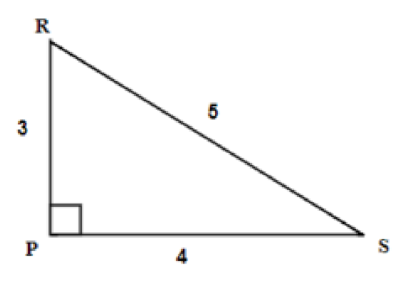
$\tan \theta = \frac{z}{x}$

In Example 2, our reference angles can be $\angle E$ or $\angle F$. Using the definition of trigonometric ratios, we can write sinE = $\frac{e}{d}$, cosE = $\frac{f}{d}$, and tanE = $\frac{e}{f}$.

When calculating we will usually round the ratios to four decimal places and at the end our final answer to one decimal place unless stated otherwise.

EXAMPLE 3

For the given triangle find the sine, cosine and tangent ratios. If necessary round to four decimal places.



Solution

We have two possible reference angles: R an S.

Using the definitions, the trigonometric ratios for angle R are:

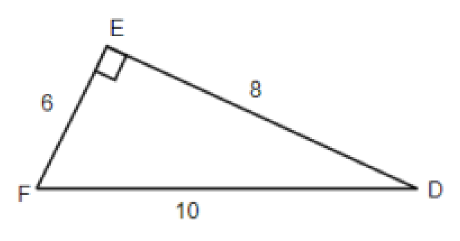
- sin R= $\frac{4}{5}$ = 0.8
- cos R= $\frac{3}{5}$ = 0.6
- tan R=$\frac{4}{3}$ = 1.3333…

Using the definitions, the trigonometric ratios for angle S:

- sin S = $\frac{3}{5}$ = 0.6
- cos S = $\frac{4}{5}$ = 0.8
- tan S = $\frac{3}{4}$ = 0.75

TRY IT 3

For the given triangle find the sine, cosine, and tangent ratios. If necessary round to four decimal places.



Show answer

- sin F = $\frac{8}{10}$ = 0.8
- cos F = $\frac{6}{10}$ =0.6
- tan F = $\frac{8}{6}$ = 1.3333…
- sin D =$\frac{6}{10}$ = 0.6
- cos D =$\frac{8}{10}$ = 0.8
- tan D =$\frac{6}{8}$ = 0.75

Now, let us use a scientific calculator to find the trigonometric ratios. Can you find the sin, cos, and tan buttons on your calculator? To find the trigonometric ratios make sure your calculator is in Degree Mode.

EXAMPLE 4

Using a calculator find the trigonometric ratios. If necessary, round to 4 decimal places.

a) sin 30°

b) cos 45°

c) tan 60°

Solution

Make sure your calculator is in Degree Mode.

a) Using a calculator find that sin 30° = 0.5

b) Using a calculator find that cos 45° = 0.7071 Rounded to 4 decimal places.

c) Using a calculator find that tan 60° = 1.7321 Rounded to 4 decimal places.

TRY IT 4

Find the trigonometric ratios. If necessary, round to 4 decimal places.

a) sin 60°

b) cos 30°

c) tan 45°

Show answer

a) sin 60° = 0.8660

b) cos 30° = 0.8660

c) tan 45° = 1

Finding Missing Sides of a Right Triangle

In this section you will be using trigonometric ratios to solve right triangle problems. We will adapt our problem solving strategy for trigonometry applications. In addition, since those problems will involve the right triangle, it is helpful to draw it (if the drawing is not given) and label it with the given information.We will include this in the first step of the problem solving strategy for trigonometry applications.

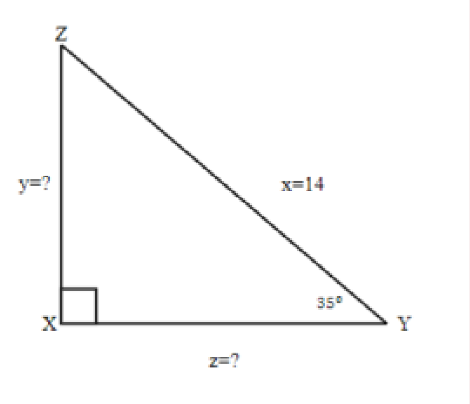
HOW TO: Solve Trigonometry Applications

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
2. **Identify** what we are looking for.
3. **Label** what we are looking for by choosing a variable to represent it.
4. **Find** the required trigonometric ratio.
5. **Solve** the ratio using good algebra techniques.
6. **Check** the answer by substituting it back into the ratio in step 4 and by making sure it makes sense in the context of the problem.
7. **Answer** the question with a complete sentence.

In the next few examples, having given the measure of one acute angle and the length of one side of the right triangle, we will solve the right triangle for the missing sides.

EXAMPLE 5

Find the missing sides. Round your final answer to two decimal places

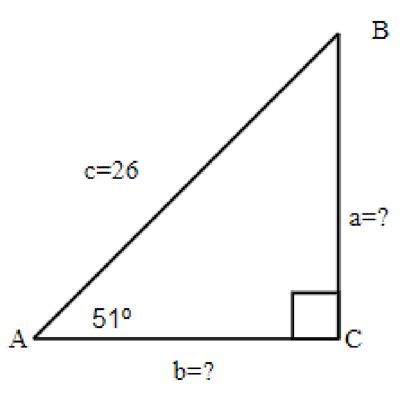


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle Y is our reference angle, y is opposite side, z is adjacent side, and x=14 is the hypotenuse.	
2. **Identify** what we are looking for.	a) the opposite side	b) adjacent side
3.**Label** what we are looking for by choosing a variable to represent it.	y=?	z=?
4. **Find** the required trigonometric ratio.	$\sin 35° = \frac{y}{14}$	$\cos 35° = \frac{z}{14}$
5. **Solve** the ratio using good algebra techniques.	14 sin 35° = y 8.03 = y	14 cos 35° = z 11.47 = z
6. **Check** the answer in the problem and by making sure it makes sense.	$0.57 \overset{?}{=} 8.03 \div 14$ 0.57 = 0.57 ✓	$0.82 \overset{?}{=} 11.47 \div 14$ 0.82 = 0.82 ✓
7. **Answer** the question with a complete sentence.	The opposite side is 8.03	The adjacent side is 11.47

TRY IT 5

Find the missing sides. Round your final answer to one decimal place.



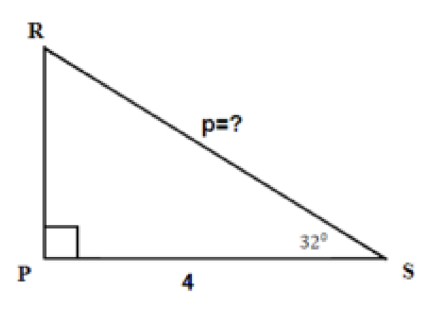
Show answer

a = 20.2

b = 16.4

EXAMPLE 6

Find the hypotenuse. Round your final answer to one decimal place.

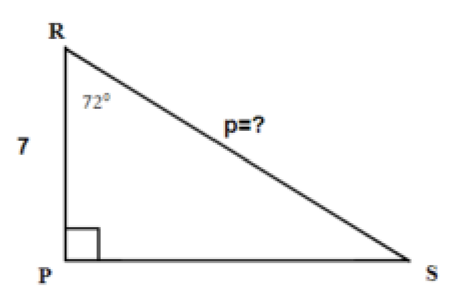


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle S is our reference angle, s is opposite side, r = 4 is the adjacent side, and p is the hypotenuse
2. **Identify** what we are looking for.	the hypotenuse
3.**Label** what we are looking for by choosing a variable to represent it.	p=?
4. **Find** the required trigonometric ratio.	$\cos 32° = \frac{4}{p}$
5. **Solve** the ratio using good algebra techniques.	$0.8480 = \frac{4}{p}$ p = 4.7170 Rounding the ratios to 4 decimal places
6. **Check** the answer in the problem and by making sure it makes sense.	$0.8480 \overset{?}{=} \frac{4}{4.7170}$ 0.8480 = 0.8480 ✓
7. **Answer** the question with a complete sentence.	The hypotenuse is 4.7 Round my final answer to one decimal place.

TRY IT 6

Find the hypotenuse. Round your final answer to one decimal place.



Show answer

p = 22.7

Finding Missing Angles of a Right Triangle

Sometimes we have a right triangle with only the sides given. How can we find the missing angles? To find the missing angles, we use the inverse of the trigonometric ratios. The inverse buttons sin^{-1}, cos^{-1}, and tan^{-1} are on your scientific calculator.

EXAMPLE 7

Find the angles. Round your final answer to one decimal place.

a) sin A = 0.5

b) cos B = 0.9735

c) tan C = 2.89358

Solution

Use your calculator and press the 2nd FUNCTION key and then press the SIN, COS, or TAN key

a) A = sin^{-1}0.5

$\angle A$ = 30°

b) B = cos^{-1}0.9735

$\angle B$ = 13.2° Rounded to one decimal place

c) C = tan^{-1}2.89358

$\angle C$ = 70.9° Rounded to one decimal place

TRY IT 7

Find the angles. Round your final answer to one decimal place.

a) sin X = 1

b) cos Y = 0.375

c) tan Z = 1.676767

Show answer

a) $\angle X$ = 90°

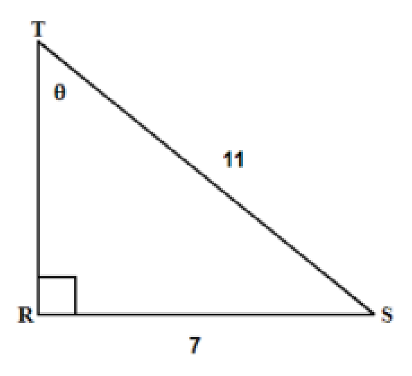
b) $\angle Y$ = 68°

c) $\angle Z$ = 59.2°

In the example below we have a right triangle with two sides given. Our acute angles are missing. Let us see what the steps are to find the missing angles.

EXAMPLE 8

Find the missing $\angle T$. Round your final answer to one decimal place.

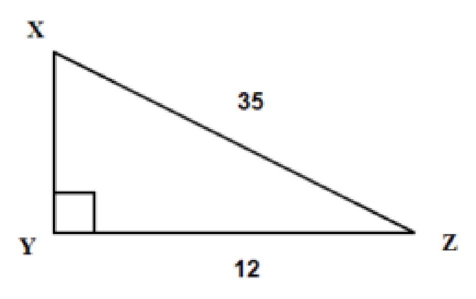


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle T is our reference angle, t = 7 is the opposite side, s is adjacent side, and r =11 is the hypotenuse
2. **Identify** what we are looking for.	angle T
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle T$ =?
4. **Find** the required trigonometric ratio.	sin T = $\frac{7}{11}$
5. **Solve** the ratio using good algebra techniques.	sin T = 0.6364 T = $\sin^{-1}$0.6364 $\angle T$ = 39.5239°
6. **Check** the answer in the problem and by making sure it makes sense.	sin 39.5239° $\overset{?}{=}$ 0.6364 0.6364 = 0.6364 ✓
7. **Answer** the question with a complete sentence.	The missing angle T is 39.5°.

TRY IT 8

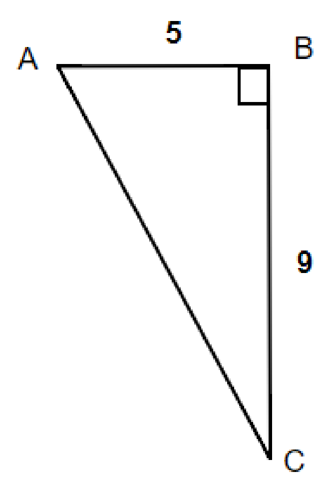
Find the missing angle X. Round your final answer to one decimal place.



Show answer
20.1°

EXAMPLE 9

Find the missing angle A. Round your final answer to one decimal place.

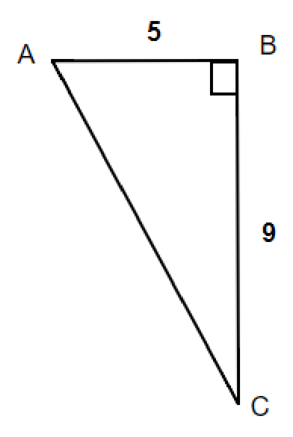


Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle A is our reference angle, $a = 9$ is the opposite side, $c = 5$ is the adjacent side, and b is the hypotenuse
2. **Identify** what we are looking for.	angle A
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle A$ =?
4. **Find** the required trigonometric ratio.	$\tan A = \frac{9}{5}$
5. **Solve** the ratio using good algebra techniques.	$\tan A = 1.8$ $A = \tan^{-1} 1.8$ $\angle A = 60.9°$
6. **Check** the answer in the problem and by making sure it makes sense.	$\tan 60.9° \overset{?}{=} 1.8$ $1.8 = 1.8$ ✓
7. **Answer** the question with a complete sentence.	The missing angle A is 60.9°.

TRY IT 9

Find the missing angle C. Round your final answer to one decimal place.



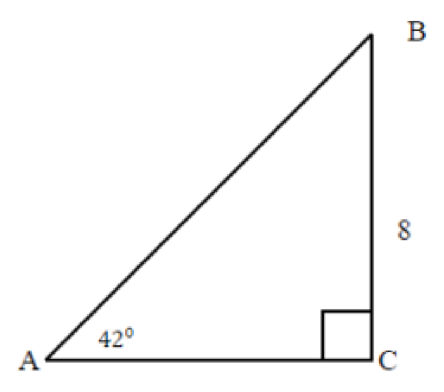
Show answer
29.1°

Solving a Right Triangle

From the section before we know that any triangle has three sides and three interior angles. In a right triangle, when all six parts of the triangle are known, we say that the right triangle is solved.

EXAMPLE 10

Solve the right triangle. Round your final answer to one decimal place.



Solution

Since the sum of angles in any triangle is 180°, the measure of angle B can be easy calculated.

$\angle B$ = 180° − 90° − 42°

$\angle B$ = 48°

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Angle A is our reference angle, a = 8 is the opposite side, b is the adjacent side, and c is the hypotenuse.	
2. **Identify** what we are looking for.	a) adjacent side	b) hypotenuse
3.**Label** what we are looking for by choosing a variable to represent it.	b = ?	c = ?
4. **Find** the required trigonometric ratio.	tan 42° = $\frac{8}{b}$	sin 42° = $\frac{8}{c}$
5. **Solve** the ratio using good algebra techniques.	0.9004 = $\frac{8}{b}$ 0.9004 b = 8 b = 8.8849	0.6691 = $\frac{8}{c}$ 0.6691 c = 8 c = 11.9563
6. **Check** the answer in the problem and by making sure it makes sense.	tan 42 ° $\stackrel{?}{=} \frac{8}{8.8849}$ 0.9 = 0.9 ✓	sin 42° $\stackrel{?}{=} \frac{8}{11.9563}$ 0.6691 = 0.6691 ✓
7. **Answer** the question with a complete sentence.	The adjacent side is 8.9. Rounded to one decimal place.	The hypotenuse is 12

We solved the right triangle

$\angle A$ = 42°

$\angle B$ = 48°

$\angle C$ = 90°

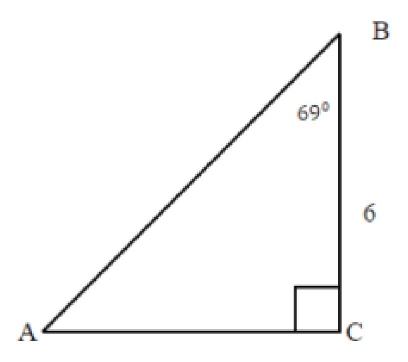
a = 8

b = 8.9

c = 12

TRY IT 10

Solve the right triangle. Round your final answer to one decimal place.



$\angle A$= 21°

$\angle B$ = 69°

$\angle C$ = 90°

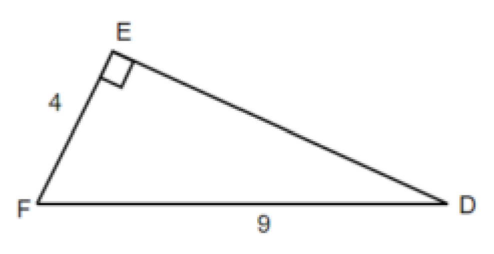
Show answer
a = 6

b = 15.6

c = 16.7

EXAMPLE 11

Solve the right triangle. Round to two decimal places.



Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	A drawing is given. Let angle D be our reference angle, d = 4 is the opposite side, f is the adjacent side, and e = 9 is the hypotenuse	
2. **Identify** what we are looking for.	a) angle D	b) adjacent
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle D$ =?	f = ?
4. **Find** the required trigonometric ratio.	sin D = $\frac{4}{9}$	$4^2 + f^2 = 9^2$
5. **Solve** the ratio using good algebra techniques.	sin D = 0.4444 D = $\sin^{-1}$0.4444 $\angle D$ = 26.3850°	$16 + f^2 = 81$ $f^2 = 81 – 16$ $f^2 = 65$ f = square root of 65 f = 8.06
6. **Check** the answer in the problem and by making sure it makes sense.	sin 26.3850° $\overset{?}{=}$ $\frac{4}{9}$ 0.4444 =0.4444 ✓	$4^2 + 8.06^2 \overset{?}{=} 9^2$ 81 = 81 ✓
7. **Answer** the question with a complete sentence.	The missing angle D is 26.39°.	The adjacent side is 8.06 Rounded to two decimal places

The missing angle F = 180° – 90° – 26.39° = 63.64°

We solved the right triangle

$\angle D$ = 26.39°

$\angle E$ = 90°

$\angle F$ = 63.61°

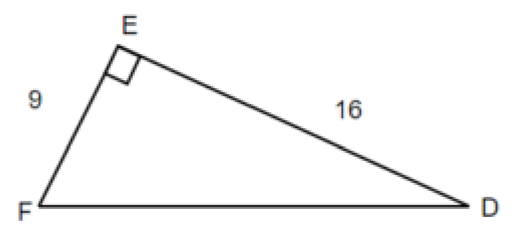
d = 4

e = 9

f = 8.06

TRY IT 11

Solve the right triangle. Round to one decimal place.



$\angle D$ = 29.3°

$\angle E$ = 90°

$\angle F$ = 60.7°

Show answer
d = 9

e = 18.4

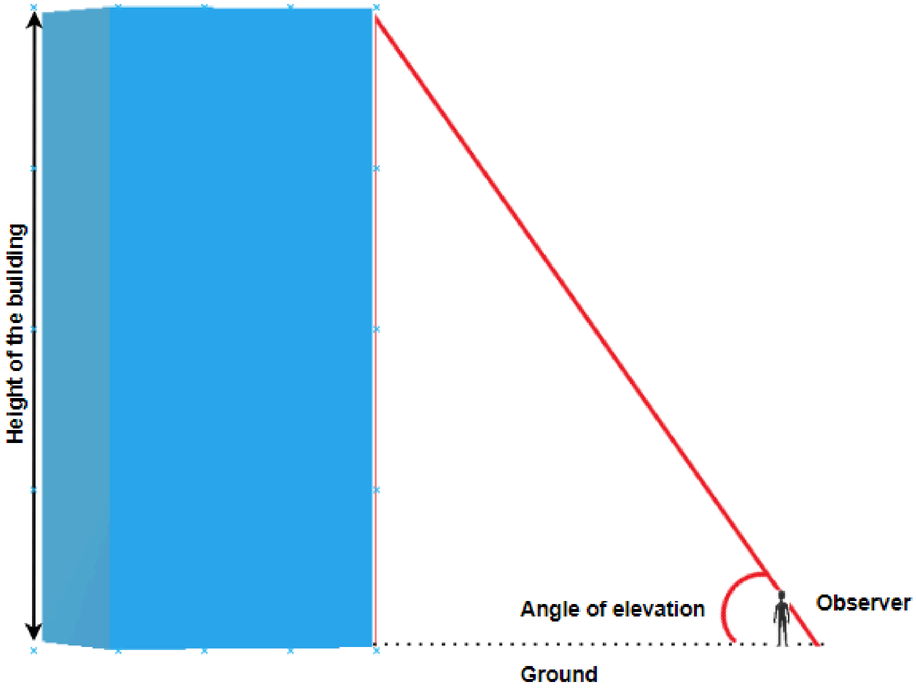
f = 16

Solve Applications Using Trigonometric Ratios

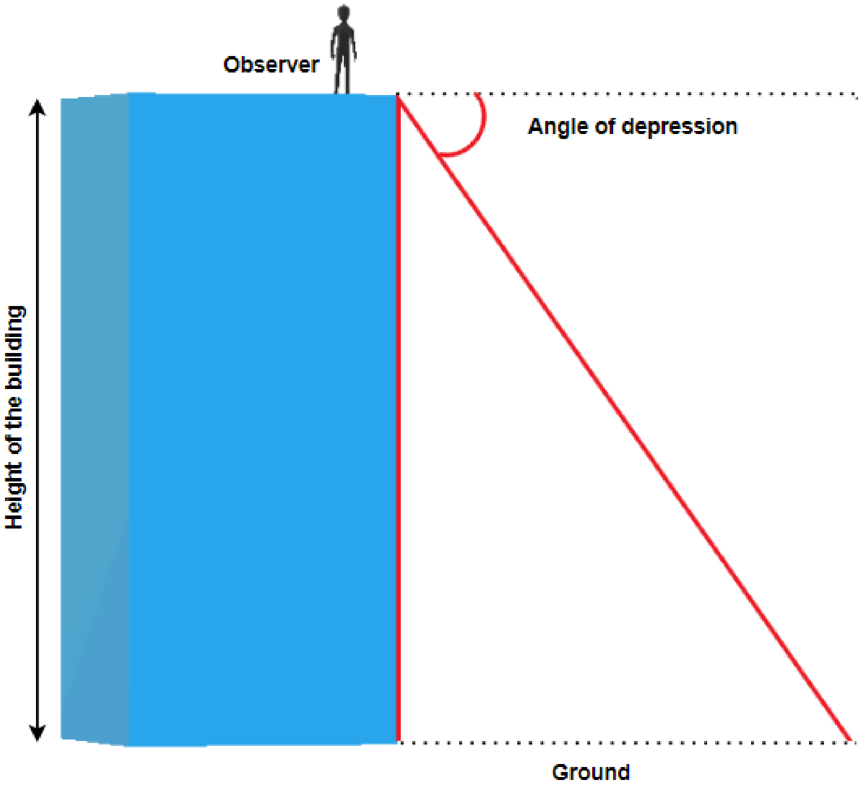
In the previous examples we were able to find missing sides and missing angles of a right triangle. Now, let’s use the trigonometric ratios to solve real-life problems.

Many applications of trigonometric ratios involve understanding of an angle of elevation or angle of depression.

The angle of elevation is an angle between the horizontal line (ground) and the observer’s line of sight.



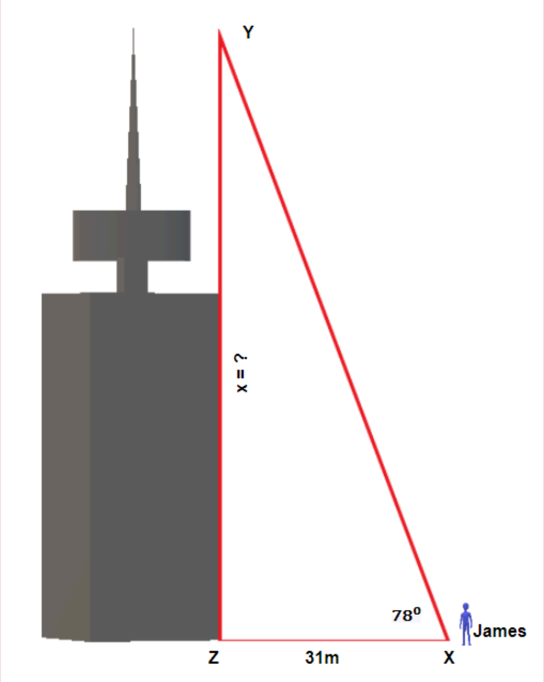
The angle of depression is the angle between horizontal line (that is parallel to the ground) and the observer's line of sight.



EXAMPLE 12

James is standing 31 metres away from the base of the Harbour Centre in Vancouver. He looks up to the top of the building at a 78° angle. How tall is the Harbour Centre?

Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	 Angle X is our reference angle, x is opposite side, y = 31 m is the adjacent side, and z is the hypotenuse.
2. **Identify** what we are looking for.	The opposite side
3.**Label** what we are looking for by choosing a variable to represent it.	x=?
4. **Find** the required trigonometric ratio.	tan 78° = $\frac{x}{31}$
5. **Solve** the ratio using good algebra techniques.	$4.7046 = \frac{x}{31}$ $x = 145.8426$
6. **Check** the answer in the problem and by making sure it makes sense.	$4.7046 \overset{?}{=} \frac{145.8426}{31}$ $4.7046 = 4.7046$ ✓
7. **Answer** the question with a complete sentence.	The Harbour Centre is 145.8426 metres or rounded to 146 metres.

TRY IT 12

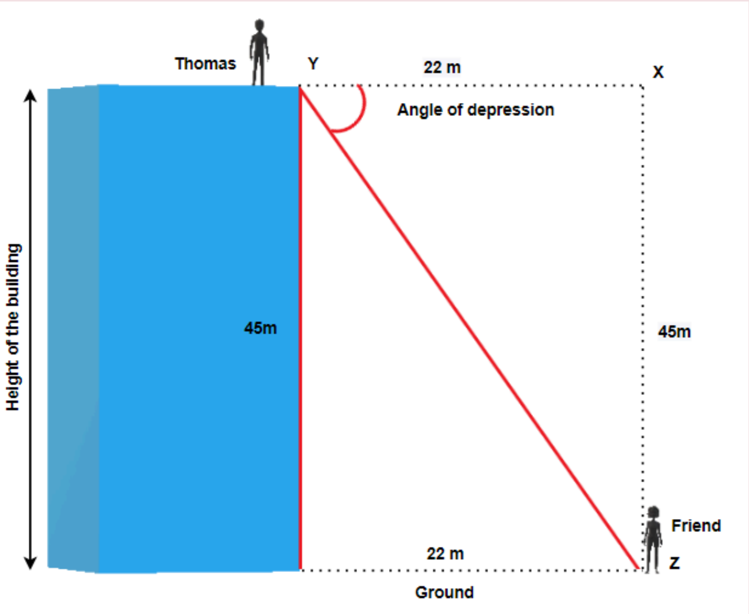
Marta is standing 23 metres away from the base of the tallest apartment building in Prince George and looks at the top of the building at a 62° angle. How tall is the building?

Show answer
43.3 metres

EXAMPLE 13

Thomas is standing at the top of the building that is 45 metres high and looks at her friend that is standing on the ground, 22 metres from the base of the building. What is the angle of depression?

Solution

1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.	 Angle Y is our reference angle, y = 45 m is the opposite side, z = 22 m is the adjacent side, and x is the hypotenuse
2. **Identify** what we are looking for.	angle Y
3.**Label** what we are looking for by choosing a variable to represent it.	$\angle Y$ =?
4. **Find** the required trigonometric ratio.	tan Y = $\frac{45}{22}$
5. **Solve** the ratio using good algebra techniques.	tan Y = 2.0455 Y = $\tan^{-1}2.0455$ $\angle Y$ = 63.9470°
6. **Check** the answer in the problem and by making sure it makes sense.	tan 63.9470° $\overset{?}{=}$ 2.0455 2.0455 = 2.0455 ✓
7. **Answer** the question with a complete sentence.	The angle of depression is 63.9470° or 64° rounded to one decimal place.

TRY IT 13

Hemanth is standing on the top of a cliff 250 feet above the ground and looks at his friend that is standing on the ground, 40 feet from the base of the cliff. What is the angle of depression?

Show answer
80.9°

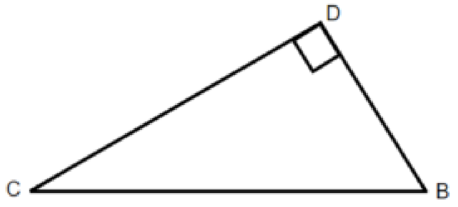
Key Concepts

- Three Basic Trigonometric Ratios: (Where θ is the measure of a reference angle measured in degrees.)
    - sine θ = $\frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}$
    - cosine θ = $\frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}$
    - tangent θ = $\frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}$
- **Problem-Solving Strategy for Trigonometry Applications**
    1. **Read** the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
    2. **Identify** what we are looking for.
    3. **Label** what we are looking for by choosing a variable to represent it.
    4. **Find** the required trigonometric ratio.
    5. **Solve** the ratio using good algebra techniques.
    6. **Check** the answer by substituting it back into the ratio solved in step 5 and by making sure it makes sense in the context of the problem.
    7. **Answer** the question with a complete sentence.

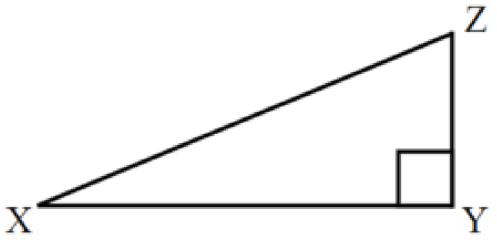
5.2 Exercise Set

Label the sides of the triangle.

1



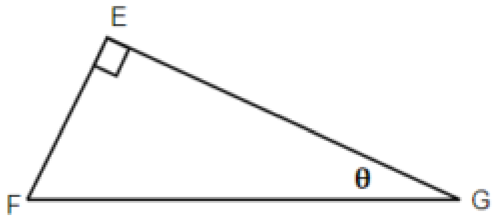
2.



3. If the reference angle in Question 1 is B, Find the adjacent ?
4. If the reference angle in Question 2 is Z, find the opposite?

Label the sides of the triangle and find the hypotenuse, opposite and adjacent.

5.



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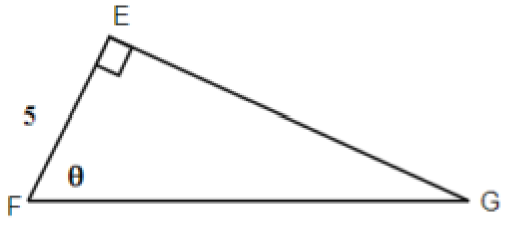
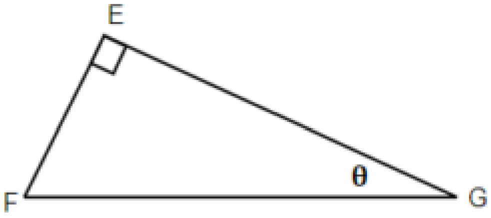


Use your calculator to find the given ratios. Round to four decimal places if necessary:

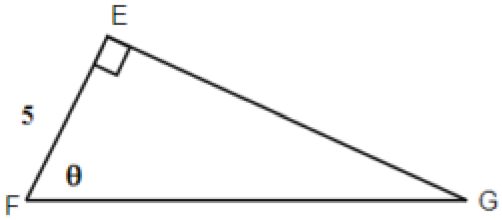
7. $\sin 47^\circ$
8. $\cos 82^\circ$
9. $\tan 12^\circ$
10. $\sin 30^\circ$

For the given triangles, find the sine, cosine and tangent of the θ.

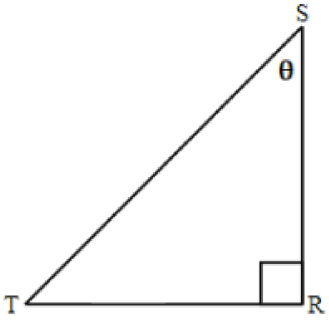
11.



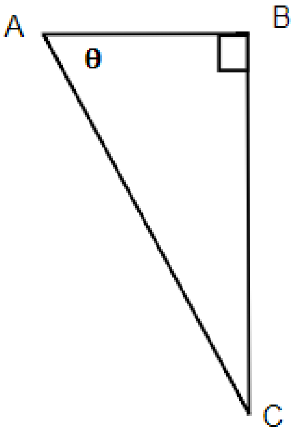
12.



13.

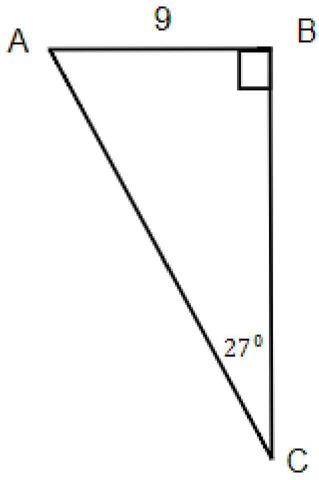


14.

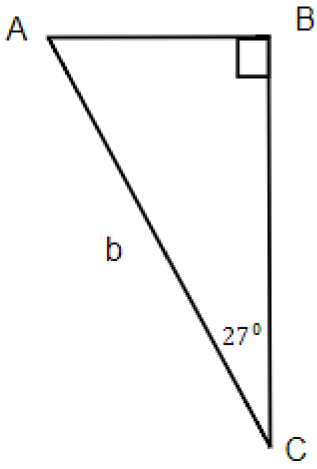


For the given triangles, find the missing side. Round it to one decimal place.

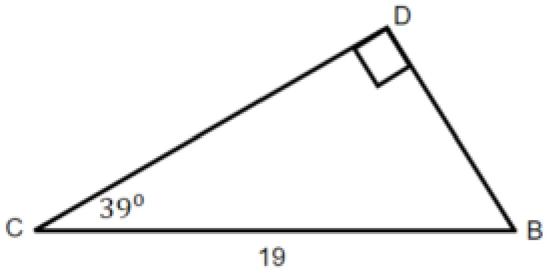
15. Find the hypotenuse.



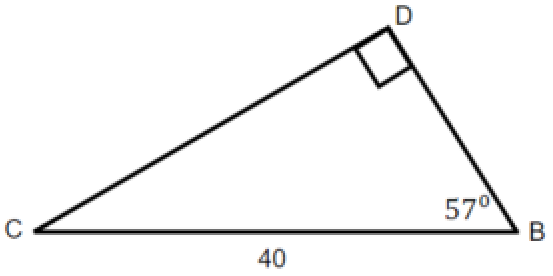
16. Find b.



17. Find the opposite.

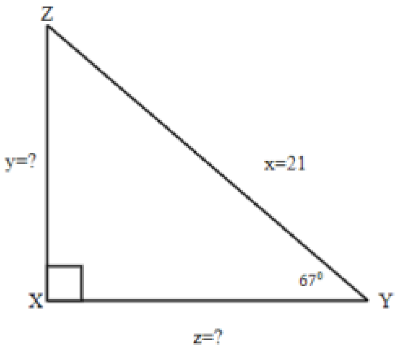


18. Find the adjacent.

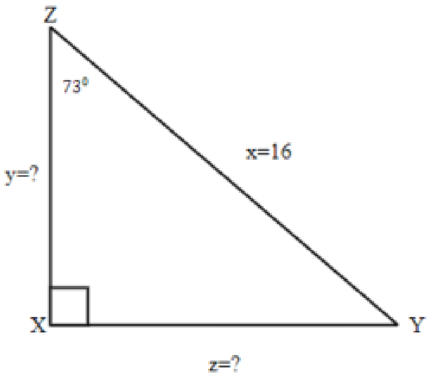


For the given triangles, find the missing sides. Round it to one decimal place.

19.

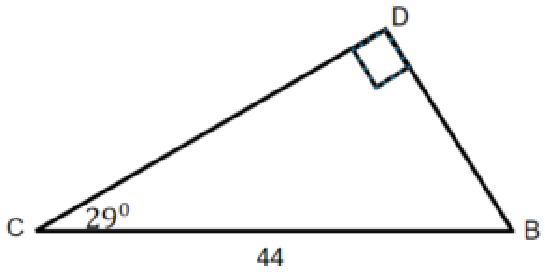


20.



Solve the triangles. Round to one decimal place.

21.



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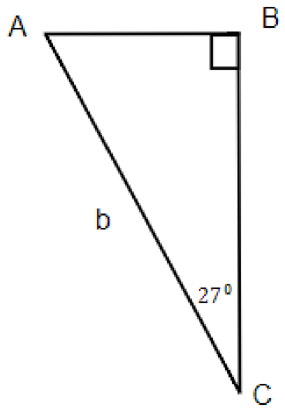
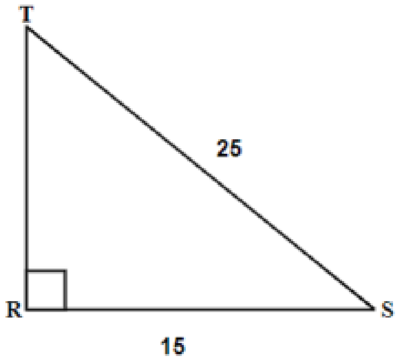
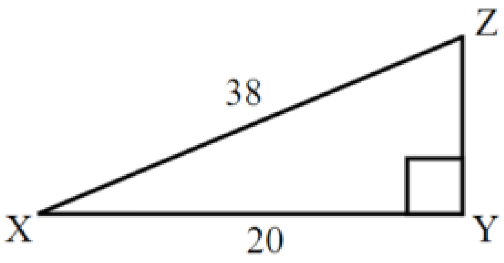


23.

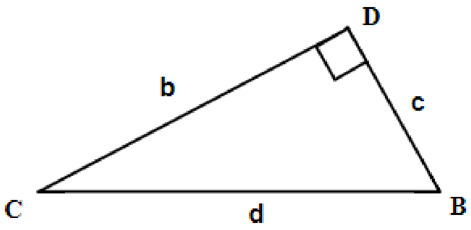
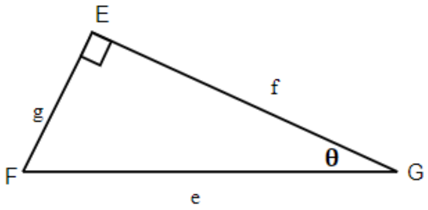
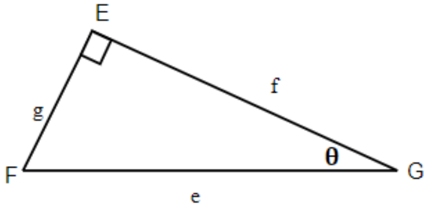


24.



25. Kim stands 75 metres from the bottom of a tree and looks up at the top of the tree at a 48° angle. How tall is the tree?
26. A tree makes a shadow that is 6 metres long when the angle of elevation to the sun is 52°. How tall is the tree?
27. A ladder that is 15 feet is leaning against a house and makes a 45° angle with the ground. How far is the base of the ladder from the house?
28. Roxanne is flying a kite and has let out 100 feet of string. The angle of elevation with the ground is 38°. How high is her kite above the ground?
29. Marta is flying a kite and has let out 28 metres of string. If the kite is 10 metres above the ground, what is the angle of elevation?
30. An airplane takes off from the ground at the angle of 25°. If the airplane traveled 200 kilometres, how high above the ground is it?

Answers:

1. 	3. c	5.  g is opposite , f is adjacent, and e is hypotenuse
7. 0.7314	9. 0.2126	11.  sin θ $= \frac{g}{e}$, cos θ $= \frac{f}{e}$, tan θ $= \frac{g}{f}$

13. sin θ $= \frac{s}{r}$, cos θ $= \frac{t}{r}$, tan θ $= \frac{s}{t}$
15. b = 19.8
17. c = 12
19. y = 19.3, z = 8.2
21. $\angle B$ = 61°$\angle C$ = 29°$\angle D$ = 90°b = 38.5c = 21.3d = 44
23. $\angle T$ = 36.9°$\angle R$ = 90°$\angle S$ = 53.1°t = 15r = 25s = 20
25. 83.3 m
27. 10.6 ft
29. 20.9°

Attribution:

This chapter has been adapted from “Solve Applications: Sine, Cosine and Tangent Ratios” in

6. Health Option

Mathematics is a tool frequently used to solve real-life problems.

Young smiling doctor standing in hospital with colleagues talking to patient in background.

In this chapter, understanding of measurement conversion, ratio, rate, proportion, and percent will help us to solve some health related problems.

6.1 Measurement; Health Applications

Learning Objectives

By the end of this section it is expected that you will be able to:

- Make unit conversions in the imperial system
- Make unit conversions in the metric system
- Convert between the imperial and the metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

Make Unit Conversions in the Imperial System

There are two systems of measurement commonly used around the world. Metric system used by most countries and imperial system used by the United States. Even though since 1970, Canada tried to switch to the metric system, the imperial system of measurements is still used in Canada. We will look at the imperial system first.

The imperial system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart, and gallons. Both the imperial system and the metric system measure time in seconds, minutes, and hours.

The equivalencies of measurements are shown in the table below. The table also shows, in parentheses, the common abbreviations for each measurement.

Imperial System of Measurement

Length	1 foot (ft.) = 12 inches (in.) 1 yard (yd.) = 3 feet (ft.) 1 mile (mi.) = 5,280 feet (ft.)	Volume	3 teaspoons (t) = 1 tablespoon (T) 16 tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl. oz.) 1 pint (pt.) = 2 cups (C) 1 quart (qt.) = 2 pints (pt.) 1 gallon (gal) = 4 quarts (qt.)
Weight	1 pound (lb.) = 16 ounces (oz.) 1 ton = 2000 pounds (lb.)	Time	1 minute (min) = 60 seconds (sec) 1 hour (hr) = 60 minutes (min) 1 day = 24 hours (hr) 1 week (wk) = 7 days 1 year (yr) = 365 days

In many medical applications, we need to convert between units of measurement, such as feet and inches, minutes and seconds, pounds and ounces, etc. We will use the identity property of multiplication to do these conversions. We'll restate the identity property of multiplication here for easy reference.

Identity Property of Multiplication

For any real number a:

$a \cdot 1 = a$

$1 \cdot a = a$

1 is the **multiplicative identity**.

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to change inches to feet. We know that 1 foot is equal to 12 inches, so we will write 1 as the fraction $\frac{1\text{ foot}}{12\text{ inches}}$. When we multiply by this fraction we do not change the value, but just change the units.

But $\frac{12\text{ inches}}{1\text{ foot}}$ also equals 1. How do we decide whether to multiply by $\frac{1\text{ foot}}{12\text{ inches}}$ or $\frac{12\text{ inches}}{1\text{ foot}}$? We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like factors and "divide out" common units like we do common factors. If we want to convert 66 inches to feet, which multiplication will eliminate the inches?

$$66\text{ inches} \bullet \frac{1\text{ foot}}{12\text{ inches}} \quad \text{or} \quad \xcancel{66\text{ inches} \bullet \frac{12\text{ inches}}{1\text{ foot}}}$$

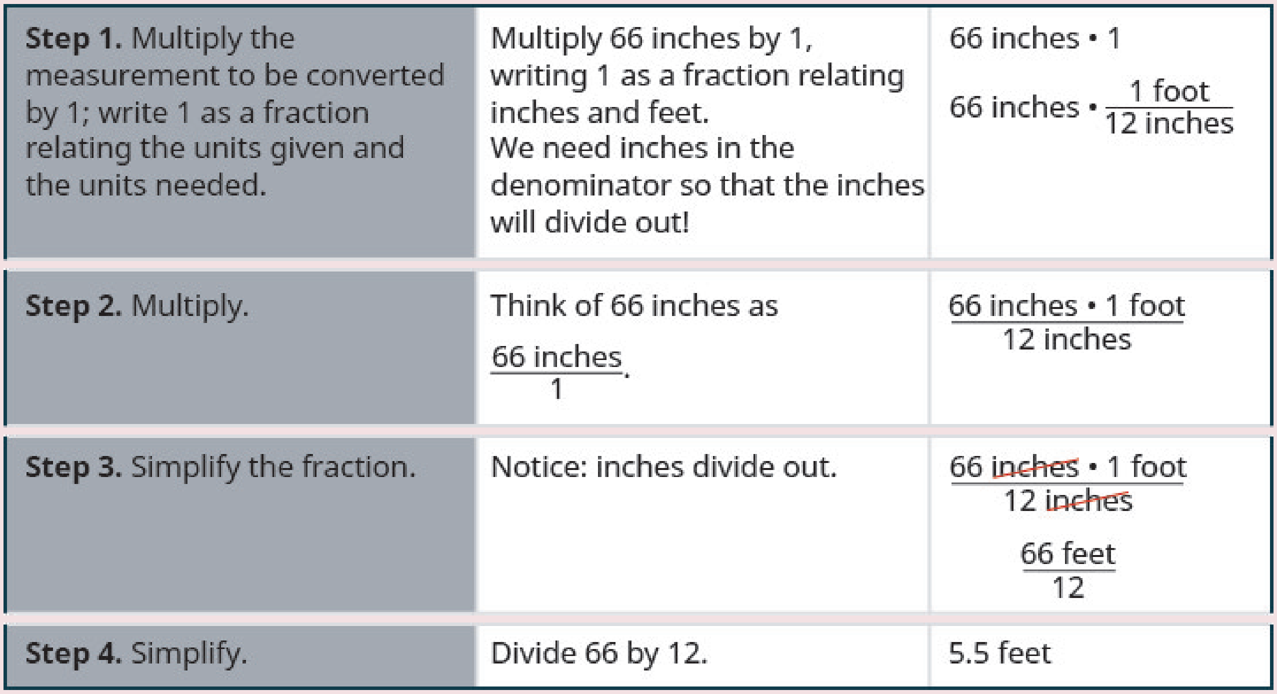
The first form works since $66\ \cancel{\text{inches}} \bullet \frac{1\text{ foot}}{12\ \cancel{\text{inches}}}$.

The inches divide out and leave only feet. The second form does not have any units that will divide out and so will not help us.

EXAMPLE 1

MaryAnne is 66 inches tall. Convert her height into feet.

Solution



Step 1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.	Multiply 66 inches by 1, writing 1 as a fraction relating inches and feet. We need inches in the denominator so that the inches will divide out!	$66 \text{ inches} \cdot 1$ $66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$
Step 2. Multiply.	Think of 66 inches as $\frac{66 \text{ inches}}{1}$.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$
Step 3. Simplify the fraction.	Notice: inches divide out.	$\frac{66 \cancel{\text{inches}} \cdot 1 \text{ foot}}{12 \cancel{\text{inches}}}$ $\frac{66 \text{ feet}}{12}$
Step 4. Simplify.	Divide 66 by 12.	5.5 feet

TRY IT 1

Lexie is 30 inches tall. Convert her height to feet.

Show answer
2.5 feet

HOW TO: Make unit conversions

1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
2. Multiply.
3. Simplify the fraction.
4. Simplify.

EXAMPLE 2

Eli’s six months son is 102.4 ounces. Convert his weight to pounds.

Solution

To convert ounces into pounds we will multiply by conversion factors of 1.

	102.4 ounces
Write 1 as $\frac{1\text{ pond}}{16\text{ ounces}}$	$102.4\text{ ounces}\times\frac{1\text{ pond}}{16\text{ ounces}}$
Divide out the common units.	$102.4\text{ ounces}\times\frac{1\text{ pond}}{16\text{ ounces}}$
Simplify the fraction.	$\frac{102.4\text{ pounds}}{16}$
Simplify.	6.4 pounds

Eli’s six months son weights 6.4 pounds.

TRY IT 2

One year old girl weights 11 pounds. Convert her weight to ounces.

Show answer
176 ounces.

When we use the identity property of multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator. Sometimes, to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

EXAMPLE 3

How many ounces are in 1 gallon?

Solution
We will convert gallons to ounces by multiplying by several conversion factors. Refer to the table on Imperial Systems of Measurement.

	1 gallon
Multiply the measurement to be converted by 1.	$\frac{1 \text{ gallon}}{1} \cdot \frac{4 \text{ quarts}}{1 \text{ gallon}} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} \cdot \frac{8 \text{ ounces}}{1 \text{ cup}}$
Use conversion factors to get to the right unit. Simplify.	$\frac{1 \text{ gallon}}{1} \cdot \frac{4 \text{ quarts}}{1 \text{ gallon}} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} \cdot \frac{8 \text{ ounces}}{1 \text{ cup}}$
Multiply.	$\frac{1 \cdot 4 \cdot 2 \cdot 2 \cdot 8 \text{ ounces}}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}$
Simplify.	128 ounces There are 128 ounces in a gallon.

TRY IT 3

How many teaspoons are in 1 cup?

Show answer
48 teaspoons

Make Unit Conversions in the Metric System

In the metric system, units are related by powers of 10. The roots words of their names reflect this relation. For example, the basic unit for measuring length is a metre. One kilometre is 1,000 metres; the prefix *kilo* means *thousand*. One centimetre is $\frac{1}{100}$ of a metre, just like one cent is $\frac{1}{100}$ of one dollar.

The equivalencies of measurements in the metric system are shown in the table below. The common abbreviations for each measurement are given in parentheses.

Metric System of Measurement

Length	**Mass**	**Capacity**
1 kilometre (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kilolitre (kL) = 1,000 L
1 hectometre (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectolitre (hL) = 100 L
1 dekametre (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekalitre (daL) = 10 L
1 metre (m) = 1 m	1 gram (g) = 1 g	1 litre (L) = 1 L
1 decimetre (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 decilitre (dL) = 0.1 L
1 centimetre (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centilitre (cL) = 0.01 L
1 millimetre (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 millilitre (mL) = 0.001 L
	1 microgram (mcg) = 0.000001 g	
1 metre = 100 centimetres	1 gram = 100 centigrams	1 litre = 100 centilitre s
1 metre = 1,000 millimetres	1 gram = 1,000 milligrams	1 litre = 1,000 millilitre s

To make conversions in the metric system, we will use the same technique we did in the Imperial system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

EXAMPLE 4

Samadia took 800mg of Ibuprofen for her inflammation. How many grams of Ibuprofen did she take?

Solution
We will convert milligrams to grams using the identity property of multiplication.

	800 milligrams
Multiply the measurement to be converted by 1.	800 milligrams × 1
Write 1 as a fraction relating kilometres and metres.	$800 \text{ milligrams} \times \frac{1 \text{ gram}}{1000 \text{ milligrams}}$
Simplify.	$800 \text{ milligrams} \times \frac{1 \text{ gram}}{1000 \text{ milligrams}}$
Multiply.	0.8 grams
	Samadia took 0.8 grams of Ibuprofen.

TRY IT 4

Klaudia took 0.125 grams of Ibuprofen for his headache. How many milligrams of the medication did she take?

Show answer
125 milligrams

EXAMPLE 5

Eleanor’s newborn baby weighed 3,200 grams. How many kilograms did the baby weigh?

Solution
We will convert grams into kilograms.

	$3{,}200 \text{ grams}$
Multiply the measurement to be converted by 1.	$3{,}200 \text{ grams} \cdot 1$
Write 1 as a function relating kilograms and grams.	$3{,}200 \text{ grams} \cdot \frac{1 \text{ kg}}{1{,}000 \text{ grams}}$
Simplify.	$3{,}200 \cancel{\text{grams}} \cdot \frac{1 \text{ kg}}{1{,}000 \cancel{\text{grams}}}$
Multiply.	$\frac{3{,}200 \text{ kilograms}}{1,000}$
Divide.	3.2 kilograms The baby weighed 3.2 kilograms.

TRY IT 5

Kari’s newborn baby weighed 2,800 grams. How many kilograms did the baby weigh?

Show answer
2.8 kilograms

As you become familiar with the metric system you may see a pattern. Since the system is based on multiples of ten, the calculations involve multiplying by multiples of ten. We have learned how to simplify these calculations by just moving the decimal.

To multiply by 10, 100, or 1,000, we move the decimal to the right one, two, or three places, respectively. To multiply by 0.1, 0.01, or 0.001, we move the decimal to the left one, two, or three places, respectively.

We can apply this pattern when we make measurement conversions in the metric system. In Example 8, we changed 3,200 grams to kilograms by multiplying by $\frac{1}{1000}$ (or 0.001). This is the same as moving the decimal three places to the left.

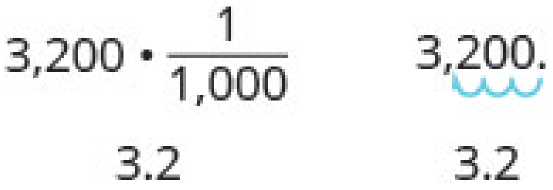


Figure.1

EXAMPLE 6

The volume of blood coursing throughout an adult human body is about 5 litres. Convert it to millilitres.

Solution

We will convert litres to millilitres. In the Metric System of Measurement table, we see that

$1 \text{ litre} = 1{,}000 \text{ millilitres.}$

	5 L
Multiply by 1, writing 1 as a fraction relating litres to millilitres.	$5 \text{ L} \cdot \frac{1000 \text{ mL}}{1\text{L}}$
Simplify.	$5\text{L} \cdot \frac{1000 \text{ mL}}{1\text{L}}$
Multiply.	5000 mL

TRY IT 6

Convert 6.3 L to millilitre

Show answer

6,300 millilitres

Convert Between the Imperial and the Metric Systems of Measurement

As Canada uses both system of measurement, we need to be able to convert between the two systems.

The table below shows some of the most common conversions.

Conversion Factors Between Imperial and Metric Systems

Length	**Mass**	**Capacity**
1 in. = 2.54 cm 1 ft. = 0.305 m 1 yd. = 0.914 m 1 mi. = 1.61 km 1 m = 3.28 ft.	1 lb. = 0.45 kg 1 oz. = 28 g 1 kg = 2.2 lb.	1 qt. = 0.95 L 1 fl. oz. = 30 mL 1 L = 1.06 qt.

(Figure.2) shows how inches and centimetres are related on a ruler.

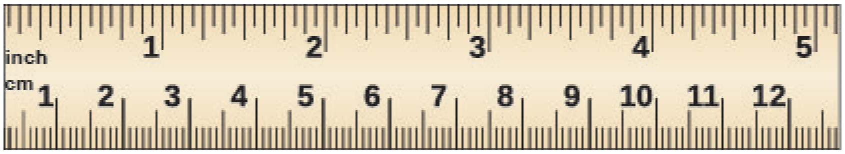


Figure.2

(Figure.3) shows the ounce and millilitre markings on a measuring cup.

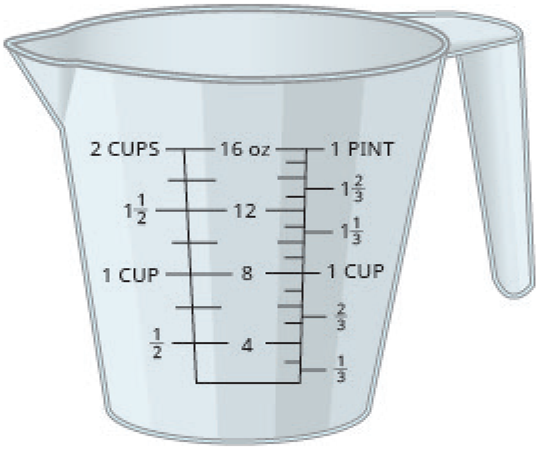


Figure.3

(Figure.4) shows how pounds and kilograms marked on a bathroom scale.

Figure.4

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

EXAMPLE 7

Plastic bag used for transfusion holds 500 mL of packed red cells. How many ounces are in the bag? Round to the nearest tenth of an ounce.

Solution

	500 mL
Multiply by a unit conversion factor relating mL and ounces.	$500\text{ millilitres}\cdot\dfrac{1\text{ ounce}}{30\text{ millilitres}}$
Simplify.	$\dfrac{50\text{ ounce}}{30}$
Divide.	16.7 ounces.
	The plastic bag has 16.7 ounces of packed red cells.

TRY IT 7

Adam donated 450 ml of blood. How many ounces is that?

Show answer
15 ounces.

EXAMPLE 8

A human brain weights about 3 pounds. How many kilograms is that? Round to the nearest tenth of a kilogram.

Solution

	3 pounds
Multiply by a unit conversion factor relating km and mi.	$3 \text{ pounds} \cdot \dfrac{1 \text{ kilogram}}{2.2 \text{pounds}}$
Simplify.	$\dfrac{3 \text{ kilograms}}{2.2}$
Divide.	1.4 kilograms
	A human brain weights around 1.4 kilograms.

TRY IT 8

A human liver normally weights approximately 1.5 kilograms. Convert it to pounds.

Show answer
3.3 pounds

Convert between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for $71°\text{F}$ what does that mean?

The Canadian and imperial systems use different scales to measure temperature. The Canadian system uses degrees Celsius, written °C. The imperial system uses degrees Fahrenheit, written°F. (Figure.5) shows the relationship between the two systems.
The diagram shows normal body temperature, along with the freezing and boiling temperatures of water in degrees Fahrenheit and degrees Celsius.

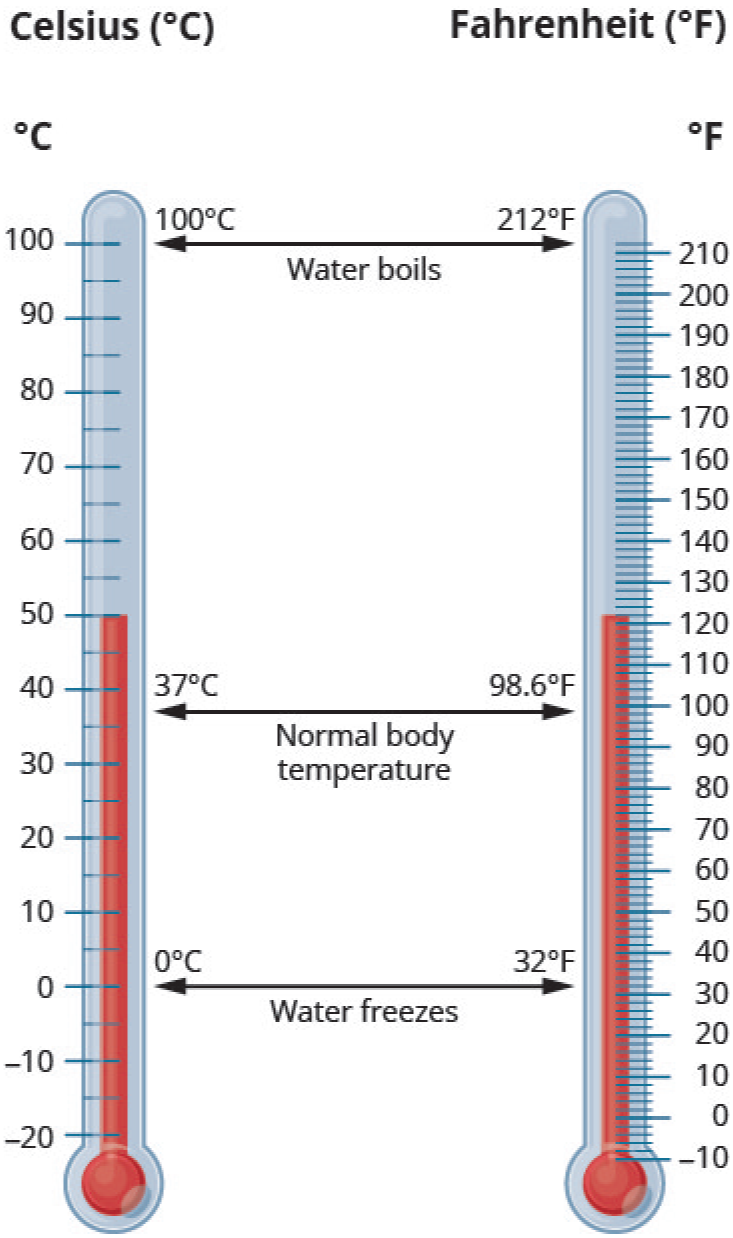


Figure.5

Temperature Conversion

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula

$$C = \frac{5}{9}\,(F - 32).$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula

$$F = \frac{9}{5}C + 32.$$

EXAMPLE 9

Before mixing, the Pfizer-BioNTech COVID-19 vaccine may be stored in an ultra-cold freezer between -112°F and -76°F. Convert the temperatures into degrees Celsius.

Solution

We will substitute a) -112°F and b) -76°F into the formula to find C.

a) Substitute -112 for F	$C=\frac{5}{9}\left(-112-32\right)$
Simplify in parentheses.	$C=\frac{5}{9}\left(-144\right)$
Multiply.	C= – 80 So we found that -112°F is equivalent to -80°C
b) Substitute -76 for F	$C=\frac{5}{9}\left(-76-32\right)$
Simplify.	$C=\frac{5}{9}\left(-108\right)$ C= – 60 So we found that -76°F is equivalent to – 60°C.

TRY IT 9

Convert the Fahrenheit temperature to degrees Celsius: 59° Fahrenheit.

Show answer°C
15°C

EXAMPLE 10

Once mixed, the Pfizer-BioNTech COVID-19 vaccine can be left at room temperature 2°C to 25°C. Convert the temperatures into degrees Fahrenheit.

Solution

We will substitute a) 2°C and b) 25°C into the formula to find F.

a) Substitute 2 for C	$F = \frac{9}{5} \cdot 2 + 32$
Simplify.	F= 35.6 So we found that 2°C is equivalent to 35.6°F.
b) Substitute 25 for C	$F = \frac{9}{5} \cdot 25 + 32$
Simplify.	F= 77
	So we found that 25°C is equivalent to 77°F.

TRY IT 10

Patient with high fever had a temperature of 39° Celsius. Convert it to Fahrenheit.

Show answer
102.2°F

Key Concepts

- **Metric System of Measurement**
  - **Length**

$$\begin{array}{lcl} 1 \text{ kilometre (km)} & = & 1{,}000 \text{ m} \\ 1 \text{ hectometre (hm)} & = & 100 \text{ m} \\ 1 \text{ dekametre (dam)} & = & 10 \text{ m} \\ 1 \text{ metre (m)} & = & 1 \text{ m} \\ 1 \text{ decimetre (dm)} & = & 0.1 \text{ m} \\ 1 \text{ centimetre (cm)} & = & 0.01 \text{ m} \\ 1 \text{ millimetre (mm)} & = & 0.001 \text{ m} \\ 1 \text{ metre} & = & 100 \text{ centimetres} \\ 1 \text{ metre} & = & 1{,}000 \text{ millimetres} \end{array}$$

  - **Mass**

1 kilogram (kg) = 1,000 g
1 hectogram (hg) = 100 g
1 dekagram (dag) = 10 g
1 gram (g) = 1 g
1 decigram (dg) = 0.1 g
1 centigram (cg) = 0.01 g
1 milligram (mg) = 0.001 g
1 gram = 100 centigrams
1 gram = 1,000 milligrams
1 gram = 1,000000 micograms

- **Capacity**
  1 kilolitre (kL) = 1,000 L
  1 hectolitre (hL) = 100 L
  1 dekalitre (daL) = 10 L
  1 litre (L) = 1 L
  1 decilitre (dL) = 0.1 L
  1 centilitre (cL) = 0.01 L
  1 millilitre (mL) = 0.001 L
  1 litre = 100 centilitre s
  1 litre = 1,000 millilitre s

- **Temperature Conversion**
  - To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula $\mathrm{C} = \frac{5}{9}(\mathrm{F} - 32)$
  - To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula $\mathrm{F} = \frac{9}{5}\mathrm{C} + 32$

6.1 Exercise Set

In the following exercises, convert the units.

1. A basketball player is 6 feet tall. Convert his height to inches.
2. Kelvin is 18 inches tall. Convert his height to feet.
3. Doctor recommended a patient to walk 1.5 miles every other day. Convert the distance to feet.
4. Misty’s surgery lasted $1\frac{1}{2}$ hours. Convert the time to seconds.
5. How many teaspoons are in a pint?
6. Jon is 6 feet 4 inches tall. Convert his height to inches.

7. Baby Preston weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.

In the following exercises, convert the units.

8. Ryan ran 5 kilometres. Convert the length to metres.
9. Emily is 1.55 metres tall. Convert her height to centimetres.
10. June’s multivitamin contains 1,500 milligrams of calcium. Convert this to grams.
11. One stick of butter contains 91.6 grams of fat. Convert this to milligrams.
12. Dimitri’s daughter weighed 3.8 kilograms at birth. Convert this to grams
13. A bottle of medicine contained 300 millilitres. Convert this to litres.

In the following exercises, solve.

14. Matthias is 1.8 metres tall. His son is 89 centimetres tall. How much taller is Matthias than his son?
15. One glass of orange juice provides 560 milligrams of potassium. Linda drinks one glass of orange juice every morning. How many grams of potassium does Linda get from her orange juice in 30 days?
16. Jonas drinks 200 millilitres of water 8 times a day. How many litres of water does Jonas drink in a day?

In the following exercises, make the unit conversions. Round to the nearest tenth.

17. Bill is 75 inches tall. Convert his height to centimetres.
18. Kathryn is 1.6 metres tall. Convert her height to feet

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

19. 86° Fahrenheit
20. 104° Fahrenheit
21. 72° Fahrenheit
22. 0° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

23. 5° Celsius
24. -10° Celsius
25. 22° Celsius
26. 43° Celsius

Everyday Math

27. **Nutrition** Julian drinks one can of soda every day. Each can of soda contains 40 grams of sugar. How many kilograms of sugar does Julian get from soda in 1 year?

Answers

1. 72 inches
2. 1.5 feet
3. 7,920 feet
4. 5,400 s
5. 96 teaspoons
6. 76 in.
7. 115 ounces
8. 5,000 metres
9. 155 centimetres
10. 1.5 grams
11. 91,600 milligrams
12. 3,800 grams
13. 0.3 litres
14. 91 centimetres
15. 16.8 grams
16. 1.6 litres
17. 190.5 centimetres
18. 5.2 feet
19. 30°C
20. 40°C
21. 22.2°C
22. -17.8°C
23. 41°F
24. 14°F
25. 71.6°F
26. 109.4°F
27. 14.6 kilograms

Attributions

1. This chapter has been adapted from "Systems of Measurement" in *Elementary Algebra* (OpenStax) by Lynn Marecek and MaryAnne Anthony-Smith, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.
2. OER.hawaii.edu
3. COVID-19 Government of Canada website https://www.canada.ca/en/health-canada/news/2021/05/health-canada-authorizes-more-flexible-storage-conditions-for-pfizer-biontech-covid-19-vaccine.html

6.2 Ratio, Rate, and Percent; Health Applications

Learning Objectives

By the end of this section it is expected that you will be able to:

- Write a ratio as a fraction
- Application of ratio
- Write ratio as a fraction
- Find unit rate
- Use the definition of percent
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

Write a Ratio as a Fraction

Ratios

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written $a \text{ to } b$, $\frac{a}{b}$, or a:b.

In this section, we will use the fraction notation. When a ratio is written in fraction form, the fraction should be simplified. If it is an improper fraction, we do not change it to a mixed number. Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

EXAMPLE 1

Write each ratio as a fraction: a) $15 \text{ to } 27$ b) $45 \text{ to } 18$.

Solution

a)

	$15 \text{ to } 27$
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{15}{27}$
Simplify the fraction.	$\frac{5}{9}$

We leave the ratio in b) as an improper fraction.

b)

	$45 \text{ to } 18$
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{45}{18}$
Simplify.	$\frac{5}{2}$

TRY IT 1

Write each ratio as a fraction: a) $21 \text{ to } 56$ b) $48 \text{ to } 32$.

Show answer

a. $\frac{3}{8}$
b. $\frac{3}{2}$

Applications of Ratios

One real-world application of ratios that affects many people involves measuring cholesterol in blood. The ratio of total cholesterol to HDL cholesterol is one way doctors assess a person’s overall health. A ratio of less than 5 to 1 is considered good.

EXAMPLE 2

Hector’s total cholesterol is 249 mg/dl and his HDL cholesterol is 39 mg/dl. a) Find the ratio of his total cholesterol to his HDL cholesterol. b) Assuming that a ratio less than 5 to 1 is considered good, what would you suggest to Hector?

Solution

a) First, write the words that express the ratio. We want to know the ratio of Hector's total cholesterol to his HDL cholesterol.

Write as a fraction.	$\frac{\text{total cholesterol}}{\text{HDL cholesterol}}$
Substitute the values.	$\frac{249}{39}$
Simplify.	$\frac{83}{13}$

b) Is Hector's cholesterol ratio ok? If we divide 83 by 13 we obtain approximately 6.4, so $\frac{83}{13} \approx \frac{6.4}{1}$. Hector's cholesterol ratio is high! Hector should either lower his total cholesterol or raise his HDL cholesterol.

TRY IT 2

Find the patient's ratio of total cholesterol to HDL cholesterol using the given information.

Total cholesterol is 185 mg/dL and HDL cholesterol is 40 mg/dL.

Show answer

$\frac{37}{8}$

Write a Rate as a Fraction

Frequently, using rate, we compare two different types of measurements. Examples of rates are 120 kilometres in 2 hours, 160 words in 4 minutes, and $\$5$ dollars per 64 ounces.

Rate

A rate compares two quantities of different units. A rate is usually written as a fraction.

When writing a fraction as a rate, we put the first given amount with its units in the numerator and the second amount with its units in the denominator. When rates are simplified, the units remain in the numerator and denominator.

EXAMPLE 3

A healthy heart has a rate around 72 beats per 60 seconds. Write this rate as a fraction.

Solution

	$72 \text{ beats in } 60 \text{ seconds}$
Write as a fraction, 72 beats in the numerator and 60 seconds in the denominator.	$\frac{72 \text{ beats}}{60 \text{ seconds}}$
	$\frac{6 \text{ beats}}{5 \text{ seconds}}$

So 72 beats in 60 seconds is equivalent to $\frac{6 \text{ beats}}{5 \text{ seconds}}$.

TRY IT 3

Write the rate as a fraction: 70 heartbeats in 60 seconds.

Show answer
$\frac{7 \text{ heartbeats}}{6 \text{ seconds}}$

Find Unit Rate

In the last example, we calculated that a healthy heart beats at a rate of $\frac{6 \text{ beats}}{5 \text{ seconds}}$. This tells us that every 5 seconds there are 6 heart beats. This is correct, but not very useful. We usually want the rate to reflect the number of beats in one second. A rate that has a denominator of 1 unit is referred to as a unit rate.

Unit Rate

A unit rate is a rate with denominator of 1 unit.

To convert a rate to a unit rate, we divide the numerator by the denominator. This gives us a denominator of 1.

EXAMPLE 4

Marta had 74 heartbeats in8 minutes. What is Marta’s heartbeat rate?

Solution

Start with a rate of heartbeats to minutes. Then divide.	$74 \text{ heartbeats in } 8 \text{ minutes}$
Write as a rate.	$\frac{74}{8 \text{ minutes}}$
Divide the numerator by the denominator.	$\frac{2}{1 \text{ minute}}$
Rewrite as a rate.	$2/\text{minute}$

Marta’s heartbeat rate is 2 per minute.

TRY IT 4

Find the unit rate: 16 heartbeats in 12 minutes.

Show answer
\68 heartbeats/minute

Use the Definition of Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A percent is a ratio whose denominator is 100. We use the percent symbol $\%$, to show percent.

Percent

A percent is a ratio whose denominator is 100.

According to the data from Statistics Canada (2009), 57% of 6-11 year olds have or have had a cavity. This means 57 out of every 100 of 6-11 year olds have or have had a cavity. As (Figure 1) shows out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.

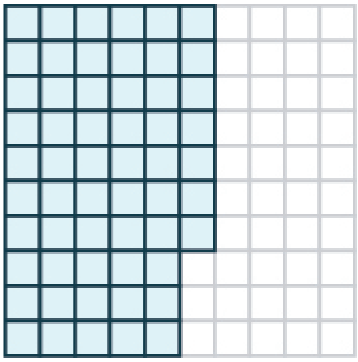


Figure 1

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, “one hundred percent” means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

EXAMPLE 5

According to a Government of Canada report updated on \left(July 9, 2021\right \text{,}\phantom{\rule{0.2em}{0ex}}\text{68\%}\) of total population received at least on dose of COVID-19 vaccine. Write this percent as a ratio.

Solution

The amount we want to convert is 68%.	68%
Write the percent as a ratio. Remember that *percent* means per 100.	$\frac{68}{100}$

TRY IT 5

Write the percent as a ratio.

According to the report from example 5, 32% of total population is partially vaccinated.

Show answer

$\frac{32}{100}$

Convert Percents to Fractions and Decimals

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

Convert a Percent to a Fraction.

1. Write the percent as a ratio with the denominator 100.
2. Simplify the fraction if possible.

EXAMPLE 6

According to the report from example 5, 36% of total population is fully vaccinated.Convert the percent to a fraction:

Solution

	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify.	$\frac{9}{25}$

TRY IT 6

According to the report from example 5, just slightly over 40% of 12 years and older population is fully vaccinated.Convert the percent to a fraction:

Show answer
$\frac{2}{5}$

To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

HOW TO: Convert a Percent to a Decimal

1. Write the percent as a ratio with the denominator 100.
2. Convert the fraction to a decimal by dividing the numerator by the denominator.

EXAMPLE 7

a) Adult males typically are composed of about 60 % water. Convert the percent to a decimal.

b) Adult females typically are compostedof 55% water. Convert the percent to a decimal

Solution
Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

a)	
	60%
Write as a ratio with denominator 100.	$\frac{60}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.60

b)	
	55%
Write as a ratio with denominator 100.	$\frac{55}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.55

TRY IT 7

Convert each percent to a decimal:

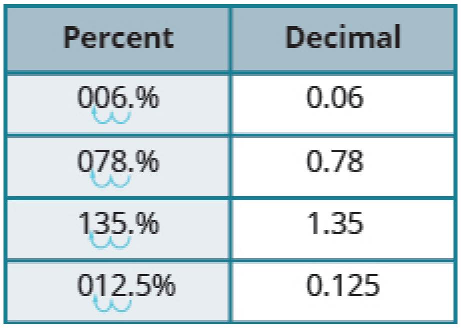
a. 9%
b. 87%

Show answer

a. 0.09
b. 0.87

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the $\%$ sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

(Figure 2) uses the percents in the table above and shows visually how to convert them to decimals by moving the decimal point two places to the left.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure 2

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

HOW TO: Convert a Decimal to a Percent

1. Write the decimal as a fraction.
2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
3. Write this ratio as a percent.

EXAMPLE 8

Convert each decimal to a percent: a) 0.05 b) 0.83

Solution

a)	
	0.05
Write as a fraction. The denominator is 100.	$\frac{5}{100}$
Write this ratio as a percent.	5%

b)	
	0.83
The denominator is 100.	$\frac{83}{100}$
Write this ratio as a percent.	83%

TRY IT 8

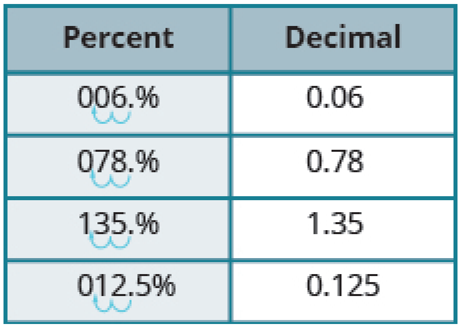
Convert each decimal to a percent: a)0.04 b)0.41

Show answer

- a. 4%
- b. 41%

To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

(Figure.3) uses the decimal numbers in the table above and shows visually to convert them to percents by moving the decimal point two places to the right and then writing the $\%$ sign.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure. 3

Now we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

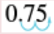
HOW TO: Convert a Fraction to a Percent

1. Convert the fraction to a decimal.
2. Convert the decimal to a percent.

EXAMPLE 9

Convert each fraction or mixed number to a percent: a) $\frac{3}{4}$ b) $\frac{11}{8}$ c) $2\frac{1}{5}$

Solution
To convert a fraction to a decimal, divide the numerator by the denominator.

a)	
Change to a decimal.	$\frac{3}{4}$
Write as a percent by moving the decimal two places.	
	75%

b)	
Change to a decimal.	$\frac{11}{8}$
Write as a percent by moving the decimal two places.	
	137.5%

c)	
Write as an improper fraction.	$2\frac{1}{5}$
Change to a decimal.	$\frac{11}{5}$
Write as a percent.	
	220%

Notice that we needed to add zeros at the end of the number when moving the decimal two places to the right.

TRY IT 9

Convert each fraction or mixed number to a percent: a) $\frac{5}{8}$ b) $\frac{11}{4}$ c) $3\frac{2}{5}$

Show answer

a. 62.5%
b. 275%
c. 340%

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place. For most other cases in this book we will round the number to the nearest thousandth, so the percent will be rounded to the nearest tenth.

EXAMPLE 10

Convert $\frac{5}{7}$ to a percent.

Solution
To change a fraction to a decimal, we divide the numerator by the denominator.

	$\frac{5}{7}$
Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

TRY IT 10

Convert the fraction to a percent: $\frac{3}{7}$

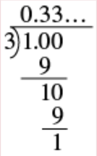
Show answer
42.9%

When we first looked at fractions and decimals, we saw that fractions converted to a repeating decimal. When we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as $1.\overline{3}$. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

EXAMPLE 11

Statistics Canada reported in 2018 that approximately $\frac{1}{3}$ of Canadian adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Solution

	$\frac{1}{3}$
Change to a decimal.	
Write as a repeating decimal.	$0.333\ldots$
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

TRY IT 11

Convert the fraction to a percent:

According to the Canadian Census 2016, about $\frac{33}{50}$ people within the population of Canada are between the ages of 15 and 64.

Show answer

$66.\overline{6}\%$, or $11\frac{6}{25}\%$

Access to Additional Online Resources

- Ratios
- Write Ratios as a Simplified Fractions Involving Decimals and Fractions
- Write a Ratio as a Simplified Fraction
- Rates and Unit Rates
- Unit Rate for Cell Phone Plan

Glossary

ratio
A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or $a : b$.

rate
A rate compares two quantities of different units. A rate is usually written as a fraction.

unit rate
A unit rate is a rate with denominator of 1 unit.

percent
A percent is a ratio whose denominator is 100.

Key Concepts

- **Convert a percent to a fraction.**
  1. Write the percent as a ratio with the denominator 100.
  2. Simplify the fraction if possible.
- **Convert a percent to a decimal.**
  1. Write the percent as a ratio with the denominator 100.
  2. Convert the fraction to a decimal by dividing the numerator by the denominator.
- **Convert a decimal to a percent.**
  1. Write the decimal as a fraction.
  2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
  3. Write this ratio as a percent.
- **Convert a fraction to a percent.**
  1. Convert the fraction to a decimal.
  2. Convert the decimal to a percent.

6.2 Exercise Set

In the following exercises, write each ratio as a fraction.

1. 20 to 36
2. 42 to 48
3. 49 to 21
4. 84 to 36

5. 0.56 to 2.8
6. 28 ounces to 84 ounces
7. 12 feet to 46 feet
8. 246 milligrams to 45 milligrams
9. total cholesterol of 175 to HDL cholesterol of 45
10. 27 inches to 1 foot

In the following exercises, find the unit rate. Round to two decimal places, if necessary.

11. 140 calories per 12 ounces
12. total cholesterol is 204 mg/dL and HDL cholesterol is 38 mg/dL
13. 584 beats in 8 minutes
14. 43 pounds in 16 weeks
15. 46 beats in 0.5 minute
16. A popular fast food burger weighs 7.5 ounces and contains 540 calories, 29 grams of fat, 43 grams of carbohydrates, and 25 grams of protein. Find the unit rate of:
    a. calories per ounce
    b. grams of fat per ounce
    c. grams of carbohydrates per ounce
    d. grams of protein per ounce. Round to two decimal places.

In the following exercises, write each percent as a ratio.

17. A patient health insurance covers 60% of the cost of his medication.
18. 57 out of 100 nursing candidates received their degree at a community college.

In the following exercises, convert each percent to a fraction and simplify all fractions.

19. 4%
20. 52%
21. 125%
22. 37.5%

In the following exercises, convert each percent to a decimal.

23. 5%
24. 63%
25. 150%
26. 21.4%
27. COVID-19 vaccines, the Pfizer and Moderna have the hightest efficiency at around 95%
28. A couple plans to have two children. The probability they will have two girls is 25%.

In the following exercises, convert each decimal to a percent.

29. 0.18
30. 1.35
31. 3
32. 0.009

In the following exercises, convert each fraction to a percent.

33. $\frac{3}{8}$
34. $\frac{5}{12}$

35. $\frac{3}{7}$
36. According to the Government of Canada, in $2017, \frac{16}{25}$ of Canadian adults were overweight or obese.

Answers

1. $\frac{5}{9}$
2. $\frac{7}{8}$
3. $\frac{7}{3}$
4. $\frac{7}{3}$
5. $\frac{1}{5}$
6. $\frac{1}{3}$
7. $\frac{6}{23}$
8. $\frac{82}{15}$
9. $\frac{35}{9}$
10. $\frac{9}{4}$
11. 11.67 calories/ounce
12. 2.73 lbs./sq. in.
13. 73 beats/minute
14. 2.69 lbs./week
15. 92 beats/minute
    - a. 72 calories/ounce
    - b. 3.87 grams of fat/ounce
    - c. 5.73 grams carbs/once
    - d. 3.33 grams protein/ounce
16. $\frac{60}{100}$
17. $\frac{57}{1000}$
18. $\frac{1}{25}$
19. $\frac{13}{25}$
20. $\frac{5}{4}$
21. $\frac{3}{8}$
22. 0.05
23. 0.63
24. 1.5
25. 0.214
26. 0.95
27. 0.25
28. 18%
29. 135%
30. 300%
31. 0.9%
32. 37.5%
33. 41.7%
34. 42.9%

Attributions

1. This chapter has been adapted from “Understand Percent” in *Prealgebra* (OpenStax) by Lynn Marecek, MaryAnne Anthony-Smith, and Andrea Honeycutt Mathis, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.
2. OER.hawaii.edu
3. Wikipedia
4. Government of Canada Statistics

6.3 Proportions; Health Applications

Learning Objectives

By the end of this section it is expected that you will be able to:

- Use the definition of proportion
- Solve proportions
- Solve applications using proportions
- Write percent equations as proportions
- Translate and solve percent proportions

Use the Definition of Proportion

When two ratios or rates are equal, the equation relating them is called a proportion.

Proportion

A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.

The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8".

If we compare quantities with units, we have to be sure we are comparing them in the right order.

EXAMPLE 1

Write the sentence as a proportion:

72 heartbeats in 1 minute is the same as 216 heartbeats in 3 minutes.

Solution

	72 is to 1 as 216 is to 3.
Write as a proportion.	$\frac{72\text{ heartbeats}}{1\text{ minute}} = \frac{216\text{ heartbeats}}{3\text{ minutes}}$

TRY IT 1

Write the sentence as a proportion:

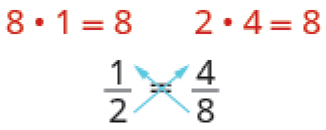
5 is to 9 as 20 is to 36.

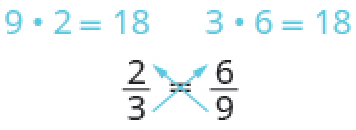
Show answer

$\frac{5}{9} = \frac{20}{36}$

Look at the proportions $\frac{1}{2} = \frac{4}{8}$ and $\frac{2}{3} = \frac{6}{9}$. From our work with equivalent fractions we know these equations are true. But how do we know if an equation is a proportion with equivalent fractions if it contains fractions with larger numbers?

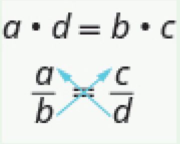
To determine if a proportion is true, we find the **cross products** of each proportion. To find the cross products, we multiply each denominator with the opposite numerator (diagonally across the equal sign). The results are called a cross products because of the cross formed. The cross products of a proportion are equal.





Cross Products of a Proportion

For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$, its cross products are equal.



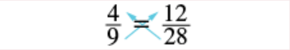
Cross products can be used to test whether a proportion is true. To test whether an equation makes a proportion, we find the cross products. If they are the equal, we have a proportion.

EXAMPLE 2

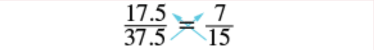
Determine whether each equation is a proportion:

- a. $\frac{4}{9} = \frac{12}{28}$
- b. $\frac{17.5}{37.5} = \frac{7}{15}$

Solution
To determine if the equation is a proportion, we find the cross products. If they are equal, the equation is a proportion.

a)	
	$\frac{4}{9} = \frac{12}{28}$
Find the cross products.	$28 \cdot 4 = 112 \qquad 9 \cdot 12 = 108$ 

Since the cross products are not equal, $28 \cdot 4 \neq 9 \cdot 12$, the equation is not a proportion.

b)	
	$\frac{17.5}{37.5} = \frac{7}{15}$
Find the cross products.	$15 \cdot 17.5 = 262.5 \qquad 37.5 \cdot 7 = 262.5$ 

Since the cross products are equal, $15 \cdot 17.5 = 37.5 \cdot 7$, the equation is a proportion.

TRY IT 2

Determine whether each equation is a proportion:

- a. $\frac{7}{9} = \frac{54}{72}$
- b. $\frac{24.5}{45.5} = \frac{7}{13}$

Show answer

- a. no
- b. yes

Solve Proportions

To solve a proportion containing a variable, we remember that the proportion is an equation. All of the techniques we have used so far to solve equations still apply. In the next example, we will solve a proportion by multiplying by the Least Common Denominator (LCD) using the Multiplication Property of Equality.

EXAMPLE 3

Solve: $\frac{x}{63} = \frac{4}{7}$.

Solution

	$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.	$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.	$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.	$x = 36$
Check: To check our answer, we substitute into the original proportion.	
	$\frac{x}{63} = \frac{4}{7}$
Substitute $x = 36$	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓

TRY IT 3

Solve the proportion: $\frac{n}{84} = \frac{11}{12}$.

Show answer

77

When the variable is in a denominator, we'll use the fact that the cross products of a proportion are equal to solve the proportions.

We can find the cross products of the proportion and then set them equal. Then we solve the resulting equation using our familiar techniques.

EXAMPLE 4

Solve: $\frac{144}{a} = \frac{9}{4}$.

Solution
Notice that the variable is in the denominator, so we will solve by finding the cross products and setting them equal.

	$\frac{144}{a} = \frac{9}{4}$
Find the cross products and set them equal.	$4 \cdot 144 = a \cdot 9$
Simplify.	$576 = 9a$
Divide both sides by 9.	$\frac{576}{9} = \frac{9a}{9}$
Simplify.	$64 = a$
Check your answer:	
	$\frac{144}{a} = \frac{9}{4}$
Substitute a = 64	$\frac{144}{64} \overset{?}{=} \frac{9}{4}$
Show common factors.	$\frac{9 \cdot 16}{4 \cdot 16} \overset{?}{=} \frac{9}{4}$
Simplify.	$\frac{9}{4} = \frac{9}{4}$ ✓

Another method to solve this would be to multiply both sides by the LCD, $4a$. Try it and verify that you get the same solution.

TRY IT 4

Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

Show answer
65

EXAMPLE 5

Solve: $\frac{52}{91} = \frac{-4}{y}$.

Solution

Find the cross products and set them equal.	$\frac{52}{91} = \frac{-4}{y}$
	$y \cdot 52 = 91(-4)$
Simplify.	$52y = -364$
Divide both sides by 52.	$\frac{52y}{52} = \frac{-364}{52}$
Simplify.	$y = -7$
Check:	
	$\frac{52}{91} = \frac{-4}{y}$
Substitute y = −7	$\frac{52}{91} \overset{?}{=} \frac{-4}{-7}$
Show common factors.	$\frac{13 \cdot 4}{13 \cdot 4} \overset{?}{=} \frac{-4}{-7}$
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓

TRY IT 5

Solve the proportion: $\frac{84}{98} = \frac{-6}{x}$.

Show answer
−7

Solve Applications Using Proportions

When we set up the proportion, we must make sure the units are correct—the units in the numerators match and the units in the denominators match.

EXAMPLE 6

When pediatricians prescribe acetaminophen to children, they prescribe 5 millilitres (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many millilitres of acetaminophen will her doctor prescribe?

Solution

Identify what you are asked to find.	How many ml of acetaminophen the doctor will prescribe
Choose a variable to represent it.	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion.	$\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$
Substitute given values—be careful of the units.	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by 80.	$80 \cdot \frac{5}{25} = 80 \cdot \frac{a}{80}$
Multiply and show common factors.	$\frac{16 \cdot 5 \cdot 5}{5 \cdot 5} = \frac{80a}{80}$
Simplify.	$16 = a$
Check if the answer is reasonable.	
Yes. Since 80 is about 3 times 25, the medicine should be about 3 times 5.	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

You could also solve this proportion by setting the cross products equal.

TRY IT 6

Pediatricians prescribe 5 millilitres (ml) of acetaminophen for every 25 pounds of a child's weight. How many millilitres of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Show answer
12 ml

EXAMPLE 7

The doctor prescribes for you to take 60 milligrams of the medication that is found in a liquid cough syrup. The label on the syrup reads 100 mg /5 mL. How much cough syrup should you take?

Solution

Identify what you are asked to find.	How much cough syrup you should take?
Choose a variable to represent it.	Let $x =$ amount of cough syrup.
Write a sentence that gives the information to find it.	If there is 100 mg in 5 mL, then 60 mg is in what amount?
Translate into a proportion.	$\frac{mg}{mL} = \frac{mg}{mL}$
Substitute given values.	$\frac{100}{5} = \frac{60}{x}$
Use the cross product.	$100x = 5 \cdot \ 60$
Divide by 100.	x = 3
Check if the answer is reasonable.	
Yes. Since 60 mg is less then 100 mg, than 3 mL is less that 5 mL.	
Write a complete sentence.	You should take 3 mL of cough syrup.

TRY IT 7

The doctor prescribes 30 milligrams of the medication found in the liquid cough syrup to your daughter. The label on the syrup reads 100 mg /5 mL. How much cough syrup you should give to your daughter?

Show answer

1.5 mL

EXAMPLE 8

For adults over 18, the recommended daily allowance for protein is 0.8 grams per 1 kilogram of body weight. How many grams of protein is an adult allowed to consume per day if your weight is 68 kilograms?

Solution

Identify what you are asked to find.	How many grams of protein is an adult allowed to consume per day?
Choose a variable to represent it.	Let p = number of grams of protein.
Write a sentence that gives the information to find it.	If 0.8 gr is allowed for 1 kg, then how many grams is allowed for 68 kg?
Translate into a proportion.	$\frac{protein}{weight} = \frac{protein}{weight}$
Substitute given values.	$\frac{0.8}{1} = \frac{p}{68}$
The variable is in the denominator, so find the cross products and set them equal.	$p \cdot \ 1 = 0.8 \cdot \ 68$
Simplify.	p = 54.4
Check if the answer is reasonable.	
Yes, if the allowance would be 1 gram per 1 kilogram, it would be 68 grams a day.	
Write a complete sentence.	An adult over 18 is allowed to consume 54.4 grams of protein.

TRY IT 8

Base on example 8, how many grams of protein is Nicole allowed if her weight is 52 kilograms?

Show answer
41.6 grams

Write Percent Equations As Proportions

The proportion method for solving percent problems involves a percent proportion. A **percent proportion** is an equation where a percent is equal to an equivalent ratio.

For example, $60\% = \frac{60}{100}$ and we can simplify $\frac{60}{100} = \frac{3}{5}$. Since the equation $\frac{60}{100} = \frac{3}{5}$ shows a percent equal to an equivalent ratio, we call it a percent proportion. Using the vocabulary we used earlier:

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$

$$\frac{3}{5} = \frac{60}{100}$$

Percent Proportion

The amount is to the base as the percent is to 100.

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$

If we restate the problem in the words of a proportion, it may be easier to set up the proportion:

$$\text{The amount is to the base as the percent is to one hundred.}$$

We could also say:

$$\text{The amount out of the base is the same as the percent out of one hundred.}$$

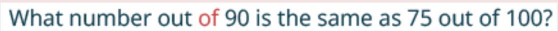
First we will practice translating into a percent proportion. Later, we'll solve the proportion.

EXAMPLE 9

Translate to a proportion. What number is 75% of $90?$

Solution

If you look for the word "of", it may help you identify the base.

Identify the parts of the percent proportion.	 $\underbrace{\text{What number}}_{\text{amount}}$ is $\underbrace{75\%}_{\text{percent}}$ of $\underbrace{90?}_{\text{base}}$
Restate as a proportion.	 What number out of 90 is the same as 75 out of 100?
Set up the proportion. Let $n = \text{number}.$	$\frac{n}{90} = \frac{75}{100}$

TRY IT 9

Translate to a proportion: What number is 60% of $105?$

Show answer

$$\frac{n}{105} = \frac{60}{100}$$

EXAMPLE 10

Translate to a proportion. 19 is 25% of what number?

Solution

Identify the parts of the percent proportion.	$\underbrace{19}_{\text{amount}}$ is $\underbrace{25\%}_{\text{percent}}$ of $\underbrace{\text{what number ?}}_{\text{base}}$
Restate as a proportion.	19 out of what number is the same as 25 out of 100?
Set up the proportion. Let $n = \text{number.}$	$\frac{19}{n} = \frac{25}{100}$

TRY IT 10

Translate to a proportion: 36 is 25% of what number?

Show answer

$\frac{36}{n} = \frac{25}{100}$

EXAMPLE 11

Translate to a proportion. What percent of 27 is 9?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What percent}}_{\text{percent}}$ of $\underbrace{27}_{\text{base}}$ is $\underbrace{9?}_{\text{amount}}$
Restate as a proportion.	9 out of 27 is the same as what number out of 100?
Set up the proportion. Let $p = \text{percent.}$	$\frac{9}{27} = \frac{p}{100}$

TRY IT 11

Translate to a proportion: What percent of 52 is 39?

Show answer

$\frac{n}{100} = \frac{39}{52}$

Translate and Solve Percent Proportions

Now that we have written percent equations as proportions, we are ready to solve the equations.

EXAMPLE 12

A human body is made up of mostly water. As we age, total body water content also diminishes so that by the time we are in our eighties the percent of water in our bodies has decreased to around 45%. If your grandfather weights 80 kg, what number is the 45% of his weight?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What number}}_{\text{amount}}\ \text{is}\ \underbrace{45\%}_{\text{percent}}\ \text{of}\ \underbrace{80?}_{\text{base}}$
Restate as a proportion.	What number out of 80 is the same as 45 out of 100?
Set up the proportion. Let n $=$ number.	$\frac{n}{80} = \frac{45}{100}$
Find the cross products and set them equal.	$100 \cdot n = 80 \cdot 45$
Simplify.	$100n = 3{,}600$
Divide both sides by 100.	$\frac{100n}{100} = \frac{3{,}600}{100}$
Simplify.	$n = 36$
Check if the answer is reasonable.	
Yes. 45 is a little less than half of 100 and 36 is a little less than half 80.	
Write a complete sentence that answers the question.	36 kg is 45% of 80 kg.

TRY IT 12

Adult male typically are composed of about 60% water. If a male weights 86 kg, what number is 60% of $86?$

Show answer
51.6 kg

In the next example, the percent is more than 100, which is more than one whole. So the unknown number will be more than the base.

EXAMPLE 13

According to Wikipedia, the red blood cells of an average adult human male store collectively about 2.5 grams of iron. This represents 65% of the total iron contained in the body. Find the total amount of iron contained in the body. Round your answer to the nearest tenth.

Solution

Identify the parts of the percent proportion.	65% = percent, 2.5 grams = amount, base = ?
Restate as a proportion.	65% of what number is 2.5
Set up the proportion. Let $n =$ number.	$\frac{65}{100} = \frac{2.5}{n}$
Find the cross products and set them equal.	$65n = 100 \cdot\ 2.5$
Simplify.	65 n = 250
Divide both sides by 65.	$\frac{65n}{65} = \frac{250}{65}$
Simplify.	n = 3.8
Check if the answer is reasonable.	
Yes.100% is more than 65% and 3.8 is more than 2.5.	
Write a complete sentence that answers the question.	The total amount of iron in the body of an average adult human male is 3.8 grams.

TRY IT 13

Translate and solve using proportions: 70% of 64 is what number?

Show answer
44.8

EXAMPLE 14

The daily recommended intake for potassium is 4700 grams. If a medium sized banana contains 425 grams of potassium, what percent of the daily recommended intake is that? Round to one decimal place.

Solution

Identify the parts of the percent proportion.	4700 = base, 425 = amount, percent = ?
Restate as a proportion.	What percent of 4700 is 425?
Set up the proportion. Let n = number.	$\frac{425}{4700} = \frac{n}{100}$
Find the cross products and set them equal.	$4700n = 425 \cdot \ 100$
Simplify.	4700 n = 42500
Divide both sides by 4700.	$\frac{4700n}{4700} = \frac{42500}{4700}$
Simplify.	n = 9
Check if the answer is reasonable.	
Yes. 9 is less than 10 and 425 is less than 470.	
Write a complete sentence that answers the question.	425 is 9% of 4700.

TRY IT 14

Translate and solve using proportions: What percent of 72 is $27?$

Show answer
37.5%

Key Concepts

- **Proportion**
    - A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.The proportion states two ratios or rates are equal. The proportion is read “a is to b, as c

is to d".

- **Cross Products of a Proportion**
    - For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$, its cross products are equal: $a \cdot d = b \cdot c$.
- **Percent Proportion**
    - The amount is to the base as the percent is to 100. $\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$

Glossary

proportion
A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".

6.3 Exercise Set

In the following exercises, write each sentence as a proportion.

1. 4 is to 15 as 36 is to 135.
2. 12 is to 5 as 96 is to 40.

In the following exercises, determine whether each equation is a proportion.

3. $\frac{7}{15} = \frac{56}{120}$
4. $\frac{11}{6} = \frac{21}{16}$
5. $\frac{12}{18} = \frac{4.99}{7.56}$
6. $\frac{13.5}{8.5} = \frac{31.05}{19.55}$

In the following exercises, solve each proportion.

7. $\frac{x}{56} = \frac{7}{8}$
8. $\frac{49}{63} = \frac{z}{9}$
9. $\frac{5}{a} = \frac{65}{117}$
10. $\frac{98}{154} = \frac{-7}{p}$
11. $\frac{a}{-8} = \frac{-42}{48}$
12. $\frac{2.6}{3.9} = \frac{c}{3}$
13. $\frac{2.7}{j} = \frac{0.9}{0.2}$
14. $\frac{\frac{1}{2}}{1} = \frac{m}{8}$

In the following exercises, solve the proportion problem.

15. Pediatricians prescribe 5 millilitre s (ml) of acetaminophen for every 25 pounds of a child's weight. How many millilitres of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?
16. At the gym, Carol takes her pulse for 10 sec and counts 19 beats. How many beats per minute is this? Has Carol met her target heart rate of 140 beats per minute?
17. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12

ounces of the drink?

18. Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?
19. Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child's weight. How many milligrams will the doctor prescribe?
20. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds. How many beats per minute is this? Has Kevin met his target
21. One brand of microwave popcorn has 120 calories per serving. A whole bag of this popcorn has 3.5 servings. How many calories are in a whole bag of this microwave popcorn?
22. Marissa loves the Caramel Macchiato at the coffee shop. The 16 oz. medium size has 240 calories. How many calories will she get if she drinks the large 20 oz. size?
23. For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

In the following exercises, translate to a proportion.

24. What number is 35% of 250?
25. 45 is 30% of what number?
26. What percent of 85 is 17?

In the following exercises, translate and solve using proportions.

27. What number is 65% of 180?
28. 17% of what number is .65?
29. What percent of 56 is 14?

Answers

1. $\frac{4}{15} = \frac{36}{135}$
2. $\frac{12}{5} = \frac{96}{40}$
3. yes
4. no
5. no
6. yes
7. 49
8. 47
9. 9
10. -11
11. 7
12. 2
13. 0.6
14. 14.4
15. 9 ml
16. 114, no
17. 159 cal
18. $\frac{3}{4}$ cup

19. 90 mg
20. 162, no
21. 420 cal
22. 300 cal
23. 180 mg
24. $\frac{n}{250} = \frac{35}{100}$
25. $\frac{45}{n} = \frac{30}{100}$
26. $\frac{17}{85} = \frac{p}{100}$
27. 117
28. 45
29. 25%

Attributions

1. This chapter has been adapted from “Solve Proportions and their Applications” in *Prealgebra* (OpenStax) by Lynn Marecek, MaryAnne Anthony-Smith, and Andrea Honeycutt Mathis, which is under a CC BY 4.0 Licence. Adapted by Izabela Mazur. See the Copyright page for more information.
2. OER.hawaii.edu
3. Wikipedia
4. Government of Canada Statistics

7. Data Analysis I

Data is collected daily from a variety of sources for the purpose of providing information. Once you have collected data, what will you do with it? Data can be described and presented in many different formats. For example, suppose you are interested in buying a house in a particular area. You may have no clue about the house prices, so you might ask your real estate agent to give you a sample data set of prices. Looking at all the prices in the sample often is overwhelming. A better way might be to look at the median price and the variation of prices. The median and variation are just two ways that you will learn to describe data. Your agent might also provide you with a graph of the data.

In this chapter, you will study numerical and graphical ways to describe and display your data. You will learn how to calculate, and even more importantly, how to interpret these measurements and graphs. A table can be used to collect and organize data which can then be more easily analyzed to determine patterns or trends. Frequency distributions and stem-and-leaf plots provide a tabular view that can be more revealing than a basic table. A graph is a tool that helps you learn about the shape or distribution of a sample or a population. A graph can be a more effective way of presenting data than a mass of numbers because we can see where data clusters and where there are only a few data values. Newspapers and the Internet use graphs to show trends and to enable readers to compare facts and figures quickly. Statisticians often graph data first to get a picture of the data. Then, more formal tools may be applied. Some of the types of graphs that are used to summarize and organize data are the bar graph, the histogram, the frequency polygon (a type of broken line graph), the pie chart, and the box plot.

In this chapter, we will look at ways to collect, present and describe data. We will also consider how data can be presented in misleading ways.

Learning Objectives

By the end of the chapter the student should be able to:

- Present and analyze data using frequency distributions, stem-and-leaf plots, pictographs, bar graphs, line graphs, and pie charts.
- Describe and calculate the central measures of tendency: mean, median and mode
- Design a statistical experiment, collect the data and analyze the results.

7.1 Measures of Central Tendency

Learning Objectives

By the end of this section it is expected that you will be able to:

- Calculate three measures of central tendency: the mean, median and mode
- Describe what the mean, median and mode tell us about a data set

When we discuss the mountains of the world we often hear mention of the world's highest peak, Mount Everest. Aside from the highest, there are other criteria which can be used to compare mountains. Perhaps we might want to determine the average height of the mountains within a specified mountain range. We may wish to determine who has made the most number of summit attempts of a particular mountain and what that number of attempts was. These kinds of questions can be answered with statistics.

Measures of Central Tendency

In our daily lives we encounter data and statistics on a regular basis. The data can be analysed to ascertain patterns and trends or to determine measures of central tendency. Three statistics that measure the center point of a set of data are mean, median and mode. Each of these measures of central tendency serves a different purpose and provides a different perspective.

Consider a college level biology class of 44 students that receives feedback on their midterm exam. The instructor shares the following results with the class: the average score was 68%, the most common score was 48%, and the score that ranked right in the middle of the class was 75%. The three values each represent a different measure of central tendency for the exam scores. What can we conclude about how the class performed? An examination of each of the three measures will help answer this question.

Mean

In the college biology class the average score on the midterm exam was 68%. The average score is also referred to as the **mean.** The average score of 68% on the biology exam is the sum of all exam scores divided by the number of scores (44 students).

Mean

The **mean** is the sum of all data items divided by the number of data items. This can be expressed as :

$$mean = \bar{x} = \frac{\Sigma x}{n}$$

The **symbol** $\sum$ is called the **summation symbol** and indicates that all data items are to be summed. The mean or average of a set of data is calculated by adding all of the values and dividing by the number of values.

Consider a runner who is training for a 10 km race. Her coach requires that she run an average distance of 15 km a day. Over a two week period the runner logged the following distances (in km) on her daily runs.

Su	M	T	W	Th	F	Sa	Su	M	T	W	Th	F	Sa
15	16	14	22	15	10	30	0	15	20	20	24	5	32

Did the runner meet the coach’s requirement? To determine the answer we need to calculate the average or mean.

To determine the average daily distance the calculation would be:

$$\bar{x} = \frac{15+16+14+22+15+10+30+0+15+20+20+24+5+32}{14} = \frac{238}{14} = 17\,km$$

On average the runner covered 17km/day which exceeds the coach’s required 15km/day.

EXAMPLE 1

Fifteen students wrote a math test and received the following grades: 89, 45, 78, 76, 73, 98, 73, 92, 88, 73, 100, 51, 64, 80, 95.

Solution

To determine the mean:

Add the grades and divide by the number of grades (15)
(89+45+78+76+73+98+73+92+88+73+100+51+64+80+95)/15 = 1175/15 = 78.33…

The average grade, rounded to the nearest whole number, was 78.

TRY IT 1

Find the mean of the data set: 5.7, 3.4, 7.8, 9.2, 3.8, 1.6

Show answer

$$\frac{5.7+3.4+7.8+9.2+3.8+1.6}{6}=5.25$$

Data is often compiled in the form of tables and it is more efficient to calculate the mean using aggregates of each value.

EXAMPLE 2

A class of 46 students were asked to rate their instructor on a scale of 1 to 5, with 5 being the highest. The table provides the number of students assigning a score of 1, 2, 3, 4, or 5.

Score	Number of Students Assigning this Score
1	2
2	5
3	15
4	11
5	13

Solution

To determine the mean, sum the 46 scores and divide by 46:

$\frac{(1\times2)+(2\times5)+(3\times15)+(4\times11)+(5\times13)}{46} = \frac{166}{46} = 3.6$(rounded to one decimal place)

The instructor received an average score of 3.6.

TRY IT 2

A coffee outlet sells coffee in 4 different sizes: small, medium, large, and extra large. The table provides a summary of one morning's sales.

a) Calculate the total number of ounces of coffee sold.

b) Ignoring the four cup sizes, what was the average number of ounces of coffee consumed per purchase (to the nearest ounce)?

c) Based on your answer, which of the four sizes is most representative of the typical coffee purchase?

Size	**Quantity of Coffee**	**Number of Purchasers**
small	12 oz.	29
medium	15 oz.	47
large	20 oz.	52
extra large	24 oz.	11

Show answer

a) (29×12 oz) + ((47x15oz) + (52x20oz) + (11x24oz) = 2357 ounces

b) $\frac{(12\times29)(15\times47)(20\times52)(24\times11)}{29+47+52+11} = 17\,oz$ per purchase

c) This 17 oz cup is closest to a medium cup.

Median

When the values in a set of data are quite different we can consider another measure of central tendency called the **median.** In the biology class with 44 students the score that ranked right in the middle of the class was 75%. This means that half of the exam scores were lower than 75% and half of the exam scores were higher than 75%.

Median

The **median** is the data item in the middle of each set of ranked, or ordered, data. The median separates the upper half and the lower half of a data set. It is the "middle" value of the data set when it is arranged from highest to lowest values or from lowest values to highest values. If there are two middle values, the median is the average of these two values.

To find the **median** of a group of data items:

1. Arrange the data items in order, from smallest to largest or from largest to smallest.
2. If the **number of data items is odd**, the median is the data item in the middle of the list.
3. If the **number of data items is even**, the median is the mean of the two middle data items.

Returning to the runner who logged the following distances over two weeks:

Su	M	T	W	Th	F	Sa	Su	M	T	W	Th	F	Sa
15	16	14	22	15	10	30	0	15	20	20	24	5	32

The median would be determined by ranking the distances from smallest to largest:

$$\underbrace{0\quad 5\quad 10\quad 14\quad 15\quad 15\quad 15}_{7\text{ items}}\quad \underbrace{16\quad 20\quad 20\quad 22\quad 24\quad 30\quad 32}_{7\text{ items}}$$

The number of data items is **even (14 items)**, therefore the **median** is the average of the two middle items. The two middle values are 15km and 16km. The average of these is (15 + 16)/ 2 = 15.5km so the median distance is 15.5 km. This indicates that for the runner half of the distances run would be less than 15.5 km and half would be greater than 15.5 km.

If the number of data items is **odd,** the median is the data item that lies exactly in the middle of the ranked data items; it separates the data items into two halves.

EXAMPLE 3

Fifteen students wrote a math test and received the following grades: 89, 45, 78, 76, 73, 98, 73, 92, 88, 73, 100, 51, 64, 80, 95. What was the median grade?

Solution

To determine the **median**:

1. Order the data set from highest to lowest values: 100, 98, 95, 92, 89, 88, 80, 78, 76, 73, 73, 73, 64, 51, 45
2. Determine the middle value: 100, 98, 95, 92, 89, 88, 80, 78, 76, 73, 73, 73, 64, 51, 45

Note that there is an odd number of values (15) so there is exactly one middle value. The median value is the 8th value since 15 values/2 = the 7.5th or 8th value. Counting from highest to lowest or from lowest to highest values, the 8th value is 78.

In example 1 the **mean** was calculated as 78.3… In this case the mean and median are quite close. It is important to note that the mean and median may not always be so close.

TRY IT 3

Find the median of the data set: 5.7, 3.4, 7.8, 9.2, 3.8, 1.6

Show answer

$$\frac{3.8+5.7}{2}=4.75$$

When data is presented in the form of a table the median can be determined using a few different methods.

EXAMPLE 4

A class of 46 students were asked to rate their instructor on a scale of 1 to 5, with 5 being the highest possible rating. The table provides the number of students assigning a score of 1, 2, 3, 4, or 5.

Score	**Number of Students Assigning this Score**	**Total Number of Scores**
1	2	2 scores of 1
2	5	2+5= 7 scores of 1 or 2
3	15	7+15= 22 scores of 1, 2 or 3
4	11	22+11= 33 scores of 1, 2, 3, or 4
5	13	33+13= 46 scores of 1, 2, 3 ,4 or 5

Solution

To determine the median, divide the number of student scores by 2: 46/2=23 This indicates that there will be 23 scores in the top half and 23 scores in the bottom half, therefore the middle score will be the average of the 23rd and 24th scores. Working from the low score of 1 to the high score of 5, the first 22 scores were either 1, 2, or 3. The 23rd score is 4 and the 24th score is 4, therefore the median score is 4.

Note: An alternate method for determining the median would be to list every score from low to high and then count until the middle score is reached: 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4 ⇑ 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5

The 23rd and 24th scores are 4 and 4 so the median is the average of these (4+4)/2 = 4

TRY IT 4

A coffee outlet sells coffee in 4 different sizes: small, medium, large, and extra large. The table provides a summary of one morning's sales. Determine which of the four sizes, small, medium, large or extra large would be the median size purchased.

Size	**Number of Purchasers**
small (12 oz)	29
medium (15 oz.)	47
large (20 oz.)	52
extra large (24 oz.)	11

Show answer

$\frac{29+47+52+11}{2} = 69.5$. Therefore since 69 purchasers were in the bottom half and 69 were in the top half, the median purchaser was a medium cup sup of 15 oz..

Mode

For the college biology class the most common score was 48%. This grade represents the **mode** for the set of exam scores, as the **mode** is the data value that occurs **most often**. Although a mode of 48% may seem to indicate a poor overall result, this is not necessarily the case. A score of 48% could possibly have occurred only twice for the entire class if all other grades each occurred only once.

Mode

The **mode** is the most frequently occurring data value. A data set can have more than one mode. If there are two values that each occur the same number of times then the data set is **bimodal.**

EXAMPLE 5

Fifteen students wrote a math test and received the following grades: 89, 45, 78, 76, 73, 98, 73, 92, 88, 73, 100, 51, 64, 80, 95.

Solution

To determine the mode: Count the number of times each data value occurs. In this example the value that occurs three times, 73, is the mode.

TRY IT 5

Over a two week period a runner logged the following distances (in km) on her daily runs. Determine the mode (the distance that she ran most often).

Su	M	T	W	Th	F	Sa	Su	M	T	W	Th	F	Sa
15	16	14	22	15	10	30	0	15	20	20	24	5	32

Show answer
The mode is 15 km

When data is presented in tabular form the mode is often apparent.

EXAMPLE 6

A class of 46 students were asked to rate their instructor on a scale of 1 to 5, with 5 being the highest. The table provides the number of students assigning a score of 1, 2, 3, 4, or 5.

Scores	**Number of Students assigning this Score**
1	2
2	5
3	15
4	11
5	13

Solution

The mode is the value that occurs 15 times, which is 3.

It is possible to have more than one mode in a set of data. If two different data values occur most often the data is considered to be bimodal. In example 2, if the students assigning scores of 1 changed their scores to 5, then the score of 3 and the score of 5 would both occur 15 times. In this case the data set would be bimodal.

If each value in a set of data occurs the same number of times then there is said to be no mode. In Try It 1 the data set is 5.7, 3.4, 7.8, 9.2, 3.8, 1.6. There is no mode for this set of data.

TRY IT 6

A coffee outlet sells coffee in 4 different sizes: small, medium, large, and extra large. The table provides a summary of one morning's sales. Determine which of the four sizes, small, medium, large or extra large would be the mode.

Size	**Number of Purchasers**
small (12 oz.)	29
medium (15 oz.)	47
large (20 oz.)	52
extra large (24 oz.)	11

Show answer

Since 52 of the purchasers chose the large size of 20 oz. the 20 oz cup size is the mode.

Mean, Median or Mode?

The three measures of central tendency can yield very different results for one data set, as evidenced by examples one and two above. When choosing which measure to use, consideration must be given to the purpose.

In example 1 the average grade and median are both 78 but the mode is 73. Although the mean is often used for grade comparisons, the median is also useful as it serves as an indicator of the midpoint of the grade distribution (distribution will be covered in a later section of this text). Although the mode is used less often with grades, it does indicate to students that although a grade of 73 was below the average, it was the most common grade.

EXAMPLE 7

Consider again the example 6 involving the instructor evaluation. In a class of 46 students, the mean evaluation score was 3.6, the median was 4 and the mode was 3. Which measure is most useful: mean, median or mode? Why?

Solution

The difference in these values may seem slight, yet each provides a different perspective. The mean and the median are probably most useful.

The mean score of 3.6 does not truly reflect any of the possible choices since only scores of 1, 2, 3, 4, and 5 are possible but it does indicate that on average students are more happy (than not) with the instructor. The median score of 4 indicates that half of the students awarded the instructor a score of 4 or 5 so the median also provides an encouraging result. The mode indicates that out of the entire class a ranking of 3 was given most often (15 times) by one third of the students but it doesn't yield any information about the other two thirds of the evaluations.

TRY IT 7

Referring back to the coffee outlet, discuss with a classmate: which measure is most useful to the owner of the coffee outlet? mean, median or mode? Why?

Show answer
Answers may vary.

Key Concepts

- Three measures of central tendency are the **mean, median** and **mode**
- **To determine the mean:**
  Add all data values and divide by the number of data values. $mean = \bar{x} = \frac{\Sigma x}{n}$
- **To determine the median:**
  Rank the data values from smallest to largest and determine the middle value.
- **To determine the mode:**
  Count the number of occurences of each data value and determine which value occurs most often.

Glossary

mean
is the average of all data values.
median
is the data value that divides all ranked data values into two equal parts. It need not be one of the data values.
mode
is the data value that occurs most often.

7.1 Exercise Set

1. Find the mean, median and mode for the data set: 55, 45, 35, 65, 25, 75, 85
2. Find the mean, median and mode for the data set: 68, 55, 63, 68, 55, 63, 45, 68
3. Find the mean, median and mode for the data set:

1	1	1	2	2
2	2	3	3	3
3	3	4	5	6

4. Find the mean, median and mode for the data set: 25, 2, 7, 47, 56, 27, 2, 17, 56
5. A hotdog vendor sold the following number of hotdogs over a two week period.

56	72	67	85	55	59	65
32	82	49	66	52	70	44

   a. What was the total number of hotdogs sold in the two weeks?
   b. What was the average number of hotdogs sold over the two week period?
   c. What was the median number of hotdogs sold over the 2-week period?

6. A runner has four different routes that he chooses from. Over the period of one month the runner chose one circuit every day as indicated in the chart.

Circuit	Number of times chosen
Very Easy (5 km)	6
Pleasant (10 km)	12
Challenging (20 km)	8
Exhausting (40 km)	2

   a. What was the total number of kilometres run during the month?
   b. What was the average number of kilometres run per day?
   c. What was the median number of kilometres run per day?

7. The ages of all students in a precalculus math class are:

17	18	18	18	19	19	19	19
19	19	19	20	20	20	20	21
21	21	22	22	23	24	25	25
28	31	31	35	48	49	50	70

   a. Determine the mean, median and mode.
   b. If the highest and lowest ages are removed from the class, how are the mean, median and mode impacted?

8. The ages of all students in a graduate English class are:

25	25	25	28	29	29	30
30	31	31	31	34	38	42

a. Determine the mean, median and mode.

b. If the highest and lowest ages are removed from the class, how are the mean, median and mode impacted?

9. A math instructor returned a graded exam to the class of 44 students. The instructor announced that 15 students scored 63%. and 15 students scored 71%. Which measure(s) of central tendency can be determined from this information?
10. Before returning a graded exam to a class of students the instructor announced that the mean was 55% and the median was 78%. What does this indicate about how the students in the class performed on the exam?

Answers

1. mean 55; median 55; mode none
2. mean 60.625; median 63; mode 68
3. mean 2.73…; median 3; mode 3
4. mean 26.55…; median 25; mode 2, 56 – data bimodal
5.
   a. 854 hotdogs
   b. 61 hotdogs
   c. 62 hotdogs
6.
   a. 390 km
   b. 13.9 km
   c. 10 km
7.
   a. mean 25.9, median 21, mode 19
   b. The new mean is lower at 24.8 and the median and mode are the same
8.
   a. mean 30.6, median 30, bimodal 25 & 31
   b. The new mean is lower at 30.1, the median remains at 30 and there is one mode of 31
9. the mode
10. From the median we know that half of the students scored 78% or better. Since the mean was only 55% we can conclude that some scores were quite low (less than 55%).

7.2 Graphs and Tables

Learning Objectives

By the end of this section it is expected that you will be able to:

- Extract information from a table, a bar graph, a line graph or a pie graph
- Create a stem and leaf graph from a set of data
- Create a frequency distribution table from a set of data
- Create a line graph, a bar graph and a pie graph (with or without technology)
- Compare a bar graph to a histogram

We have seen how data can be represented numerically with measures such as the mean, median and mode. Data can be organized and displayed in visual formats that allow the user to more easily extract information. When we represent data graphically we can determine data clusters, make comparisons, or determine trends.

Displaying Data with Tables or Graphs

We will consider some graphical alternatives for displaying the information presented in the following paragraph:

> According to Venture Kamloops, the six largest employers in Kamloops, British Columbia, along with the number of employees in parentheses are: Interior Health Authority (3398), School Distric#73 (1924), Thompson Rivers University (1092), Highland Valley Copper Mine (1351), the City of Kamloops (761) and Bc Lottery Corporation (440) http://venturekamloops.com/pdf/DBIK-Community-Facts-Residential-April-2017.pdf

Tables

Transferring the data to a table, as in Table 1, provides greater clarity. The reader can quickly determine the names of the employers and their corresponding number of employees. It is easier to determine the employer with the greatest and least number of employees.

Table 1

Employer	**Number of Employees**
Interior Health Authority	3398
School District #73	1924
Thompson Rivers University	1092
Highland Valley Copper Mine	1351
City of Kamloops	761
BC Lottery Corporation	440

Graphs

When the data is represented visually the reader can quickly retrieve information and make comparisons. Technology can be used to easily create a wide variety of graphs. The data from the table was entered into a spreadsheet and three graphs were generated. The results are displayed below as a bar graph (Fig. 1a), a pie graph (Fig. 1b), and a waterfall graph (Fig. 1c).

Bar Graph

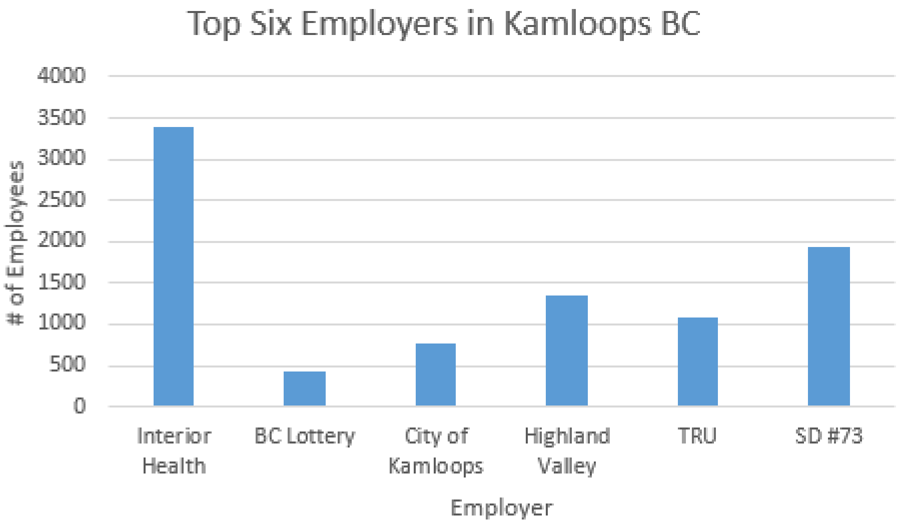


Fig. 1a

Circle or Pie Graph

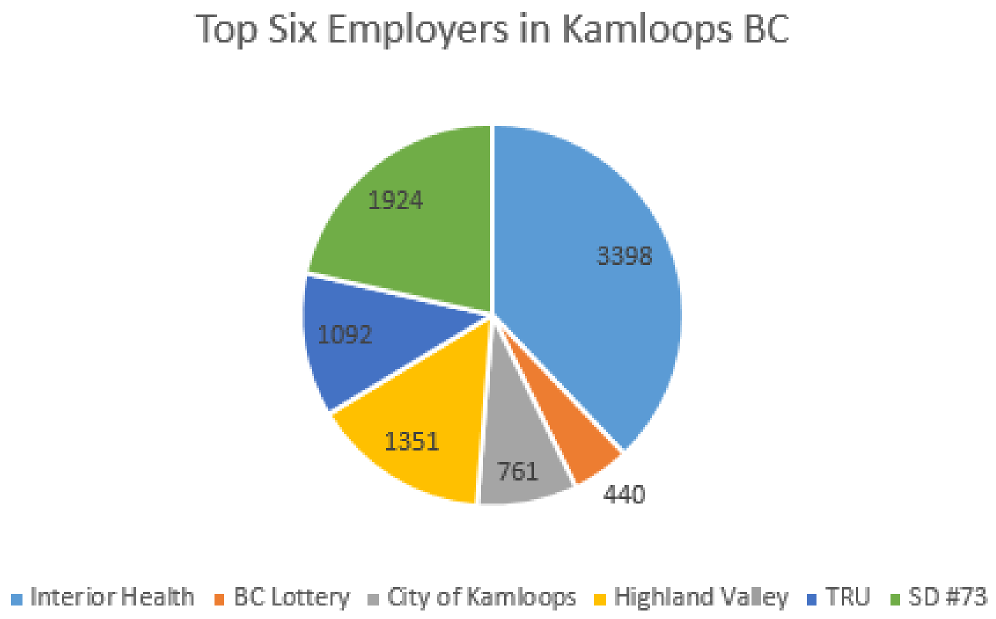


Fig. 1b

Waterfall

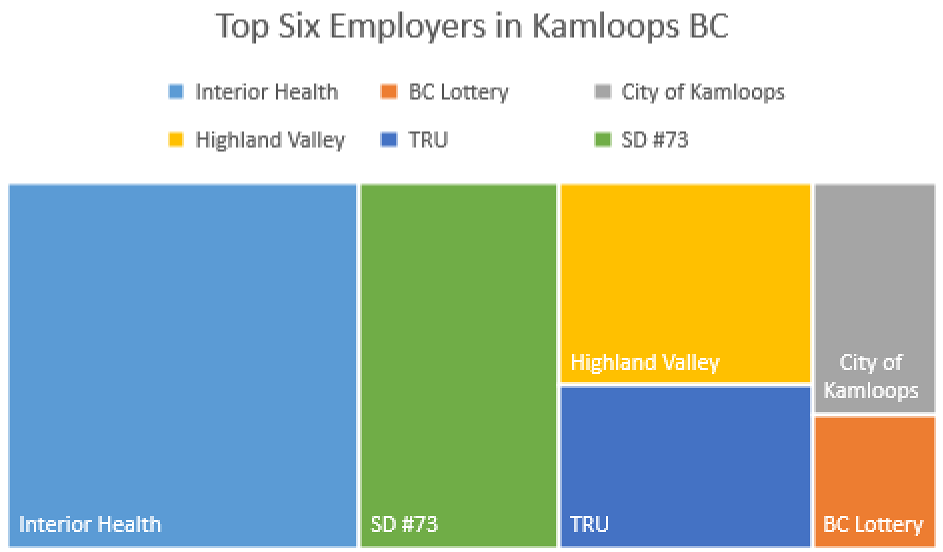


Fig. 1c

Consider each of the three graphs in Figures 1a, 1b and 1c (bar, pie and waterfall) to answer the following questions. Beside each answer indicate which of the three graph(s) provides the answer.

1. Which of the six employers has the most number of employees?
2. How many employees work for the largest employer?

3. Which of the six employers has the least number of employees?
4. How many employees work for the smallest employer?
5. Where does TRU place in the ranking of number of employees?
6. Which graph is the easiest to extract data from?

The answers to the six questions would be:

1. Interior Health has the most number of employees. This information is found in all three graphs.
2. Interior Health employs 3398 people. This information can only be determined using the pie graph
3. BC Lottery has the least number of employees. This information is found in all three graphs.
4. BC Lottery employs 440 people. This information can only be determined using the pie graph
5. TRU ranks fourth in the number of employees. This can be stated with certainty by using the bar graph or pie graph. The reader may not be so certain with the waterfall graph.
6. This depend on which information is required.

Note that there is not necessarily one form of graph that is better than the others. It is important to recognize that the way in which the information is presented will impact its use. By making one change, such as including the numerical values for the bar or waterfall graphs, the user would be able to obtain more exact information.

When choosing and creating a table or graph it is important to know what kind of information is required. A decision can then be made as to how best to depict this. Since technology provides easily accessible tools for creating tables and charts, this section will focus on the **features** of different tables and graphs rather than on the manual construction of the graphs.

We will now take a closer look at line graphs, bar graphs, and circle graphs as well as stem-and-leaf plots and frequency tables.

Stem-and-Leaf Graph

One simple graph, the stem-and-leaf graph or stemplot, is a good choice when the data sets are small. This graph indicates data clusters and can be used to determine the measures of central tendency.

A stem-and-leaf graph divides each observation of data into a stem and a leaf. The leaf consists of one final significant digit. For example, 23 has a stem of 2 and a leaf 3. The number 432 has a stem of 43 and a leaf of 2. Likewise, the number 5,432 has a stem 543 and a leaf of two. The decimal 9.3 has a stem of nine and a leaf of three.

To create the plot, write the stems in a vertical line from smallest to largest. Draw a vertical line to the right of the stems. Then write the leaves in increasing order next to their corresponding stem.

EXAMPLE 1

For Susan’s spring pre-calculus class, scores for the first exam were as follows (ranked from lowest to highest):

33; 42; 49; 49; 53; 55; 55; 61; 63; 67; 68; 68; 69; 69; 72; 73; 74; 78; 80; 83; 88; 88; 88; 90; 92; 94; 94; 94; 94; 96; 100

a) Create a stem-and-leaf graph for the data.

b) Describe where the data clusters.

c) What percentage of the students obtained a score of 90 or better?

d) What is the mean, median and mode?

Solution

a) To create the graph, rank the data from lowest to highest.

Create the column for the stems. This will be the first digit in a two digit number and the first two digits in a three digit number. The stems will start at 3 and end at 10.

For each data value, add each leaf to its corresponding stem. For the value 33. the stem is 3 and the leaf is 3. For the value 68 the stem is 6 and the leaf is 8. Since 68 occurs twice in the data set, for the stem of 6 there will be two leaves of 8.

Stem	**Leaf**
3	3
4	2 9 9
5	3 5 5
6	1 3 7 8 8 9 9
7	2 3 4 8
8	0 3 8 8 8
9	0 2 4 4 4 4 6
10	0

b) There appears to be two clusters of data. The stemplot shows that most scores fell in either the 60s or the 90’s.

c) Eight out of the 31 scores or approximately 26% were in the 90s or 100.

d) The mean is 73.5. Since there are 31 students, the median is the 16th score, which is 73. The mode is 94 as it occurs 4 times.

TRY IT 1

For the Park City basketball team, scores for the last 30 games were as follows (from lowest to highest):
32; 32; 33; 34; 38; 40; 42; 42; 43; 44; 46; 47; 47; 48; 48; 48; 49; 50; 50; 51; 52; 52; 52; 53; 54; 56; 57; 57; 60; 61

a) Construct a stem-and-leaf graph for the data.

b) In what percent of the games did the team score less than 40 points?

c) Use the graph to determine the mean, median and mode.

Show answer
a)

Stem	Leaf
3	2 2 3 4 8
4	0 2 2 3 4 6 7 7 8 8 8 9
5	0 0 1 2 2 2 3 4 6 7 7
6	0 1

b) 16.7%

c) Mean is 47.3; Median is 48; Bimodal 48 and 52

The stem-and-leaf graph presents a quick way to graph data and it gives an exact picture of the data. It also provides an opportunity to recognize outliers. An **outlier** is an observation of data that does not fit the rest of the data. It is sometimes called an **extreme value**. When you graph an outlier, it will appear not to fit the pattern of the graph. Some outliers are due to mistakes (for example, writing down 50 instead of 500) while others may indicate that something unusual is happening.

EXAMPLE 2

A restaurant was scouting for a new location. It wants to be within walking distance to theatres or performing arts facilities. It gathered data for the distances (in kilometres) between a potential new location and several theatres or arts facilities:

1.1; 1.5; 2.3; 2.5; 2.7; 3.2; 3.3; 3.3; 3.5; 3.8; 4.0; 4.2; 4.5; 4.5; 4.7; 4.8; 5.5; 5.6; 6.5; 6.7; 12.3

a) Create a stemp-and-leaf graph for the data. Note: The leaves are the digits to the right of the decimal.

b) Do the data seem to have any concentration of values? What does this indicate to the restaurant about this potential location?

c) Do there appear to be any outliers?

d) Determine the median and the mean.

e) Eliminate the outlier and recalculate the mean. What impact does the outlier have on the mean?

Solution

a)

Stem	**Leaf**
1	1 5
2	3 5 7
3	2 3 3 5 8
4	0 2 5 5 7 8
5	5 6
6	5 7
7	
8	
9	
10	
11	
12	3

b) Values appear to concentrate between three and five kilometres. This potential location might not be best as many of the theatres and arts facilities are not within walking distance.

c) The value 12.3 km appears to be an outlier.

d) The median is the 11th data value or 4.0 km The mean is 4.3 km.

e) The mean will be 3.91 km. The outlier results in a much larger mean (4.3 km rather than 3.91 km).

TRY IT 2

The following data show the distances (in kilometres) to a college from the homes of the members of the counselling department:

0.5; 0.7; 1.1; 1.2; 1.2; 1.3; 1.3; 1.5; 1.5; 1.7; 1.7; 1.8; 1.9; 2.0; 2.2; 2.5; 2.6; 2.8; 2.8; 2.8; 3.5; 3.8; 4.4; 4.8; 4.9; 5.2; 5.5; 5.7; 5.8; 8.0

a) Create a stem-and-leaf graph using the data.

b) Determine the mean, median, mode and any outliers.

Show answer
a)

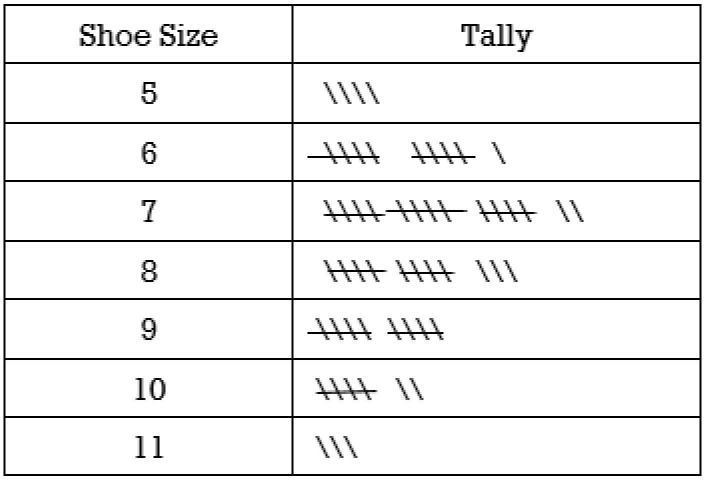
Stem	**Leaf**
0	5 7
1	1 2 2 3 3 5 5 7 7 8 9
2	0 2 5 6 8 8 8
3	5 8
4	4 8 9
5	2 5 7 8
6	
7	
8	0

b) Mean is 2.89 km; Median $\frac{2.2+2.5}{2} = 2.35km$ Mode 2.8 km Outlier 8.0 km

Frequency Distributions

Frequency is the number of occurrences of an event over a period of time. The frequency of a full moon is generally once a month. The frequency of one's birthday is once a year. A frequency distribution table illustrates the frequency or number of times that a specific outcome or data value occurs. Tally marks can be used to keep track of the number of occurences. Once the tally is complete the **frequency distribution table** can be created.

Consider a marketing survey where sixty-five females were asked their shoe size. The responses ranged from size 5 to size 11. A tally of the results is illustrated:



Shoe Size	Tally
5	\\\\
6	𝍸 𝍸 \
7	𝍸 𝍸 𝍸 \\
8	𝍸 𝍸 \\\
9	𝍸 𝍸
10	𝍸 \\
11	\\\

The tally is then easily converted to a **frequency distribution table** .

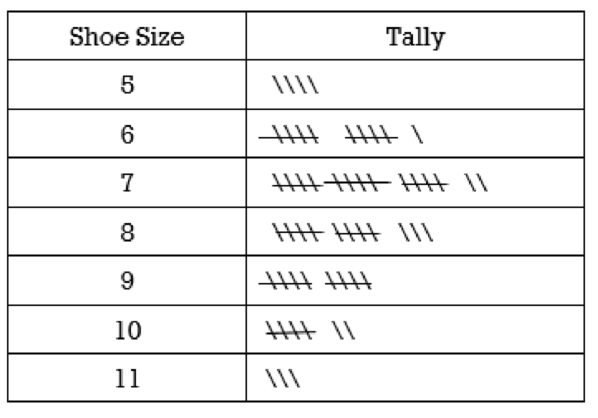
Shoe Size	Number of Females
5	4
6	11
7	17
8	13
9	10
10	7
11	3

Frequency Distribution

A **frequency distribution** can show the **absolute frequency** and the **relative frequency**. The absolute frequency is the number of occurences of a data value. The relative frequency is the ratio of the number of occurrences of a data value to the total number of data values.

EXAMPLE 3.1

a) Create a frequency distribution table to show the absolute frequency and the relative frequency for the shoe size tally of 65 females:



Shoe Size	Tally
5	\\\\
6	~~\\\\~~ ~~\\\\~~ \
7	~~\\\\~~ ~~\\\\~~ ~~\\\\~~ \\
8	~~\\\\~~ ~~\\\\~~ \\\
9	~~\\\\~~ ~~\\\\~~
10	~~\\\\~~ \\
11	\\\

b) Which shoe size was the most common? What percentage of the females wear this size?

c) Which shoe size was the least common? What percentage of the females wear this size?

Solution

a) The frequency table will require 3 columns and 8 rows:

Shoe Size	Absolute Frequency	Relative Frequency
5	4	6%
6	11	17%
7	17	26%
8	13	20%
9	10	15%
10	7	11%
11	3	5%

The **absolute frequency** is the number of females with a specific shoe size.

The **relative frequency** is the ratio of the number of females with a specific shoe size to the total number of females. Since there are 65 females in the survey, the relative frequency for shoe size 5 is 4/65 = 0.0615 = 6% Note: the relative frequencies have been converted from decimals to percentages and rounded to the nearest whole number.

b) Size 7 is the most common with 26%

c) Size 11 is the least common with 5%

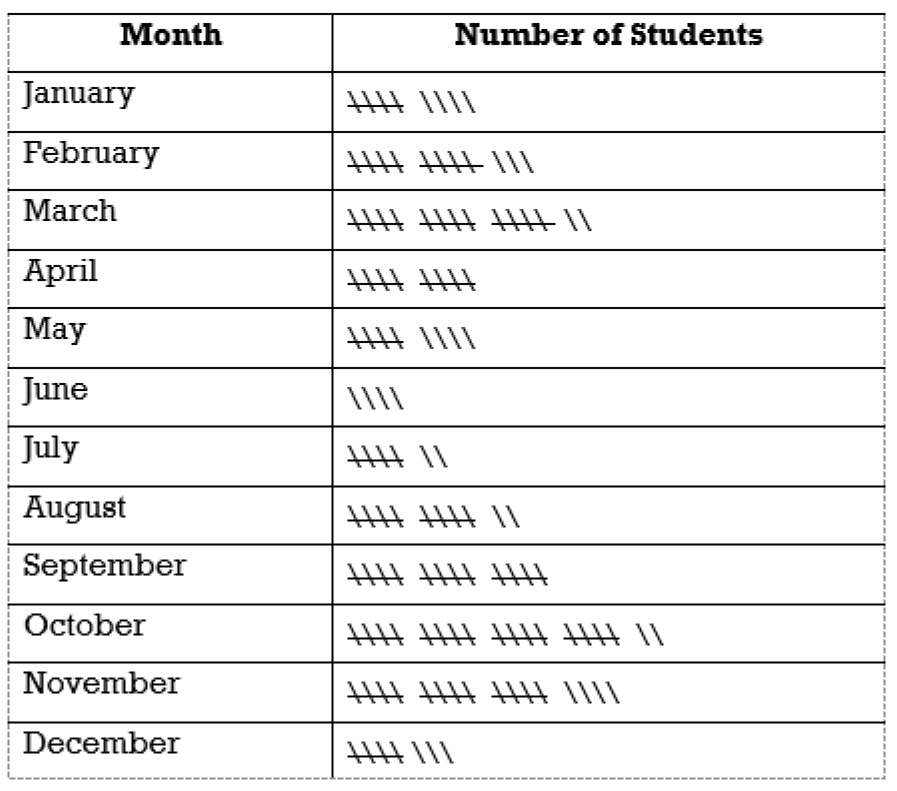
TRY IT 3

The tally of the birth months for a class of 145 students is shown in the following table.

a) Create a frequency distribution table that shows both the absolute and the relative frequencies. The **absolute frequency** is the number of birthdays. The **relative frequency** is the ratio of the number of birthdays to the total number of students. Note: Round the relative frequencies to the nearest whole number.

b) Which month is the most common? What percentage of the students had a birthday during this month?

c) Which month is the least common? What percentage of the students had a birthday during this month?



Month	Number of Students
January	𝍸 \\\\
February	𝍸 𝍸 \\\
March	𝍸 𝍸 𝍸 \\
April	𝍸 𝍸
May	𝍸 \\\\
June	\\\\
July	𝍸 \\
August	𝍸 𝍸 \\
September	𝍸 𝍸 𝍸
October	𝍸 𝍸 𝍸 𝍸 \\
November	𝍸 𝍸 𝍸 \\\\
December	𝍸 \\\

Show answer
a)

Month	**Number of Birthdays**	**Relative Frequency**
January	9	6%
February	13	9%
March	17	12%
April	10	7%
May	9	6%
June	4	3%
July	7	5%
August	12	8%
September	15	10%
October	22	15%
November	19	13%
December	8	6%

b) October is the most common birthday month with 15%.

c) June is the least common month with 3%.

Choosing an Appropriate Graph

Although a frequency distribution table provides quantitative information it does not allow the user to easily make comparisons or determine trends. The bar graph, line graph and pie (circle) graph provide quick visual representations of the data and allow the user to make comparisons and extract information. As stated earlier in this section, technology assists us with creating the graphs but it is the creater’s responsibility to determine the specifics. When creating a graph, consider the following:

- What information must be conveyed? Ranking, high and low values, trends?
- What type of graph will best suit this?   Bar, pie, line, waterfall…
- Select an appropriate title and labels for the axis. Without a title and labels the graph is virtually meaningless.
- What should the scale for each axis be? Should there be increments of 1, 10, 100, 1000….?
- How much detail or colour is useful or required? Consider whether to include numerical values (or not). Don’t go overboard with colour variations and information at the expense of neatness and conciseness.

Consider the bar graphs in Figures 2 and 3:

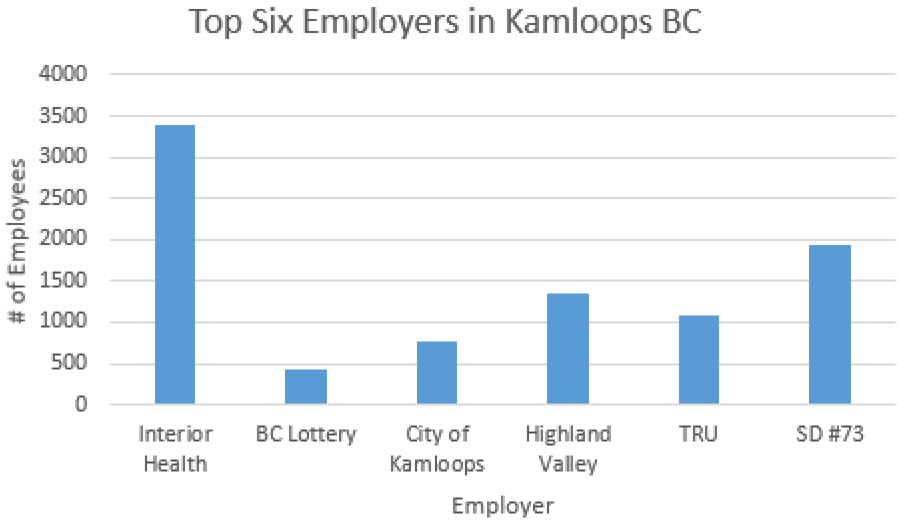


Fig. 2

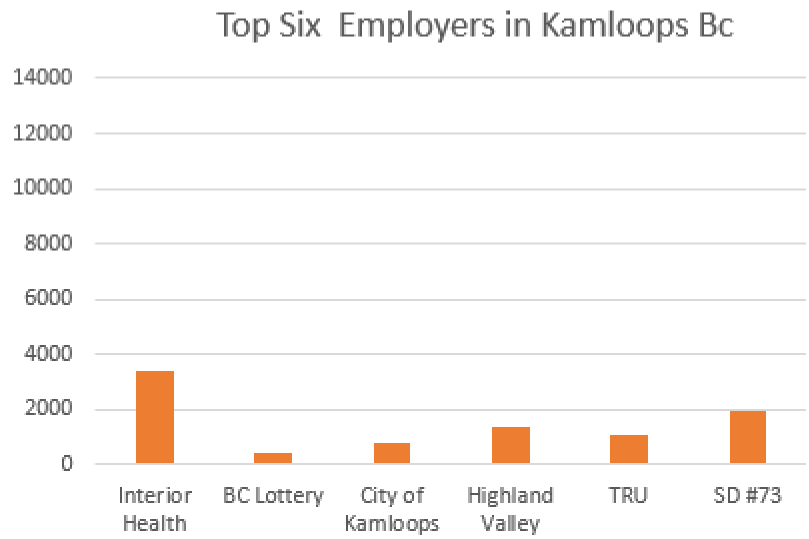


Fig. 3

Although the data values are identical for both bar graphs, it might not appear from figure 3 that Interior Health dominates as the top employer in Kamloops. This illustrates that the choice of scale is critical. In Figure 3 the graph is also missing the labels on the vertical and horizontal axes.

Consider the pie graphs in Figure 4 and Figure 5. Which is more informative?

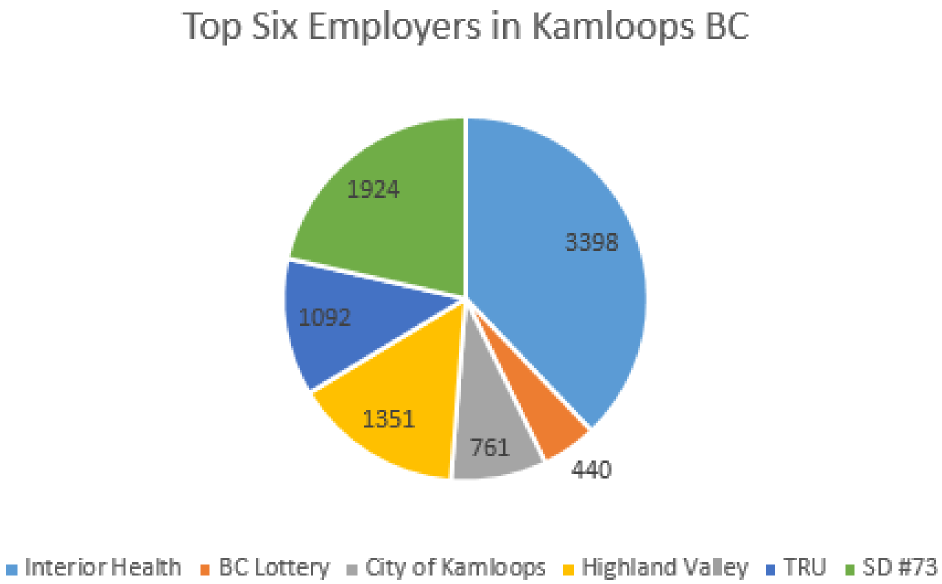


Fig. 4

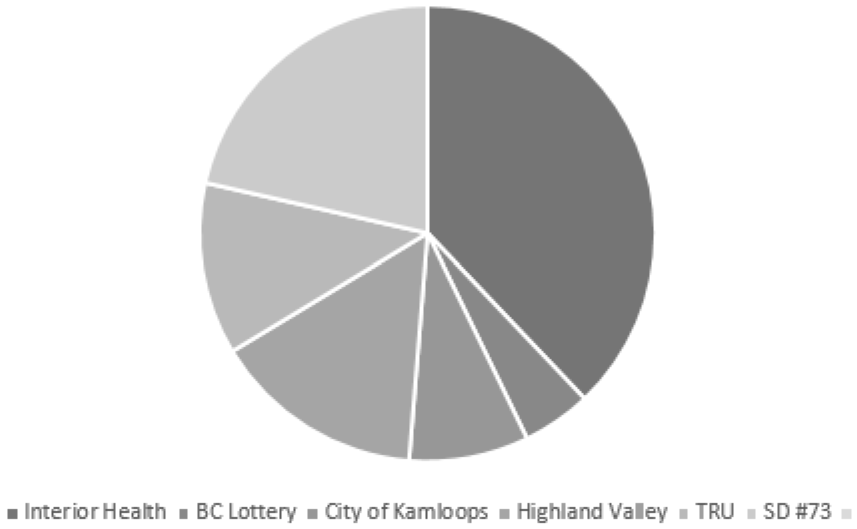


Fig. 5

When creating a graph be sure to include the title and any relevant information. The circle graph in Figure 5 is lacking a title which makes the graph meaningless. The addition of a title “Top Six Employers in Kamloops” would enable the user to determine rankings but not the actual number of employees. The addition of employee numbers as in Figure 4 would add further clarity to Figure 5. Note that although the colour in Figure 4 may make it more visually appealing, it is the title, labels and numerical values that are most informative.

Bar Graphs

A bar graph presents data using vertical or horizontal rectangular bars. Bar graphs are useful for making comparisons or for showing trends over time. One axis shows the categories and the other axis shows the values. The bar graph in Figure 6 indicates that there was a rising trend in the number of USDA (United States Department of Agriculture) certified domestic organic operations from 2005 to 2015. The reader can also make comparisons. In Figure 6 we can see that the number of certified domestic organic operations more than doubled between the years 2005 and 2015.

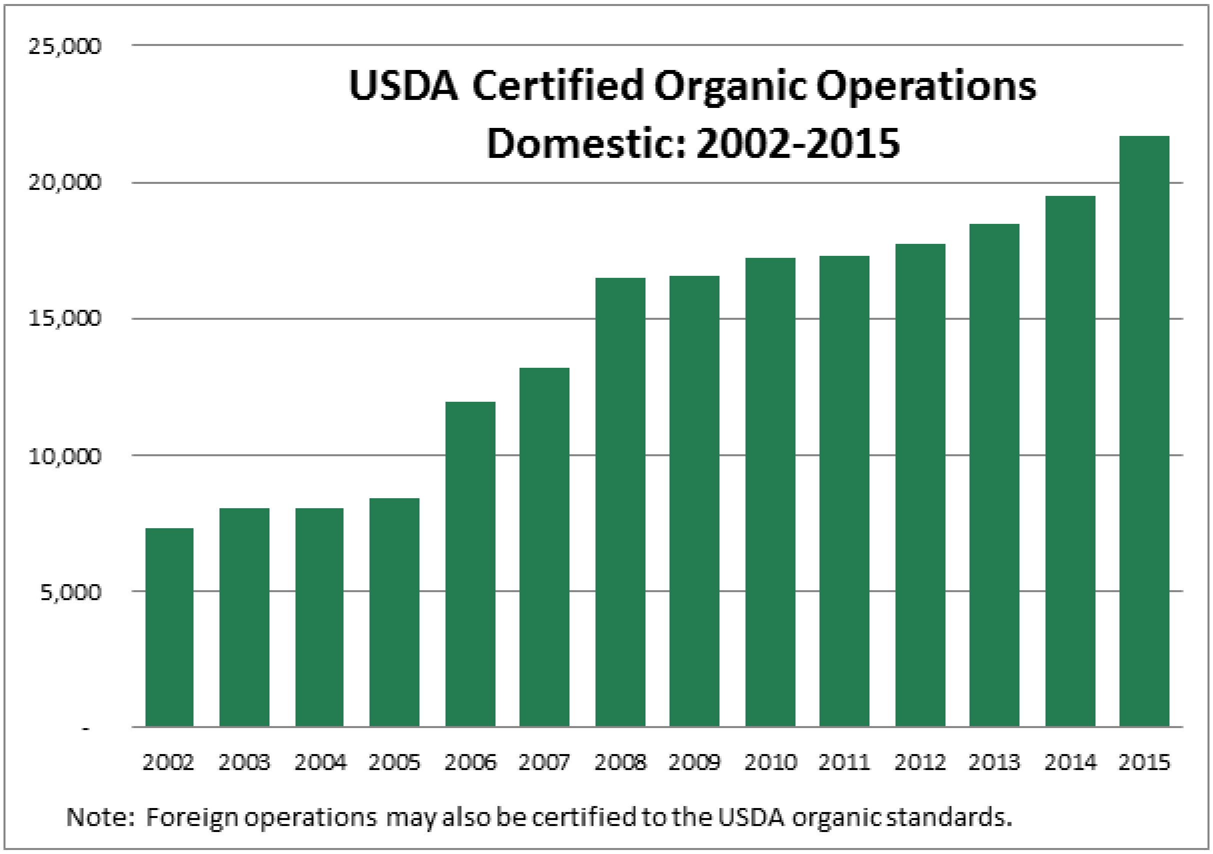


Fig. 6 "USDA Certified Organic Operations graphic" is licensed under CC BY 2.0

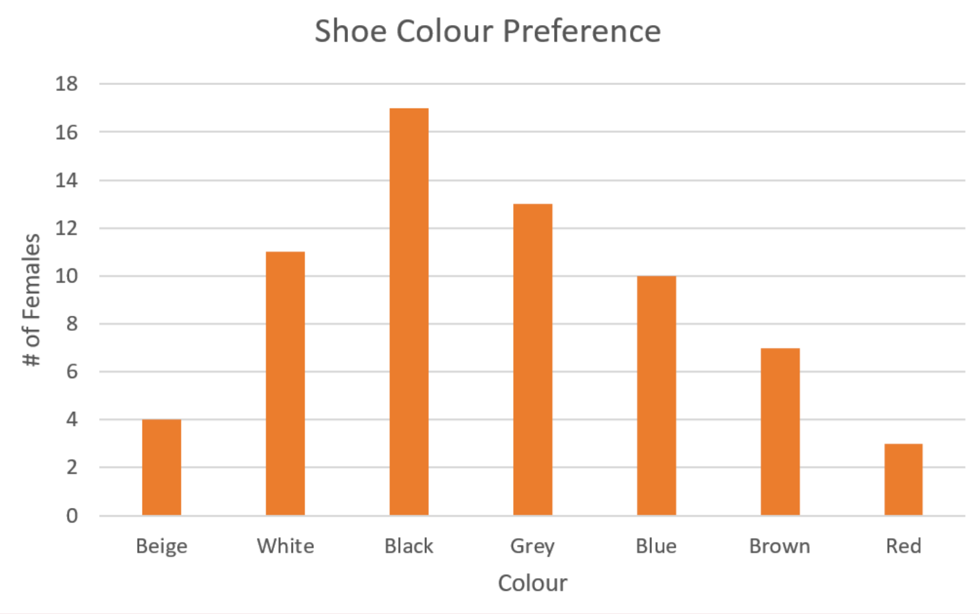
EXAMPLE 4

A retailer tracked the sale of a particular shoe style. The information in the bar graph illustrates the colour preference for one week of sales.

a) What was the most preferred colour? How many females preferred this colour?

b) What was the least preferred colour? How many females preferred this colour?

c) How many more females preferred grey over blue?



Solution

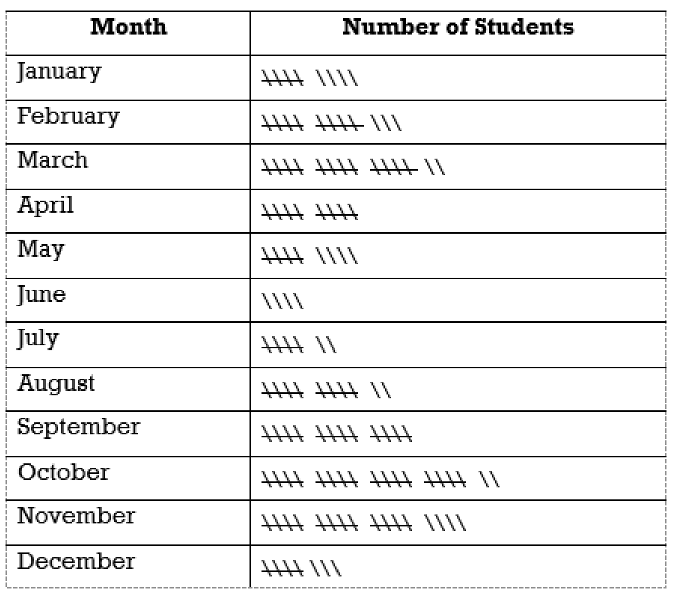
a) Black was the most preferred colour. 17 females preferred black.

b) Red was the least preferred colour. 3 females preferred red.

c) Three more preferred grey over blue.

TRY IT 4

a) Refer to the tally in TRY IT 3. Create a vertical bar graph for the distribution of birth months. Be sure to include a title, axis labels and select a reasonable scale for the values.



Month	Number of Students
January	卌 \\\\
February	卌 卌 \\\
March	卌 卌 卌 \\
April	卌 卌
May	卌 \\\\
June	\\\\
July	卌 \\
August	卌 卌 \\
September	卌 卌 卌
October	卌 卌 卌 卌 \\
November	卌 卌 卌 \\\\
December	卌 \\\

b) In which three months were there the most number of birthdays?

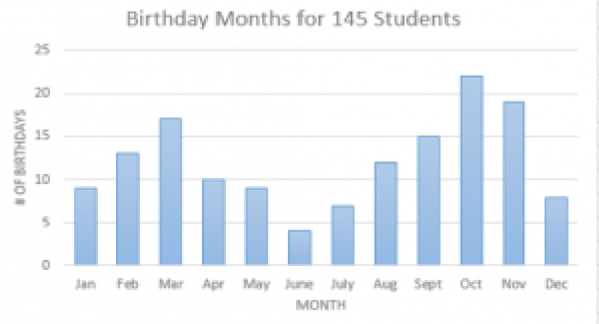
c) In which three months were there the least number of birthdays?

d) How many more birthdays were there in September as compared to April?

e) What is the trend in the number of birthdays over the course of the year?

Show answer

a)



b) October, November and March
c) June, July and December
d) 5 more in Sept. than in April
e) the no. increases in the spring and fall and decreases in the summer and winter months.

Some data sets are better represented as occuring in natural pairs. With shoe sizes or colours perhaps we might want to compare male and female responses. Bar graphs can be created to illustrate more than one category.

EXAMPLE 5

The Hornets hockey team entered the league in 2010. Each season consists of 80 games. Their win/loss record is provided in the table below.

Year	# of Wins	# of Losses
2010	33	47
2011	31	49
2012	36	44
2013	41	39
2014	44	36
2015	52	28
2016	57	23
2017	56	24
2018	59	21
2019	61	19

A bar graph provides a visual comparison of wins and losses each year.

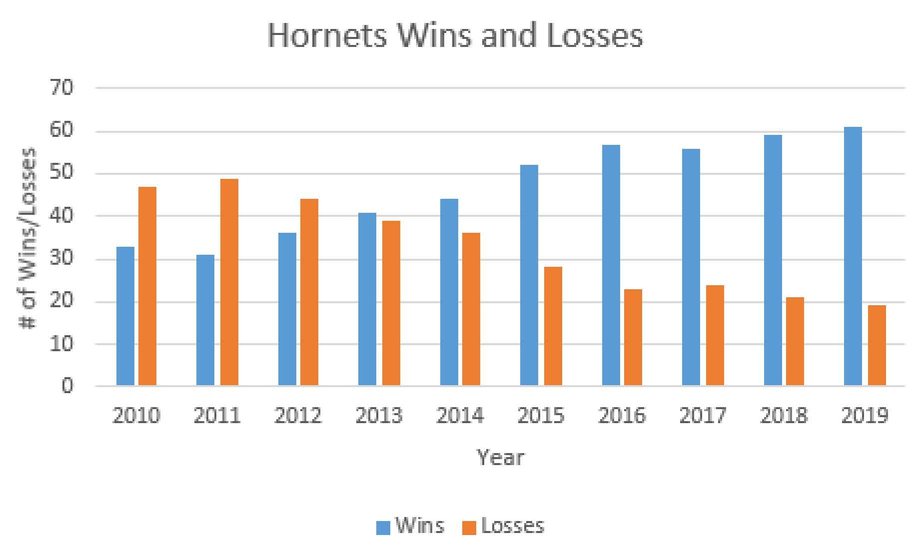
a) In which year were there the most losses? the most wins?

b) In which year were the number of wins and losses almost identical?

c) In which year did the number of wins exceed the number of losses (for the first time)?

d) Use the graph to estimate how many more wins than losses there were in 2016.

e) What was the trend in wins and losses from 2010 to 2019?



Solution

a) The team had its highest number of losses in its second year of operations 2011 and its highest number of wins in 2019.

b) 2013

c) 2013

d) 57 – 22 = 35 (note that the table indicates that it is actually 34)

e) Over the ten years, the number of wins has been increasing and the number of losses has been decreasing. The number of wins surpassed the number of loses for the first time in 2013.

Bar graphs can also be arranged in a stacked format. Refer to Figure 7. This type of bar graph illustrates the relationship between the parts and the whole. Although beyond the scope of this text it is worth illustrating.

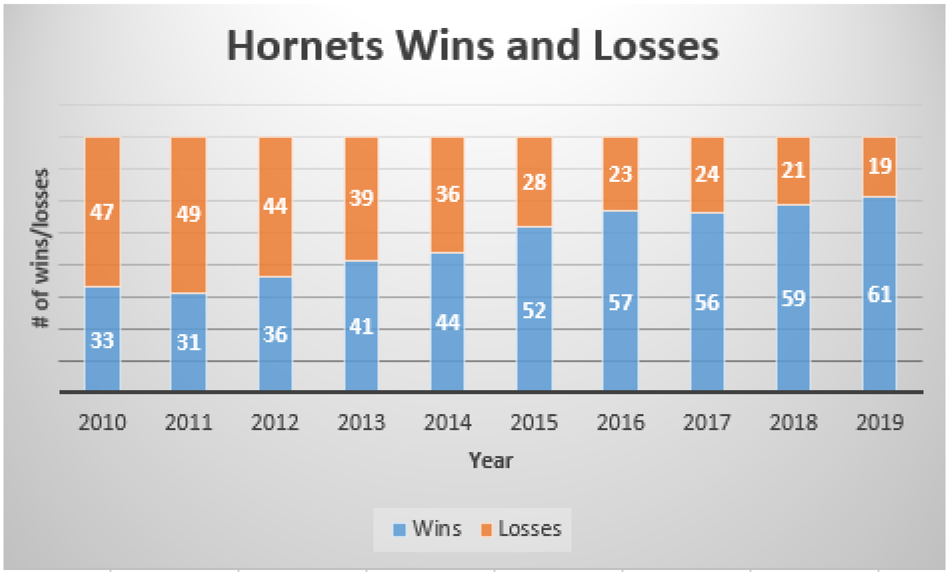


Fig. 7

TRY IT 5

The average high temperature (to the nearest degree Celsius) and the average monthly rainfall (in mm) for Phoenix Arizona are provided in the table below (Source: https://www.usclimatedata.com/climate/arizona/united-states/3172#) .

Month	**Temperature (Celsius)**	**Rainfall (mm)**
January	20	8
February	22	19
March	25	12
April	30	16
May	35	21
June	40	25
July	41	29
August	40	28
September	38	25
October	31	18
November	24	12
December	19	7

a) Create one bar graph illustrating both the average daily temperature and average rainfall for Phoenix.

b) In which month was there the most rainfall? The least rainfall?

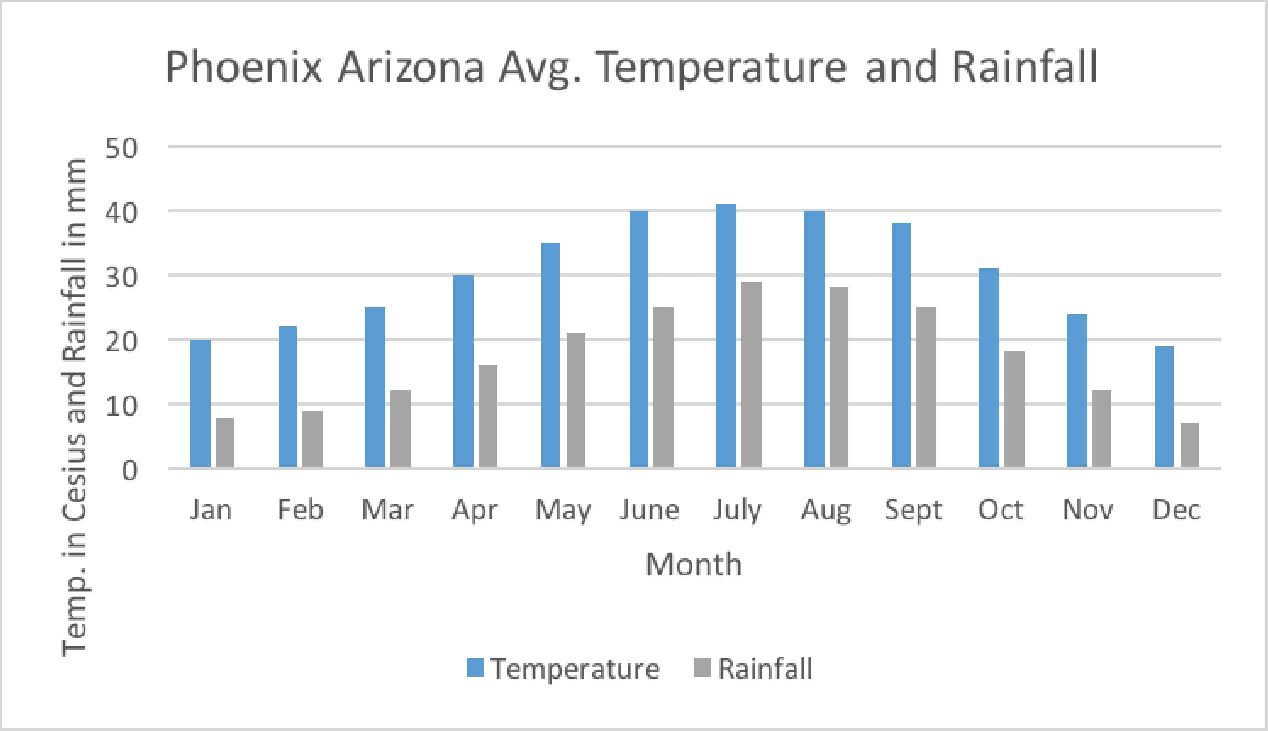
c) In which month was the average temperature the highest? the lowest?

d) What pattern is there as you compare the temperature trend with the rainfall trend?

e) Which is the better month to be in Phoenix? October or April? Why?

Show answer

a)



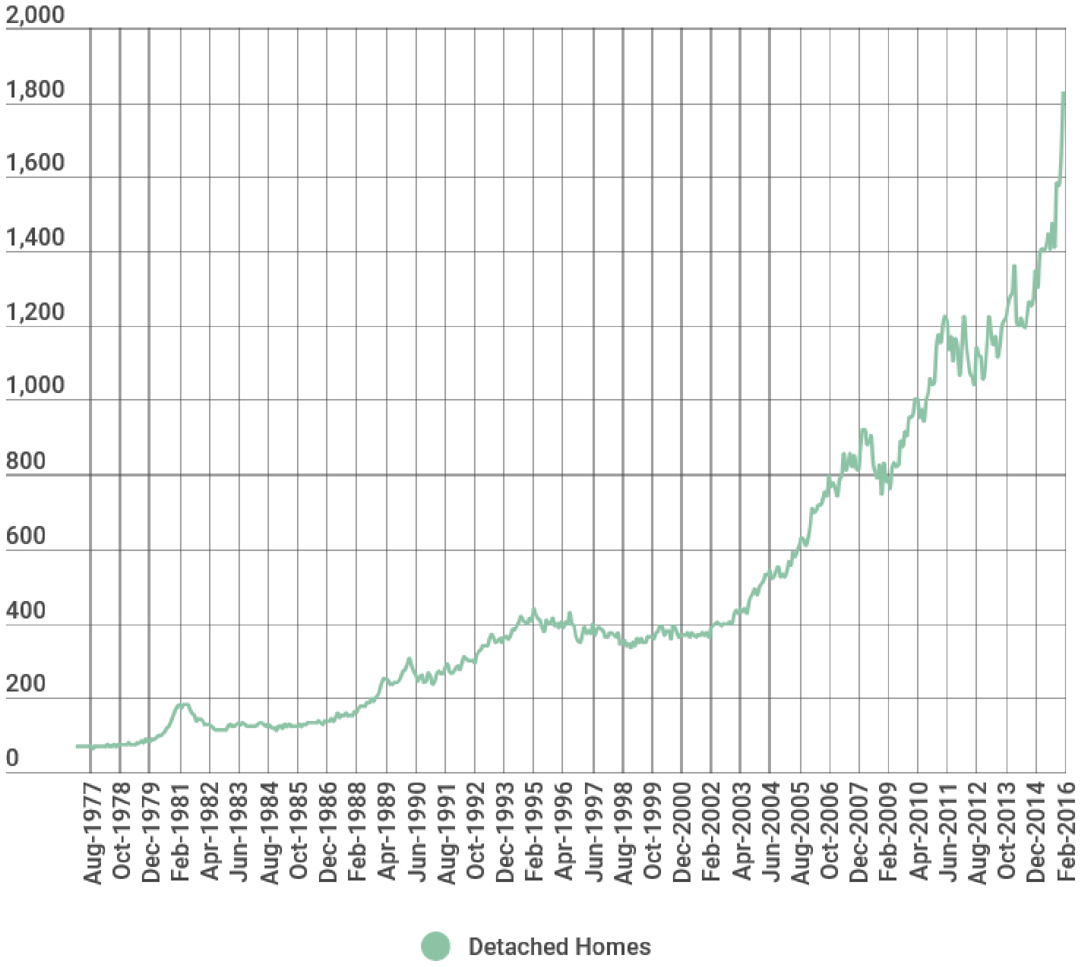
b) Most rainfall in July; least rainfall in December
c) Highest avg. temperature in July; lowest avg. temperature in December
d) As avg. temperature increases/decreases so does the rainfall
e) Both are very similar. In April it is not quite as warm and a little less rain so perhaps that might be preferred.

Line Graphs

Line graphs can be used to show data changes over time. The horizontal or x-axis represents time and the vertical or y-axis represents the data points which are plotted and joined by line segments. Trends and rates of change can be determined by considering the **slope** of the line. It is also possible to have more than one line on a graph.

Line graphs are useful for illustrating trends over time but accuracy can be lost. In Figure 8 the escalating increase in housing prices is evident but it is difficult to determine average house prices in a specific year.

Fig. 8 Average Price of Detached Homes in Vancouver BC (in $1000’s)

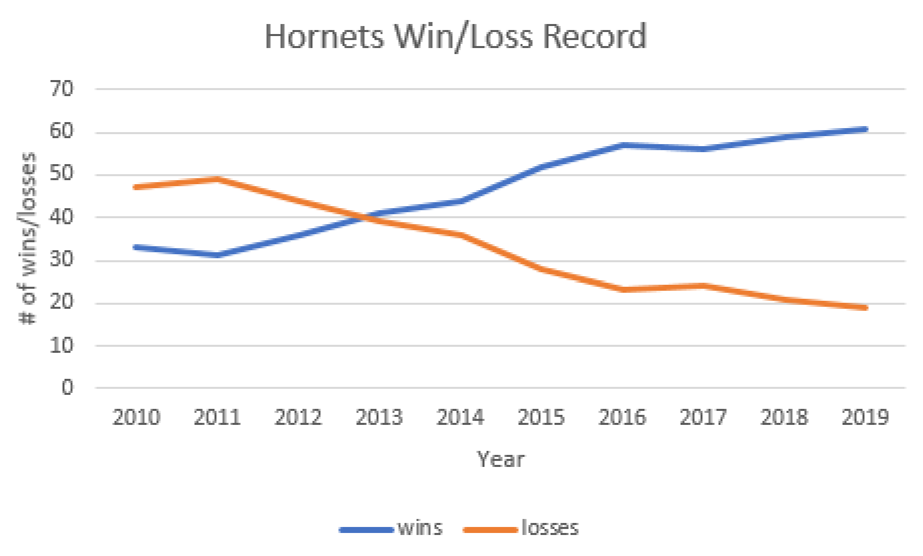


Information courtesy the Real Estate Board of Greater Vancouver

EXAMPLE 6

Consider the Hornets hockey team from Example 5. To construct a line graph, draw a horizontal axis to represent the years 2010 through 2019. The vertical axis will represent both the number of wins and the number of losses.

Solution

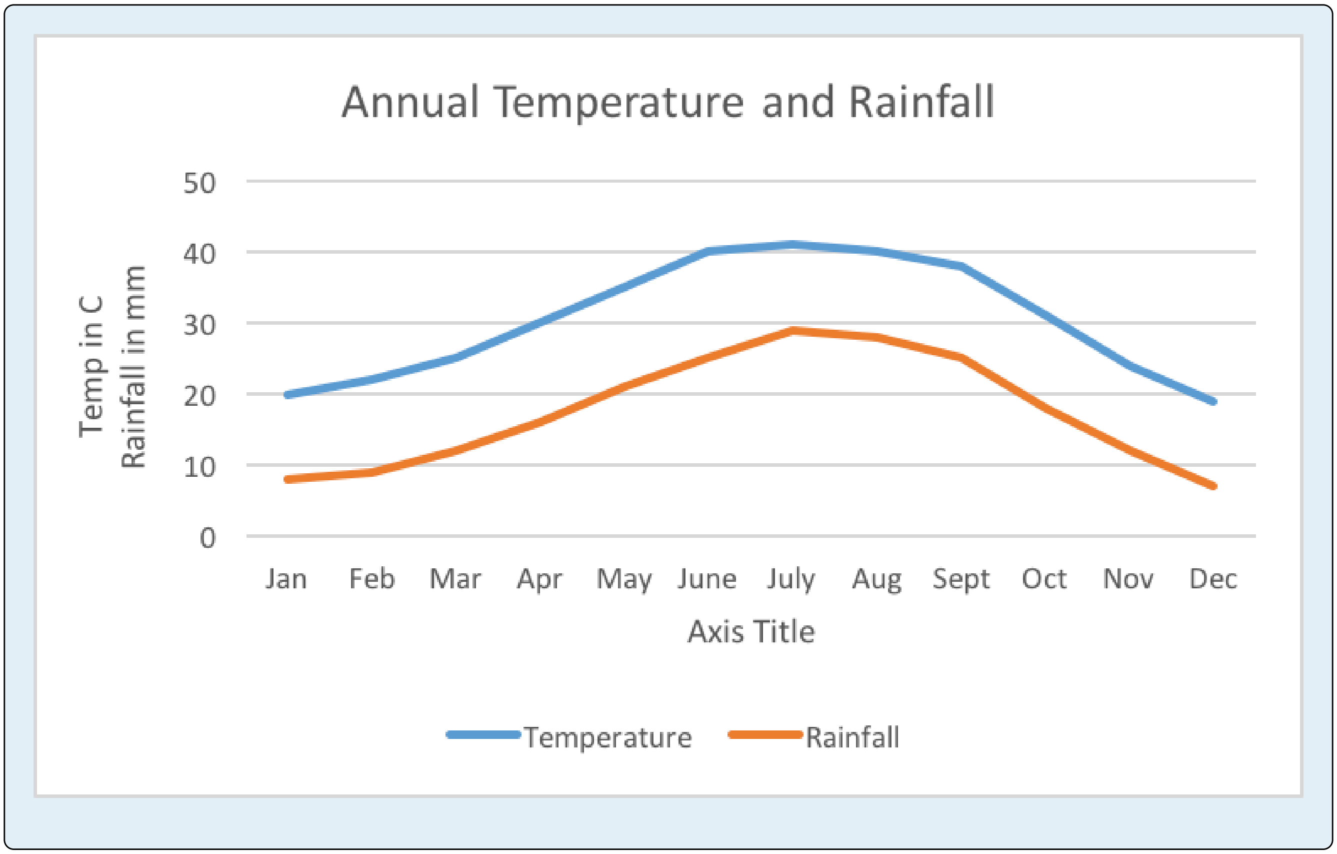


Several observations can be made from the line graph in Example 6. The number of wins increased every year except for 2010 to 2011 and 2016 to 2017. The number of wins first surpassed the number of losses in 2013 and continued to do so every year after that. The gap between the number of wins versus the number of losses was the highest in 2019. The lowest number of wins was in 2011 and the highest was in 2019. One might also make a prediction that based on the upward trend in wins that in 2020 the Hornets could have their best year ever. This is known as **extrapolating** from the data.

TRY IT 6

Use the data from Try It 5 to create a line graph representing the temperature and rainfall from January to December. Be sure to title and label the axes of your graph.

Show answer



Histograms

A different type of graph that also uses bars is the histogram. Histograms are used to illustrate the **distribution** of one specific data item such as height or temperature. In a histogram the data will be quantitative, as with income or heights. With a histogram the numerical data values are divided into "bins" or intervals. A bin could represent one data value or a range of data values. In the next example each bin represents one shoe size.

Reconsider example 3 with shoe sizes (qualitative) and example 4 with shoe colours (qualitative). Bar charts were created for both of these. A histogram could be created for the shoe sizes but not shoe colour. Refer to Figure 9. This histogram illustrates the frequency or occurrence of shoe sizes ranging from size 5 to size 11 where every bar (bin) represents one shoe size. The most frequent size is 7 and the other sizes are dispersed outward from size 7.

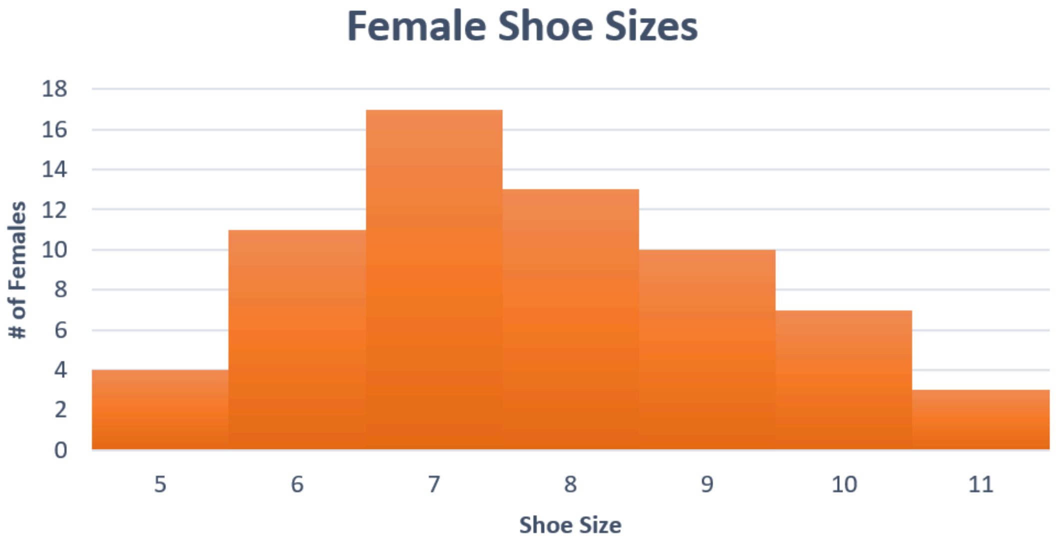


Fig. 9

Note that with a histogram there are no spaces between the bars and the bars range from low to high (or high to low). With a histogram the data values appear on the horizontal axis and the frequency (number of occurrences) appears on the vertical axis. In a histogram the data can be distinct quantities (as with shoe sizes) or it may be grouped into intervals. As an example consider a histogram representing hourly wages. The hourly wage could be distinct values: $15, $16, $17 or it could be intervals: $15-$16, $17-$18, $19-20.

Consider Figure 10 below. Every bar represents an interval that is half a unit: 0-0.5, 0.5-1, 1-1.5 and so on. From the histogram we can easily determine which interval occurs the most often and which occurs least often. We can also determine how the data values are clustered. In Figure 10 we see that the data clusters around the values -0.5 to 0.5.

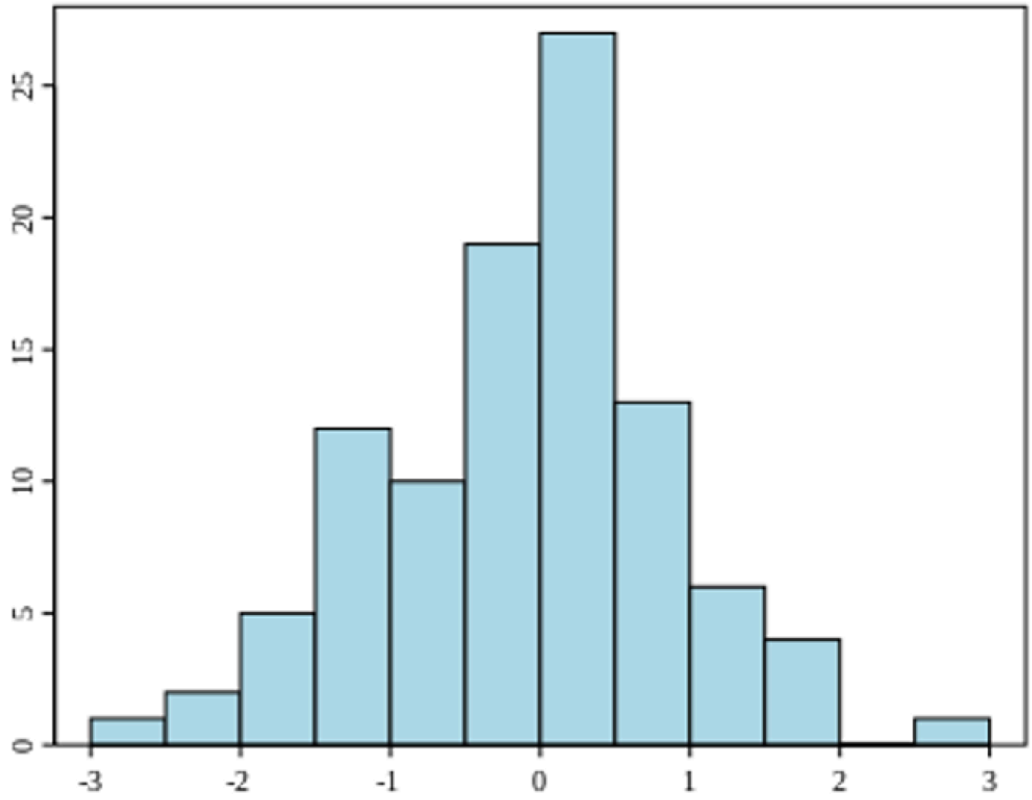


Fig. 10 Wikipedia in the public domain

Histograms are useful for representing the distribution or dispersion of data and as such will be revisited elsewhere in this book.

Glossary

frequency distribution

A table or graph that illustrates the number of times that a specific outcome or data value occurs within an interval.

histograms

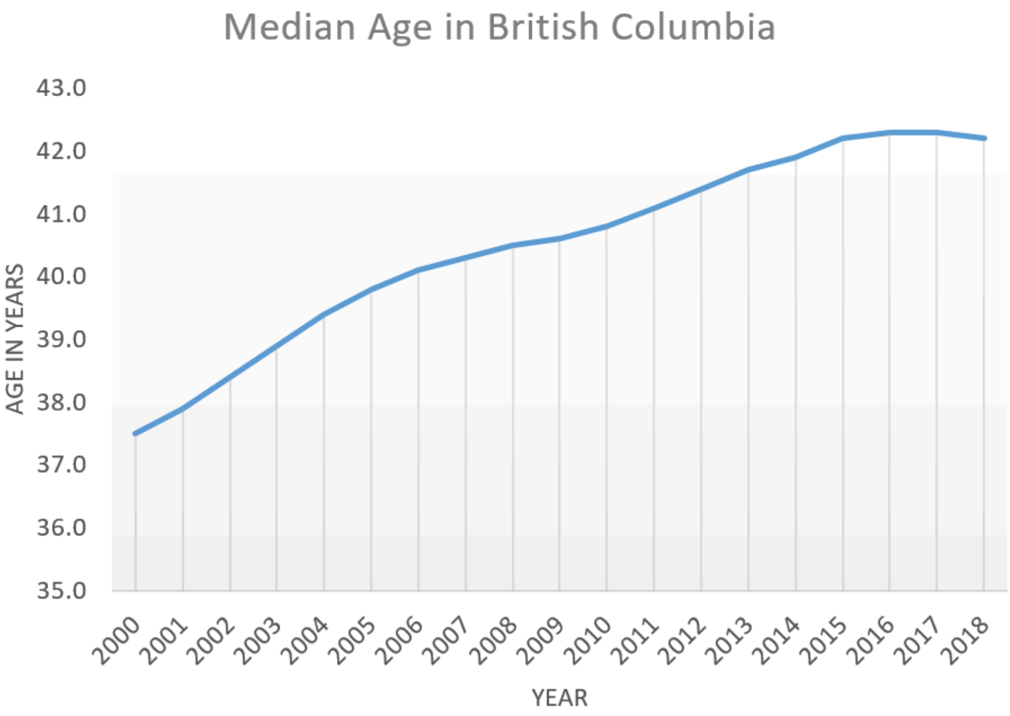
Used to illustrate the **distribution** of one specific data item such as height or temperature.

stem and leaf graph

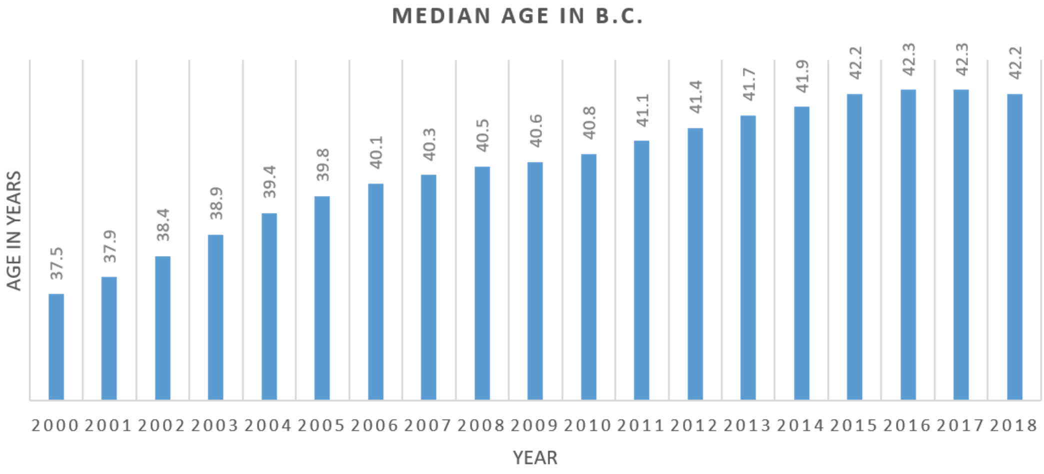
Divides each data observation into a stem and a leaf. The stem is the first digit or digits and the leaf is the last digit.

7.2 Exercise Set

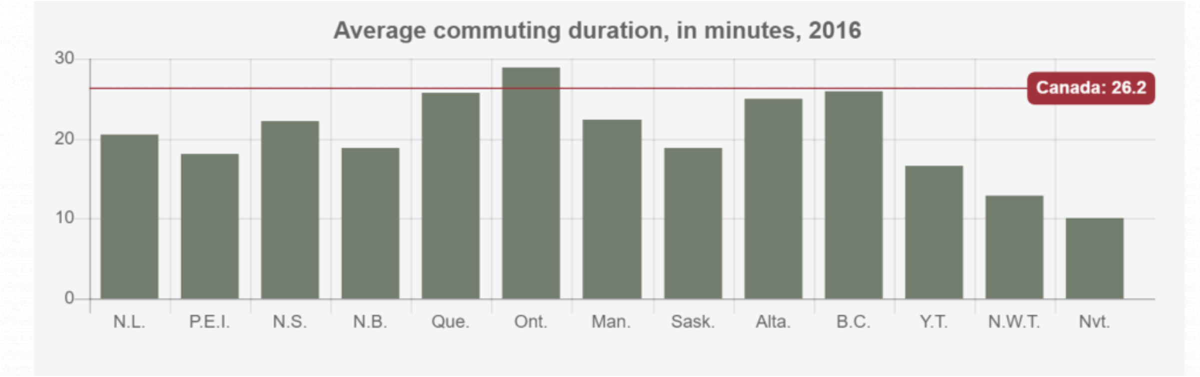
1. The two graphs below depict the median age of the population for the province of British Columbia (Source: https://www2.gov.bc.ca/gov/content/data/statistics/people-population-community/population/vital-statistics) Refer to both graphs to answer the following questions.
    a. What has been the trend from the year 2000 to 2018 for the median age in B.C.?
    b. In which year was the median age the lowest? What was the lowest median age?
    c. In which year was the median age the highest? What was the highest median age?
    d. What was the change in median age from 2000 to 2004?e) What was the change in median age from 2007 to 2011?
    e. What was the change in median age from 2014 to 2018?
    f. Which of the two graphs was more helpful in answering these questions?



Line Graph



2. Bar Graph
   The bar graph indicates the average commuting time for Canadians in 2016. (Source: Statistics Canada Census Program)



a. According to the graph, what was the average commuting time for all Canadians?

b. Which province had the highest commuting time? Estimate the time.

c. Which province had the lowest commuting time? Estimate the time.

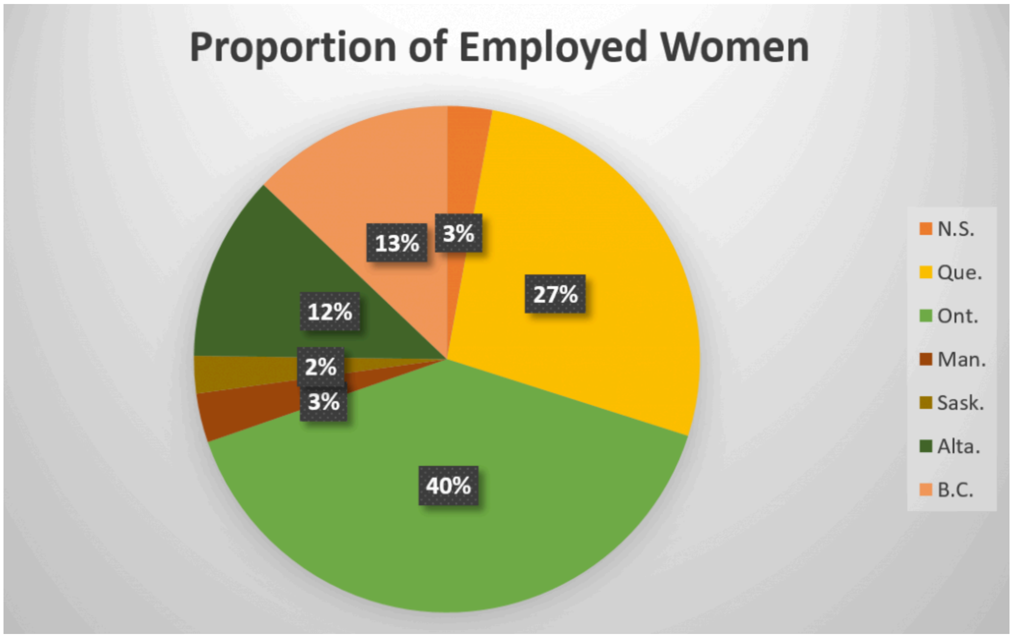
d. Which province or territory's commuting time was closest to the average?

e. Name all provinces or territories with a commuting time greater than the Canadian average.

f. Name all provinces or territories with a commuting time less than 20 minutes.

g. Which province or territory best represents the median commuting time?

3. The pie graph illustrates the proportion of women who are employed as physicians for the top seven Canadian provinces in 2016. (Source: Statistics Canada). The total number of female physicians in these 7 provinces is 25,700. Note: If you have difficulty reading the graph start at Novia Scotia (orange) and move clockwise in the pie graph. This corresponds to reading the list of provinces from top to bottom.



a. Which of the seven provinces has the highest proportion of female physicians? What is the proportion? How many female physicians are there in this province?

b. Which of the seven province has the lowest proportion of female physicians? What is the proportion? How many female physicians are there in this province?

c. What proportion of women physicians are located in the top two provinces? What might account for this?

d. Which two provinces have identical proportions of female physicians?

4. The average age of the residents at at a local seniors residence are as follows: 85, 55, 86, 57, 88, 77, 69, 79, 71, 63, 61, 92, 72, 85, 76, 65, 87, 69, 61, 74, 81, 73, 74, 66, 75, 81, 90, 56, 74, 69, 82, 64, 55, 58, 69, 90, 72, 73, 95

    a. Construct a stem plot for the data.

    b. Use the stem plot to determine the median and mode.

5. A recreational basketball league gathered information on its players. The tally for the players' heights (in feet and inches) is provided below.

Height in Feet/Inches	**Number of Players**
6/11	\\
6/10	\\
6/9	\\\\
6/8	~~\\\\~~ \
6/7	~~\\\\~~ \\
6/6	~~\\\\~~ \\\\
6/5	~~\\\\~~
6/4	~~\\\\~~ \\
6/3	~~\\\\~~ ~~\\\\~~ \
6/2	~~\\\\~~ ~~\\\\~~ \\
6/1	~~\\\\~~ \
5/5	\

a. Create a frequency distribution table that shows both the absolute and the relative frequencies.

b. Determine the mode and median.

c. Create a bar graph to illustrate this data.

d. Are there any outliers? Why does the bar graph not depict this?

6. A biker documented the daily kilometres she covered as she travelled across the Canadian prairies. Her first ten days are listed in the table below.

Day	1	2	3	4	5	6	7	8	9	10
Km	82	87	100	71	93	88	42	53	88	98

a. What was her average daily distance?
b. Create both a bar graph and a line graph.
c. What was the median daily distance?
d. On which day did she bike the furthest? the least?
e. Between which two days was there the greatest increase in distance travelled?
f. Between which two days was there the greatest decrease in distance travelled?
g. If the table were not provided, from which of the two graphs is it easier to obtain the above answers?

7. State one advantage and one disadvantage of using a bar graph, a pie graph, and a line graph.

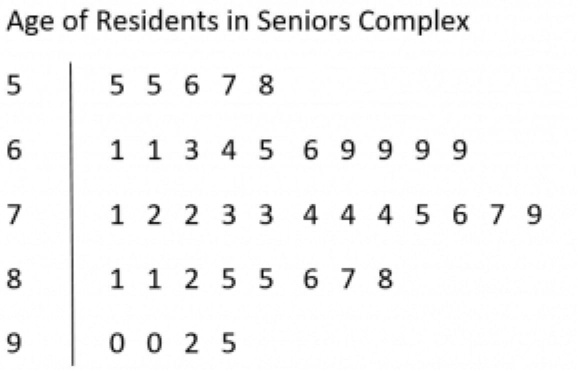
Answers

1. a. The median age increased most rapidly from 2000 to 2006. It continued to increase at a slower rate through to 2016, levelled off and decreased for the first time in 2018.
   b. In 2000 the median age was 37.5
   c. In 2016 and 2017 the median age was 42.3
   d. From 2000 to 2004 the median age increased by 1.9 years.
   e. From 2007 to 2011 the median age increased by 0.8 years.
   f. From 2014 to 2018 the median age increased by 0.3 years.
   g. Answers may vary. The bar graph provided the necessary detail but the line graph depicted the trend.
2. a. 26.2 min.
   b. Ontario 28-29 min.
   c. Nunavut 10 min.
   d. B.C.
   e. Ontario
   f. P.E.I. , N.B. , Sask. , Y.T. , N.W.T. , Nvt.
   g. Nl.
3. a. Ontario 40% 10, 280
   b. Saskatchewan 2% 514

c. 67%; These two provinces have the largest populations in Canada.

d. Nova Scotia and Manitoba

4. a. Stem plot for the data:

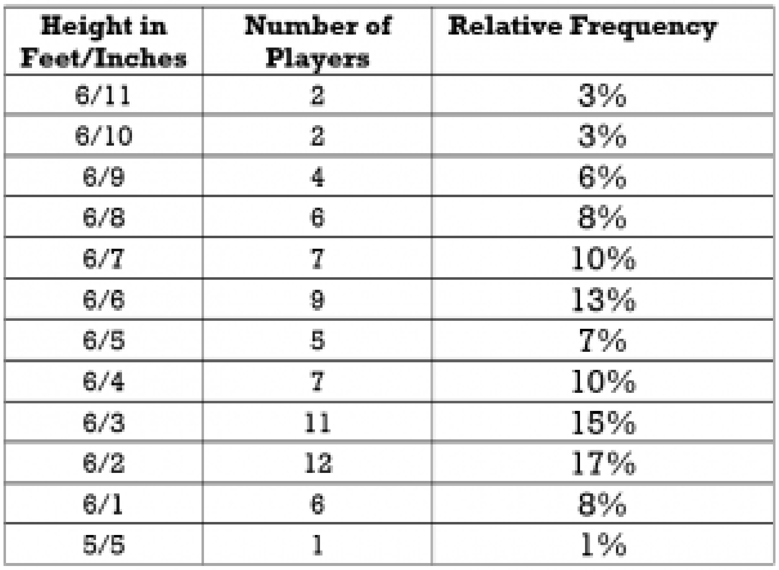


Age of Residents in Seniors Complex

5	5 5 6 7 8
6	1 1 3 4 5 6 9 9 9 9
7	1 2 2 3 3 4 4 4 5 6 7 9
8	1 1 2 5 5 6 7 8
9	0 0 2 5

b. Median is 73 and mode is 69

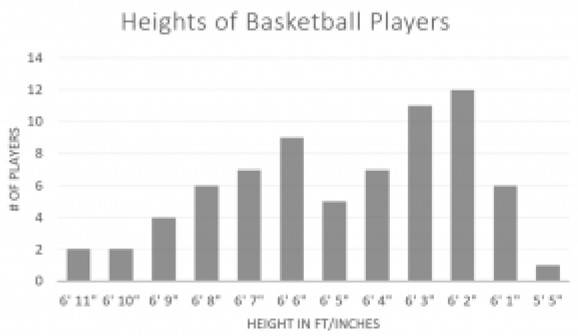
5. a. Frequency distribution table:



Height in Feet/Inches	Number of Players	Relative Frequency
6/11	2	3%
6/10	2	3%
6/9	4	6%
6/8	6	8%
6/7	7	10%
6/6	9	13%
6/5	5	7%
6/4	7	10%
6/3	11	15%
6/2	12	17%
6/1	6	8%
5/5	1	1%

b. mode is 6’2″ and median is 6’4″

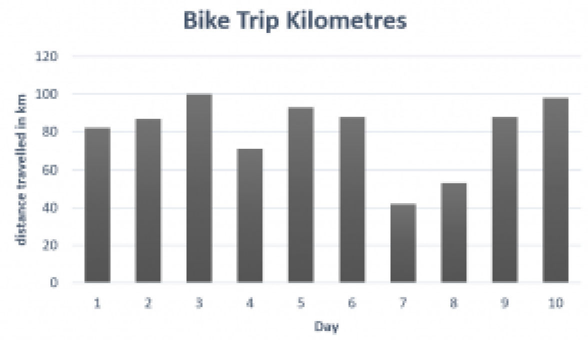
c.

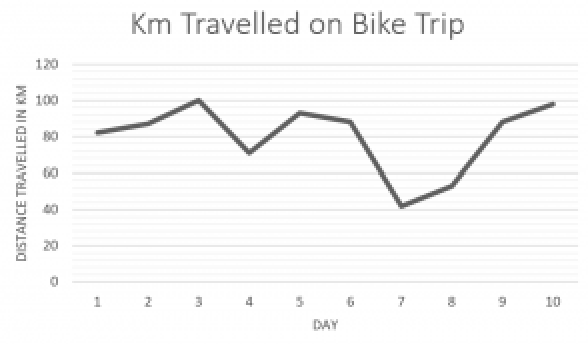


d. 5’5″ is an outlier. This is not obvious from the bar graph since the measures from 5’5″ to 6’1″ have been omitted from the graph so the gap betwen 5’5″ and 6’1″ is not apparent.

6.
   a. 80.2 km

   b.





   c. 87.5 km
   d. Day 3; Day 7
   e. From Day 8 to Day 9
   f. From Day 6 to Day 7
   g. Answers may vary

7. Answers may vary. Bar graphs provide a visual comparison of different categories (e.g. comparing the total number of wins for several different hockey teams) but they can be difficult to read accurately. Line graphs are useful for depicting trends over time but are inappropriate for comparing distinct categories (e.g. comparing the total number of wins for hockey teams). Pie graphs are useful for representing portions of a whole (e.g. voter preferences in an election) but they can be difficult to read accurately.

7.3 Collecting Data

Learning Objectives

By the end of this section it is expected that you will be able to:

- State whether data is quantitative or qualitative
- Describe the random sampling methods: simple random sampling, systematic sampling, cluster sampling and convenience sampling
- Discuss potential problems that might arise when sampling from a population

Populations and Samples

In statistics, we generally want to study a population. You can think of a population as a collection of persons, things, or objects under study. It is often not feasible or possible to study the entire population. Instead we can select a sample. The idea of sampling is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Because it takes a lot of time and money to examine an entire population, sampling is a very practical technique. If you wished to compute the overall grade point average at your school, it would make sense to select a sample of students who attend the school. The data collected from the sample would be the students' grade point averages. In elections, opinion poll samples of 1,000–2,000 people are taken. The opinion poll is supposed to represent the views of the people in the entire country.

Types of Data

Most data can be categorized as **qualitative** or **quantitative.**

Qualitative data are the result of categorizing or describing attributes of a population using our senses such as sight or touch. Hair color, blood type, ethnic group, the car model that a person drives, and the street a person lives on are examples of qualitative data. Qualitative data are generally described by words or letters. For instance, hair color might be black, dark brown, light brown, blonde, gray, or red. Blood type might be AB+, O-, or B+.

Quantitative data are always numbers. Quantitative data are the result of counting or measuring attributes of a population. Amount of money, pulse rate, weight, number of people living in your town, and number of students who take statistics are examples of quantitative data. Researchers often prefer to use quantitative data over qualitative data because it lends itself more easily to mathematical analysis. For example, it does not make sense to find an average hair color or median blood type.

EXAMPLE 1

Consider a high school math class and a sample of five student's backpacks. Determine whether the data is quantitative or qualitative.

1. One data set is the number of books students carry in their backpacks.Two students carry three books, one student carries four books, one student carries two books, and one student carries one book.

2. For the sample of five backpacks you weigh the backpacks and contents. The weights (in kilograms) of their backpacks are 3.2, 5, 4.8, 5.1, 2.3.

3. For the sample of five students you record the colour of the backpacks. The books are red, blue or black.

Solution:

1. This is quantitative data.
2. This is quantitative data.
3. This is qualitative data.

TRY IT 1

Determine the correct data type (quantitative or qualitative).

a. the number of pairs of shoes you own
b. the colour of vehicle you drive
c. the distance it is from your home to the nearest grocery store
d. the number of classes you take per school year.
e. the model of calculator you use
f. weights of sumo wrestlers
g. total number of correct answers on a quiz
h. IQ scores

Show answer
Items a, c, d, f, g and h are quantitative; items b and e are qualitative.

It is often possible to assign both qualitative and quantitative measures to one set of data.

EXAMPLE 2

You go to the supermarket and purchase three cans of soup (350 ml tomato, 400 ml lentil, and 250 ml chicken noodle), four different kinds of vegetables (broccoli, cauliflower, spinach, and carrots), and two containers pf ice cream (pistachio ice cream and vanilla ice cream).
Name the data sets that are qualitative.

Solution

The types of soups, vegetables and desserts are qualitative data because they are categorical. They are not measured or counted.

TRY IT 2

You go to the supermarket and purchase three cans of soup (350 ml tomato, 400 ml lentil, and 250 ml chicken noodle), four different kinds of vegetables (broccoli, cauliflower, spinach, and carrots), and two containers pf ice cream (pistachio ice cream and vanilla ice cream).
Name the data sets that are quantitative.

Show answer
The three cans of soup, four kinds of vegetables and two ice creams are quantitative data because you count them. The weights of the soups are quantitative because you measure weights as precisely as possible.

Sampling

Gathering information about an entire population often costs too much or is virtually impossible. Instead, we use a sample of the population. A sample should have the same characteristics as the population it is representing. There are several different methods of random sampling. This section will describe four of the most common methods. In each form of random sampling, each member of a population initially has an equal chance of being selected for the sample.

Simple Random Sampling

The easiest method to describe is called a simple random sample. Any group of 'n' individuals is equally likely to be chosen as any other group of 'n' individuals if the simple random sampling technique is used. In other words, each sample of the same size has an equal chance of being selected.

For example, suppose Lisa wants to form a four-person study group (herself and three other people) from her pre-calculus class, which has 31 members not including Lisa. To choose a simple random sample of size three from the other members of her class, Lisa could put all 31 names in a hat, shake the hat, close her eyes, and pick out three names. An alternative is for Lisa to alphabetically list the last names of the members of her class and number each with a two-digit number 01, 02, 03, 04, 05, 06,…31. Lisa can use a table of random numbers (found in many statistics books) a calculator, or a computer to generate random numbers.

EXAMPLE 3

How can Lisa determine three group mates from a numbered list of 31 students?

Solution

Lisa can generate random numbers from a calculator.

The calculator generates the first seven random numbers as follows: 0.943 0.230 0.046 0.514 0.405 0.733 0.983 Lisa reads two-digit groups until she has chosen three class members. Each random number may only contribute one class member.

The first random number 0.943 is read as the numbers 94 and 43. Neither of these corresponds to the students' assigned numbers (01 to 31).

The random number 0.230 is read as 23 and 30. Although both of these numbers corresponds to a student, only the first number, 23, will be used. The first student will be number 23.

The random number 0.046 is read as 04 and 46 which corresponds to student 04. The second student will be student number 4.

The third student will correspond to the number 14 which is read from the random number 0.514 (since there is no student numbered 51).

The three names that correspond to the two-digit numbers 23, 04 and 14 will form Lisa's group. If she needed to, Lisa could have generated more random numbers.

TRY IT 3

A fitness studio plans to purchase new equipment and wants to conduct a survey of its membership. There are over 700 members and the studio wishes to survey only a portion of this membership. Upon purchasing a membership, every member has been assigned a 3 digit membership number. Decribe how the studio can use the membership numbers to select a **simple random sample** of 80 members.

Show answer

A random number generator is used to generate a list of three digit numbers. Each random number that is generated will be compared with the membership numbers. If the number has been assignd to a member then that member will be one of the survey group. If the random number has not been assigned then the next random number is considered until 80 members have been selected.

Systematic Sampling

Systematic sampling is where the first sample member from a larger population is selected according to a random starting point. Additional sample members are then selected based on a fixed interval. The interval is calculated by dividing the population size by the desired sample size. If the population consists of 500 members and the desired sample size is 50, then the interval would be 500/50 = 10. Every tenth member of the population would be part of the sample.

EXAMPLE 4

A high school counsellor is conducting a survey of the graduating class which consists of 1243 students. Describe how the counsellor can select a systematic sample of 50 students.

Solution

The counsellor can interview 50 students. The interval is calculated as 1243 students/50 = 24.86 which rounds up to 25. This determines the interval increment as 25 so every 25th student will be in the sample.

To obtain the sample, the counsellor accesses the alphabetical list of graduates and generates a random number. Suppose the number is 03. The counsellor will interview the 3rd student on the list followed by every 25th student on the list: This will yield a sample of student 3, 28, 53, 78, and so on until 50 names have been chosen.

TRY IT 4

A fitness studio plans to purchase new equipment and wants to conduct a survey of its membership. There are over 700 members and the studio wishes to survey only a portion of this membership. Upon purchasing a membership, every member has been assigned a 4 digit membership number. Decribe how the studio can use the membership numbers to select a **systematic sample** of 80 members.

Show answer
Since 80 members are needed for the survey, the total number of members will be divided by 80. Assume there are 724 members, then 724/80 = 9.05 which rounds to an increment of 9. This determines the increment for the intervals. A list of 3-digit random numbers is generated to determine the first member in the survey group and every 9th member will be included in the survey group. If the first member has a number 546, then every 9th member counting from 546 will be chosen. When the end of the membership list is reached the increments will continue counting from the beginning of the list unil 80 members are selected.

Cluster Sampling

To choose a cluster sample, divide the population into clusters (groups) and then randomly select some of the clusters. Every member from each of the selected clusters will be in the cluster sample. This type of sampling works best in populations that can be grouped into distinct groups. In a 50 floor apartment building, each floor could represent a cluster. In a hockey league, each team could be a cluster.

EXAMPLE 5

A textbook publisher plans to conduct a survey of the faculty at a college campus. There are 23 departments at the college. Describe how the publisher can use the departments to select four cluster samples.

Solution

Let each department represents one cluster. The publisher numbers the departments from one to twenty-three and randomly selects 4 numbers which determine the four departments. Only these four departments will form the cluster sample and all faculty within the four departments (clusters) will be surveyed.

TRY IT 5

A textbook publisher plans to conduct a survey of the students at a college campus. There are 45 program areas ranging from 18 to 40 students in each program. Decribe how the publisher can use the program areas to select a **cluster sample** of at least 100 students.

Show answer
The publisher numbers the program areas from one to forty-five and generates random numbers. The first random number is used to determine the first program area (cluster). Additional random numbers are assigned to clusters until there are at least 100 students for the survey. Only the students in the selected programs (clusters) will be surveyed.

Cluster sampling can reduce the need for resources and may be more efficient. Disadvantages are that it can introduce biases or it may not represent the total population. In example 5, perhaps the textbook publisher is seeking feedback on its textbooks. If one or more of the chosen clusters does not use textbooks then the results may not be reliable.

Convenience Sampling

A type of sampling that is non-random is called convenience sampling. Convenience sampling involves using results that are readily available or convenient.

EXAMPLE 6

A computer software developer seeks to determine which of its new video games are the most popular among females. Describe how the developer can select a convenience sample.

Solution

The developer can conduct a marketing study by going to a local electronic gaming store and ask all female shoppers as they enter the store if they will participate in a 3 minute survey on video games.

TRY IT 6

A fitness studio plans to purchase new equipment and wants to conduct a survey of its membership. There are over 700 members and the studio wishes to survey 100 of its members. Decribe how the studio can select a **convenience sample** of 80 members.

Show answer
The studio owner prepares a survey and distributes it to all members who visit the studio over a 3-day period.

This form of sampling may be appealing due to its convenience but the results can be misleading. This type of surveying may be good in some cases but it can also be highly biased (favor certain outcomes) in others.

EXAMPLE 7

A study is done to determine the average tuition that undergraduate students pay per semester. Each student in the following samples is asked how much tuition he or she paid for the Fall semester. What is the type of sampling in each case? (simple random, systematic, cluster, or convenience)

a. A random number generator is used to select a student from the alphabetically numbered email listing of all undergraduate students in the Fall semester. Starting with that student, every 50th student is chosen until 75 students are included in the sample.

b. A random number generator is used to select 75 student ID numbers.
c. The freshman, sophomore, junior, and senior years are numbered one, two, three, and four, respectively. A random number generator is used to pick two of those years. All students in those two years are in the sample.
d. An administrative assistant is asked to stand in front of the library one day and to ask the first 100 undergraduate students he encounters what they paid for tuition in the Fall semester.

Solution

a. systematic
b. simple random
c. cluster
d. convenience

TRY IT 7

Determine the type of sampling used (simple random, systematic, cluster, or convenience).

a. A pollster interviews all human resource personnel in five different high tech companies.
b. A medical researcher interviews every third cancer patient from a list of cancer patients at a local hospital.
c. A high school counselor uses a computer to generate 50 random numbers and then picks students whose names correspond to the numbers.
d. A student interviews classmates in his algebra class to determine how many pairs of jeans a student owns, on the average.

Show answer

a. cluster
b. systematic
c. simple random
d. convenience

Potential Survey Issues

Users of statistical studies should be aware of the sampling method before accepting the results of the studies. Common problems to be aware of include:

1. Nonrepresentative samples: A sample must be representative of the population under study. A sample that is not representative of the population is biased. Biased samples that are not representative of the population give results that are inaccurate and not valid. An example of a biased sample would be a survey on violence in sports where only the female students in a coed high school are surveyed.
2. Self-selected samples: Surveys where responses are voluntary, such as call-in surveys, are often unreliable.
3. Sample size issues: Samples that are too small may be unreliable. Larger samples are better, if possible. In some situations, having small samples is unavoidable and can still be used to draw conclusions. Examples would include crash testing of cars or medical testing for rare conditions.
4. Undue influence: collecting data or asking questions in a way that influences the response. An example would be conducting a taste test of two sodas where one is refrigerated and the other is served at room temperature.
5. Non-response or refusal of a subject to participate: The collected responses may no longer be representative of the population. Often, people with strong positive or negative opinions may answer surveys, which can affect the results. As an example, reviewers on Internet travel sites may not be representative of the entire population.
6. Misleading use of data: Be aware of improperly displayed graphs, incomplete data, or lack of context.

Key Concepts

When conducting a survey we can choose from several sampling methods:

- Simple random sampling is where a member of the population is equally as likely to be chosen as any other member from the population.
- Systematic sampling is where the first sample member from a larger population is selected according to a random starting point. Additional sample members are then selected based on a fixed interval.
- Cluster sampling is where the population is divided into clusters (groups) and then a specific number of clusters is randomly selected. Every member from each of the selected clusters will be in the cluster sample.
- Convenience sampling is where the selection is made from a part of the population that is easy to access.

Glossary

qualitative data

are the result of categorizing or describing attributes of a population using our senses such as sight or touch.

quantitative data

are the result of counting or measuring a specific attribute of a population.

7.3 Exercise Set

1. Shoppers at a farmer's market were surveyed to determine how environmentally and market friendly they were. The survey recorded the **A)** type of bag (cloth, plastic, none, wicker, other) **B)** the number of bags (0, 1, 2, 3, more than 3) **C)** the number of market visits per year **D)** Average amount of money per visit spent at the market **E)** preferred vendor(s) . Which of A, B, C, D, E are qualitative and which are quantitative?
2. A census yields a wide variety of data. State whether each of the following questions would provide qualitative or quantitative data.
    a. What province do you live in?
    b. How many years have you lived at your current address?
    c. What type of dwelling do you live in (house, apartment, condo, mobile home, other)?
    d. How many people live in your home?
    e. How many years languages do you speak?
    f. What languages do you speak?
    g. What is your occupation?
    h. What is your annual salary?
3. Consider a typical classroom in college or university. Name two types of qualitative data and two types of quantitative data that could be collected. e.g. qualitative – score from an entrance exam; quantitative – country of birth
4. A study is done to determine the food outlet preferences for all students living on campus in the fall semester. Each student in the sample will be asked the same set of 10 questions. Four different sampling techniques are described below. What is the type of sampling in each case? (simple random, systematic, cluster, or convenience).
    a. There are 8 different student residences on campus. Two residences are randomly selected and every student living in those two residences is surveyed.
    b. The surnames of all students living on campus are arranged alphabetically and numbered from 1 to n (where n is the number of students living on campus). A

random number generator is used to determine a number between 1 and 50. This number is matched to a student with the same number. Starting with that student, every 50th student is chosen until the required number of students is chosen for the sample.

   c. One of the food outlets is chosen by drawing one outlet name. Over a four hour period one day, four helpers stop all students entering that food outlet and if they live on campus they are administered the survey.
   d. A computer is used to generate random numbers that have the same format as the students' ID numbers. Random numbers are generated until 100 random numbers are matched by student number to a student living on campus. These 100 students are contacted and arrangements are made for the interviewer to meet with the student.

5. State one advantage and one disadvantage for each of systematic sampling, cluster sampling and convenience sampling.
6. A marketing company wants to determine which is more popular – its lemonade or a competitor's lemonade. The company sets up a booth at a local arena the evening of a Professional Boxing Match. Anyone who visits the booth is asked to choose their favourite lemonade from two unmarked glasses of lemonade. The marketing company's lemonade is made onsite and served with ice and a fresh slice of lemon; the competitor's lemonade is poured straight from a bottle. All taste testers receive a chance to win a television. Name at least three problems with the methodology used for this marketing company's taste test.

Answers

1. Qualitative is A & E; Quantitative is B, C D
2. Qualitative is a, c, f, g; Quantitative is b, d, e, h
3. Answers will vary
4.
   a. Cluster
   b. Systematic
   c. Convenience
   d. Simple Random
5. Answers may vary. Systematic Sampling avoids bias but it involves a commitment in time. Cluster sampling involves less time to determine the sample but it can be biased. Convenience sampling can involve less effort but it may be non representative of the population
6. Non representative sample – attendees at a boxing match may not be interested in lemonade ; Possibly not a big enough sample; Undue Influence – the two lemonades are served up very differently; Not random but instead involves self-selection by the participants (the taste testers must choose to go to the booth); Testers might participate only for the chance to win a TV and may not provide reliable feedback.

Attribution

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7.4 Survey Creation

Learning Objectives

By the end of this section it is expected that you will be able to:

- Design a statistical experiment, collect the data, and analyze and communicate the results

This section provides an overview of the steps involved in designing and conducting a survey.

Survey Steps

1. Determine the objective of the survey and the population to be surveyed.
2. Determine the survey questions.
3. Determine the sampling technique: simple random sampling, systematic sampling, cluster or convenience. Select the sample to be surveyed.
4. Collect the data using an appropriate instrument (interview, paper questionnaire, internet,…). Record the responses in a frequency table.
5. Analyze the data and communicate the results using descriptive statistics such as measures

of central tendency and/or graphs. The graphs can be created using technology.

Math Support Survey

1. **Determine the objective of the survey and the population to be surveyed.**
    - The goal of the survey is to determine the types of math support and frequency of use by the students enrolled in upgrading math at College ABC
2. **Determine three survey questions.**
    - Three questions will be used:i. Which of the following support(s) do you access while you are enrolled in upgrading math at College ABC? Check all that apply: math learning lab, my math instructor, private tutor, internet, otherii. Which method of support do you most prefer? math learning lab, my math instructor, private tutor, internet, otheriii. How many hours a day (on average) do you work on math outside of classtime? 0; more than 0 but less than 1; 1-2; more than 2
3. **Determine the sampling technique: simple random sampling, systematic sampling, cluster or convenience. Select the sample to be surveyed.**
    - Cluster sampling will be used. There are 12 sections of upgrading math during the winter semester at College ABC. Each of these represents a cluster. The clusters will be numbered 3-14 and three clusters will be selected by rolling three die. The total of the three die will be matched with the corresponding cluster.Upon completion of this step the three selected clusters are cluster numbers 5, 9 and 14.
4. **Collect the data using an appropriate instrument (interview, paper questionnaire, internet,…). Record the responses in a frequency table.**
    - Permission to visit these three classes will be obtained by contacting the instructors via email. Each cluster will be visited on a test day when attendance tends to be most reliable. All students in each cluster will be asked to complete a three question paper survey. Three frequency tables will be created to organize the results. In total 86 students will be surveyed.a) Which of the following support(s) do you access while you are enrolled in upgrading math at College ABC? Check all that apply: math learning lab, my math instructor, private tutor, internet, other

Type of Support	**Tally**	**Frequency**
Math Learning Lab	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	43
My Math Instructor	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	28
Tutor	~~////~~ ////	9
Internet	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ //	52
Other	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ /	31

b) Which support do you prefer most?

Type of Support	**Tally**	**Frequency**
Math Learning Lab	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ //	27
My math instructor	~~////~~ ~~////~~ ~~////~~ /	16
Tutor	~~////~~ ///	8
Internet	~~////~~ ~~////~~ ~~////~~ ~~////~~ ////	24
Other	~~////~~ ~~////~~ /	11

c) How many hours per day, on average, do you work on math?

# of Hours	**Frequency**
0	31
between 0 and 1	18
1 to 2	27
More than 2	10

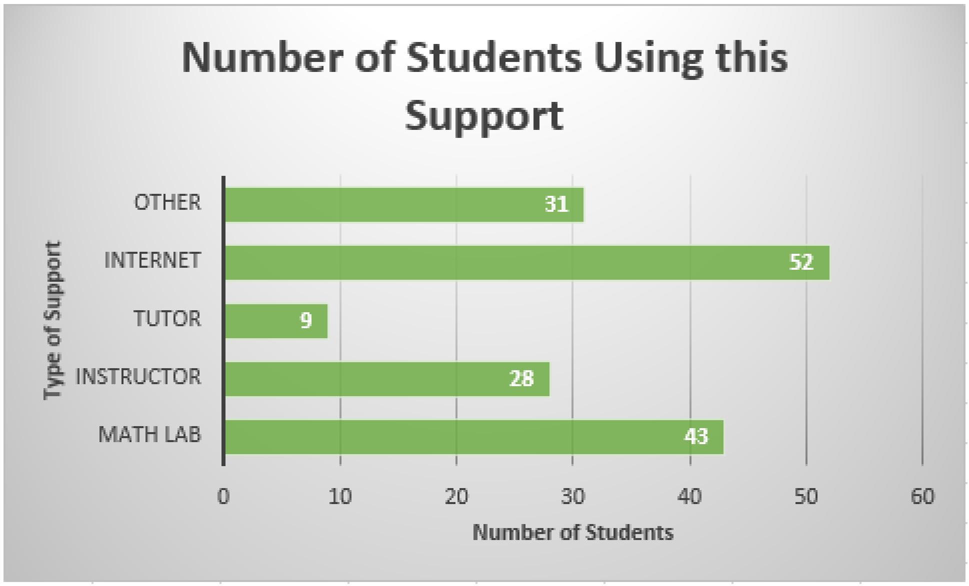
5. **Analyze the data and communicate the results. Graphs can be created using technology.**

    - Pie and bar graphs will be used to organize and summarize the results of the survey.

Bar Graphs

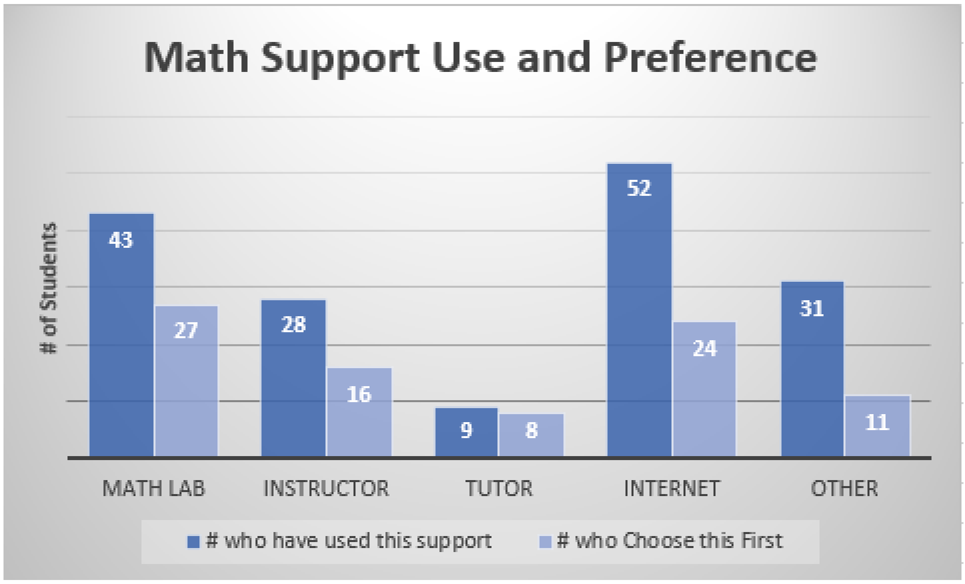
The following three graphs were created using the Chart feature in Excel.

Bar Graph#1 depicts the number of students on the horizontal axis and the type of support on the vertical axis. It is clear from the graph that the support that has been used by the most number of students is the internet and by the least number of students is a tutor.



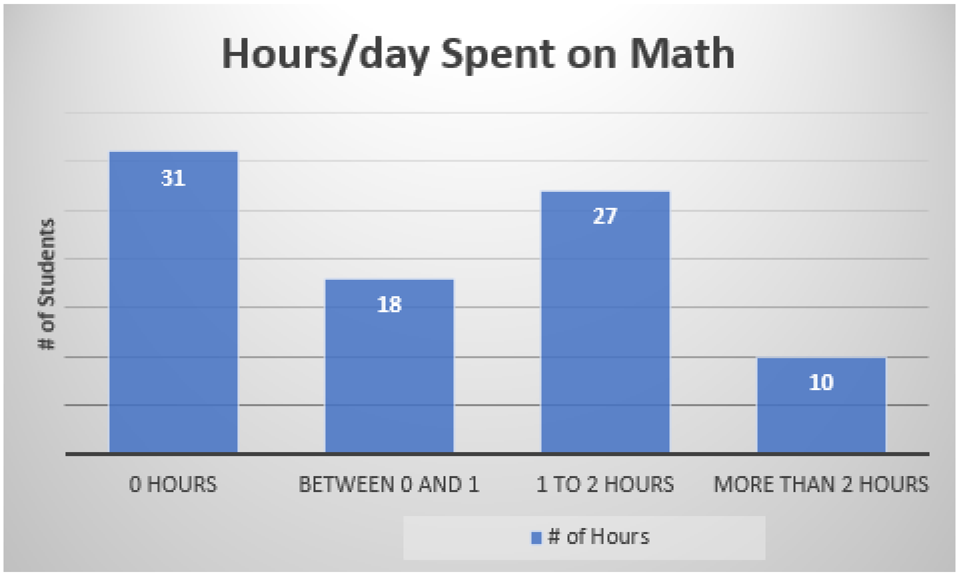
Bar Graph#1

Bar Graph#2 depicts the total number of students using the support as compared against the number of students preferring the support. Forty-three students in total have used the math lab whereas twenty-seven of these would choose the math lab first for support. Although fifty-two students have accessed the internet for support, only twenty-four would choose this type of support first. The support that is most preferred is the math lab; least preferred is the tutor.



Bar Graph#2

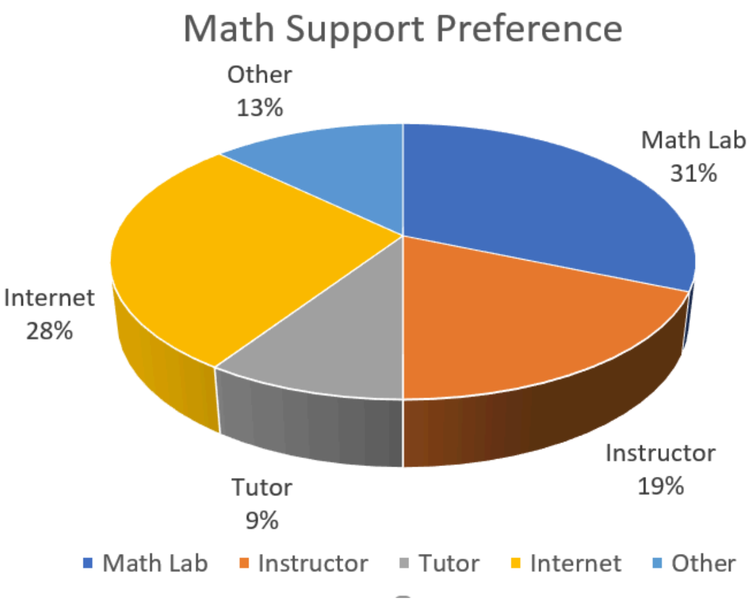
Bar Graph#3 depicts the amount of hours per day, on average, that students spend studying math.



Bar Graph#3

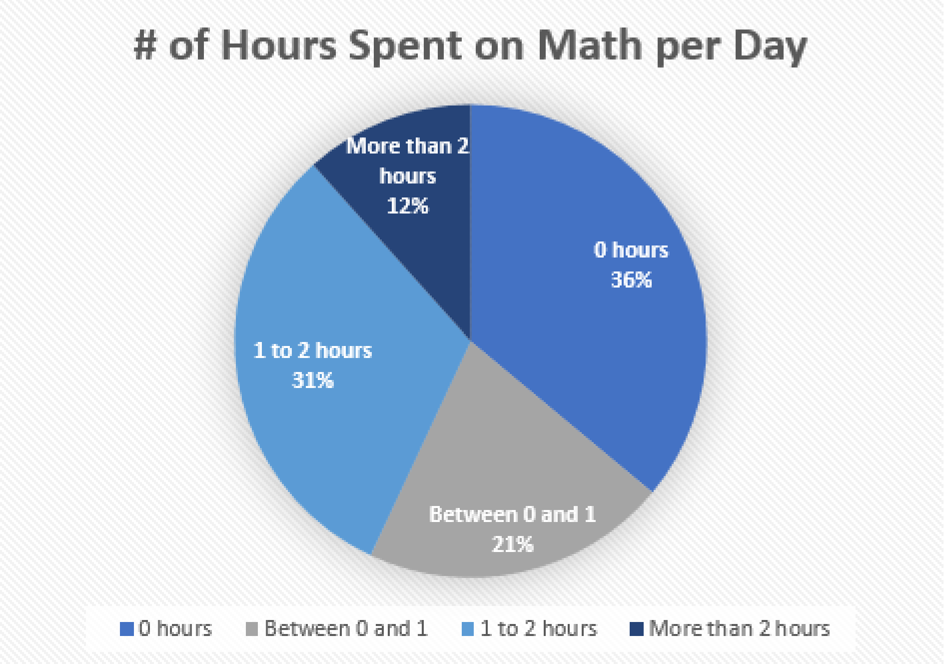
Pie Graphs

Graph#4 depicts which type of math support is most preferred by the students. The math lab is the largest sector followed closely by the Internet.



Pie Graph #4

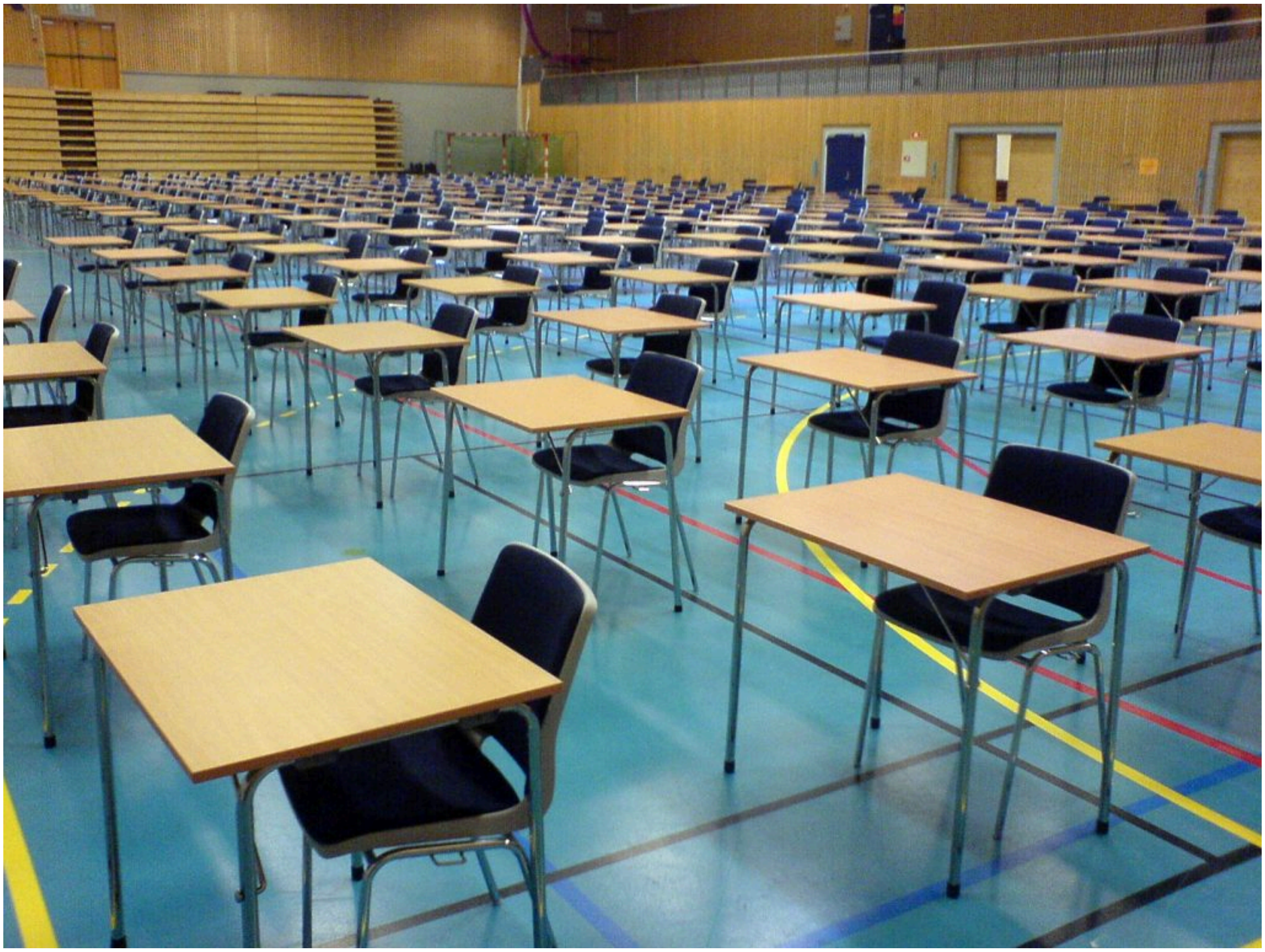
Graph#5 depicts the number of hours per day, on average, spent working on math.



Pie Graph #5

8 Data Analysis 2

8.1 Percentiles and Quartiles



Learning Objectives

By the end of this section it is expected that you will be able to:

- Describe the measures of location: percentile and quartile
- Find the percentile represented by a given data value
- Determine the first, second and third quartiles for a set of data

Measures of Central Tendency

The mean, median and mode, as **measures of central tendency,** provide us with a point of comparison. As an example, consider Company ABC where the average (mean) salary is $55,000/year. An employee earning $38,000/year might feel unjustly treated or at the very least the employee might explore the reasons for the substantial difference. If in the process the employee learns that the **median** salary at his workplace is $26,000/year the employee would learn that relative to everyone else this employee's salary is in the upper half of the employee group.

To provide additional comparison the employee could consider other measures of position or location. Two such measures are percentiles and quartiles.

Percentiles

Percentiles are useful for comparing values. If a data item is in the 75th percentile then three-quarters of the values are less than this value. This is not to be confused with a score of 75%, which is something very different. A student could score 35% on an exam but be in the 75th percentile. This means that relative to the rest of the class the student had a score that was higher than 75% of the students.

Percentiles

Percentiles divide ordered data into hundredths. A data item is said to be in the k^{th} **percentile** of a data set if k% of the data items are less than the item.

The notation P_k can be used to represent the k^{th} percentile. A data set can be divided into one hundred equal parts by ninety-nine percentiles P_1 , P_2 , P_3 , … P_{99} . The 60^{th} percentile would be denoted P_{60} . If an item is in the 60th percentile, then 60 percent of the data items are less than this item.

Consider a set of math exam scores. A student scoring in the 60th percentile achieved a score equal to or higher than 60 percent of the other students. This does not mean that the student scored 60% on the exam. Perhaps the student's score was 78%, which would mean that 60 percent of the other students in the class had exam scores less than (or equal to) 78%.

It is important to note that since percentiles divide a data set into one hundred equal parts, percentiles are best used with large data sets. Percentiles are mostly used with very large populations. For a specified percentile P_k if you were to say that k percent of the data values are less (and not the same or less) than a specified data value, it would be acceptable because removing one particular data value is not significant.

Refer again to the employee earning $38,000/year at Company ABC. If the employee learns that their salary is in the 90th percentile then 90 percent of the other employees at Company ABC have a salary less than (or possibly equal to) this salary. In relation to the other employees this salary ranks among the upper portion of the employee group.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles on entrance exams. Rather than set one value as an acceptance score, a university may set a percentile target. Perhaps all students scoring in the the 80th percentile or above will receive an acceptance letter. Every year there is likely to be a different acceptance score. Students will be accepted based on their score relative to all other applicants.

Determining Percentiles

To determine the kth percentile that is represented by a particular data item ***x*,** the following formula can be used.

$$k = \frac{\text{number of data values less than } x}{n} \times 100\%$$

Step 1: If necessary order the data values from smallest to largest.

Step 2: Determine the total number of data values, n. This will be the denominator in the formula.

Step 3: Count the number of data values that are less than the value ***x*.** This will be the value in the numerator of the formula.

Step 4: Calculate the percentile, *k*, that is associated with a score of ***x*** using the formula.

EXAMPLE 1

A class set of exam scores for 48 students are ranked from lowest to highest. Determine the percentiles associated with the scores of a) 39% b) 60% c) 94%.

39	54	59	65	75	79	84	92
42	54	60	67	76	80	86	92
43	55	60	69	76	80	88	94
48	57	60	69	77	82	88	95
51	57	63	72	77	83	89	96
51	59	65	72	78	83	91	97

Solution

a) For a score of 39%:

Step 1: The data values are already ordered from smallest to largest.

Step 2: Determine the number of data values. Since there are 48 students n = 48.

Step 3: We count 0 data values that are less than 39

Step 4: Calculate the percentile, *k*, that is associated with a score of x using the formula

$$k = \frac{\textit{number of data values less than } x}{n} \times 100\%$$

k = (0/48)*100% = 0%.

This means that the student who scored 39% is in the 0 percentile. A score of 39% is not higher than any other score.

b) For a score of 60%:

There are 13 scores lower than 60% so k = (13/48)*100% = 27%. A score of 60% is in the 27th percentile which means that 27% (or just over one-fourth) of the test scores are less than 60%.

c) For a score of 94%:

There are 44 scores less than 94% so k = (44/48)*100% = 92%. A score of 94% is in the 92nd percentile which means that 92% of the test scores are less than 94%.

TRY IT 1

A set of assignment scores for a class of 32 students are provided in the table below. Determine the percentiles associated with the scores of a) 61% b) 79% c) 98%.

72	65	85	52	61	49	65	82
55	99	58	79	98	79	58	93
88	48	97	74	65	85	71	75
99	39	60	96	80	70	54	77

Show answer

a) 61% is 28th percentile b) 79% is 59th percentile c) 98% is 91st percentile

Quartiles

Quartiles divide ordered data into quarters. Quartiles are special percentiles. The first quartile, Q_1, is the same as the 25th percentile, and the third quartile, Q_3, is the same as the 75th percentile. The median is a number that separates ordered data into halves. Half the values are the same as or smaller than the median, and half the values are the same as or larger than the median. The median can be called both the second quartile Q_2 and the 50th percentile.

Quartiles

Quartiles divide the data set into **four** equal parts.

The first quartile, Q_1, is the same as the 25th percentile, and the third quartile, Q_3, is the same as the 75th percentile. The median can be called both the second quartile, Q_2 , and the 50th percentile.

As with the median, the quartiles may or may not be part of the data set.

As indicated in Figure 1 each quartile divides a data set into four equal parts so that one-fourth of the data set is located in each part.

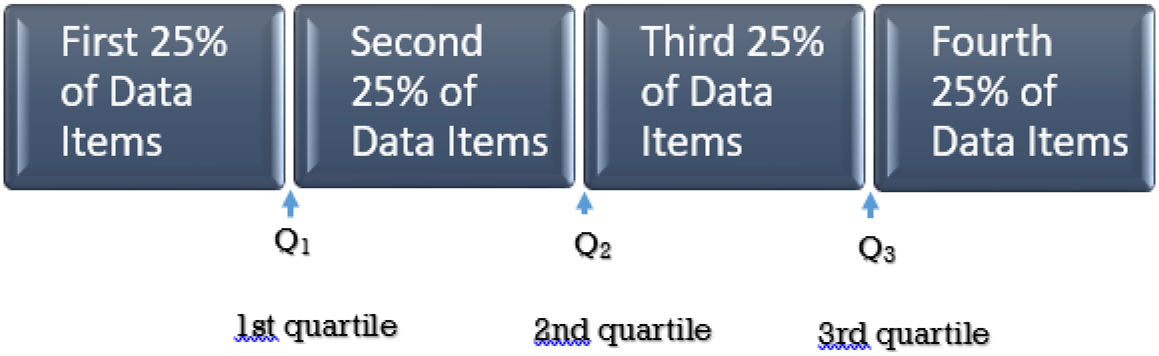


Fig. 1

Determining Quartiles

We will consider two methods for determining quartiles. As with percentiles, the data values must first be ordered from smallest to largest. The first method involves dividing the data set into four equal parts. The second method involves the use of formulas.

Determining Quartiles: Method 1

Step 1: Order the data from smallest to largest.

Step 2: Determine the number of data values **n**.

Step 3: Determine the median (Q_2) of the data set. This will divide the data set into two equal parts.

Step 4: Determine Q_1. This will divide the first half of the data set into two equal parts.

Step 5: Determine Q_3. This will divide the second half of the data set into two equal parts.

Note: The median and the quartiles may not be actual observations from the data set.

Method 1

Consider the following data set:

15 4 20 8 3 12 14 11 7 2 6 23 16

Step 1: To determine the quartiles, order the data values from smallest to largest:

2 3 4 6 7 8 11 12 14 15 16 20 23

Step 2: The number of data values is 13.

Step 3: Determine the **median**, which measures the “centre” of the data. It is the number that separates ordered data into halves. Half the observations are the same number or smaller than the median, and half the observations are the same number or larger.

$$\underbrace{2\quad 3\quad 4\quad 6\quad 7\quad 8}\quad 11\quad \underbrace{12\quad 14\quad 15\quad 16\quad 20\quad 23}$$

Since there are 13 observations, the median will be in the seventhh position. The median, and therefore the 2nd quartile Q_2 , is eleven. The median is often referred to as the “middle observation,” but it is important to note that it does not actually have to be one of the observed values.

Step 4: The first quartile, Q_1, is the **middle value of the lower half** of the data.

To determine the **first quartile**, Q_1, consider the lower half of the data observations:

2 3 4 6 7 8

Since there are six observations, the middle observation will be the average of the third and fourth data values or (4 + 6)/2 = 5 therefore Q_1 is 5

Step 5: The third quartile, Q_3, is the **middle value of the upper half** of the data.

To determine the **third quartile**, Q_3, consider the upper half of the data observations:

12 14 15 16 20 23

Since there are six observations, the middle observation will be the average of 15 and 16 , or 15.5 therefore Q_3 is 15.5.

Figure 2 illustrates the three quartiles, which divide the data set into four equal parts.

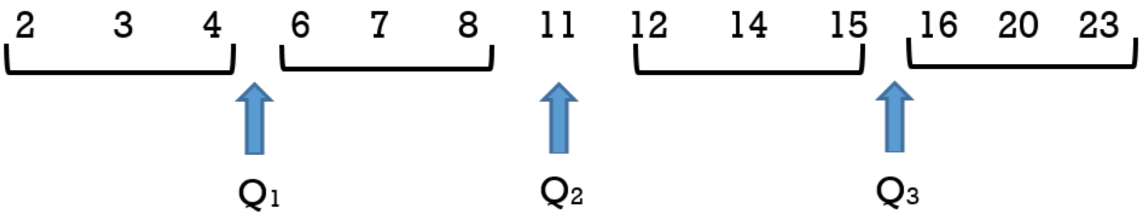


Fig. 2

The number 4.5 is the first quartile, Q_1. One-fourth of the entire set of observations lie below 4.5 and three-fourths of the data observations lie above 4.5.

The third quartile, Q_3, is 15.5. Three-fourths (75%) of the ordered data set lie below 15.5. One-fourth (25%) of the ordered data set lie above 15.5.

It is important to note that a quartile may not be a data observation. Sometimes there may be a need to average or weight the data values when determining the quartiles.

A second method for determining quartiles is to use a formula to determine the position of each quartile. This is especially useful when there is a large number of data items.

Determining Quartiles: Method 2

Quartile Formula

The following formulas, where **n** is the **number of data values**, can be used to determine the **position** of the three quartiles.

$$Q_1 = \frac{n+1}{4} \qquad Q_2 = \frac{n+1}{2} \qquad Q_3 = \frac{3(n+1)}{4}$$

It is important to note that these results indicate the **positions** of the quartiles, not the actual data obervations. If, for example, the calculation gives $Q_1=3$, this indicates that the first quartile will be the data observation in the 3rd **position.** If $Q_3 = 32$, this indicates that the third quartile will be the data observation in the 32nd **position.**

Step 1: Order the data from smallest to largest.

Step 2: Determine **n.**

Step 3: Use the formula to determine the **position** for the median (Q_2) of the data set. Count from left to right to determine the corresponding data value. If the position is a fraction then two data values will need to be weighted to determine the median value.

Step 4: Use the formula to determine the **position** for the first quartile Q_1 of the data set. Count from left to right to determine the corresponding data value. If the position is a fraction then two data values will need to be weighted to determine the value of Q_1.

Step 5: Use the formula to determine the **position** for the third quartile Q_3 of the data set. Count from left to right to determine the corresponding data value. If the position is a fraction then two data values will need to be weighted to determine the value of Q_3.

Method 2:

Consider the following data set:

15 4 20 8 3 12 14 11 7 2 6 23 16

Step 1: To determine the quartiles, order the data values from smallest to largest:

2 3 4 6 7 8 11 12 14 15 16 20 23

Step 2: The number of data values is 13.

Step 3: Use the formula to determine the **position** for the median (Q_2) of the data set.

$$Q_2 = \frac{n+1}{2} = \frac{13+1}{2} = 7th\ data\ value$$

Count from left to right to determine the corresponding data value in the 7th position. The corresponding value is 11.

$$\underbrace{2 \quad 3 \quad 4 \quad 6 \quad 7 \quad 8} \quad 11 \quad \underbrace{12 \quad 14 \quad 15 \quad 16 \quad 20 \quad 23}$$

Step 4: Use the formula to determine the **position** for the first quartile (Q_1) of the data set.

$$Q_1 = \frac{13+1}{4} = \frac{14}{4} = 3.5th\ data\ value$$

Since 3.5 is a fraction, the first quartile will be the average of the two data values that are in the 3rd and 4th positions. Count from left to right to determine the corresponding data values. The data value 4 is in the 3rd position and the data value 6 is in the 4th position so these will be averaged (4 + 6)/2 = 5. The first quartile will be 5.

Step 5: Use the formula to determine the **position** for the third quartile (Q_3) of the data set.

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(13+1)}{4} = 10.5th\,data\,value$$

Since 10.5 is a fraction, the third quartile will be the average of the two data values that are in the 10th and 11th positions. Count from left to right to determine the corresponding data values. The data value 15 is in the 10th position and the data value 16 is in the 11th position so these will be averaged (15 + 16)/2 = 15.5. The third quartile will be 15.5.

Figure 3 illustrates the three quartiles, which divide the data set into four equal parts.

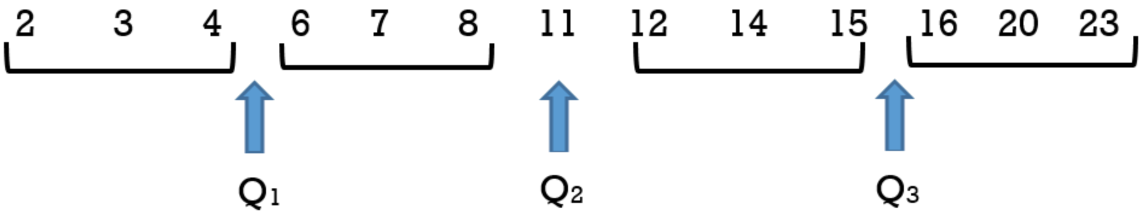


Fig. 3

EXAMPLE 2

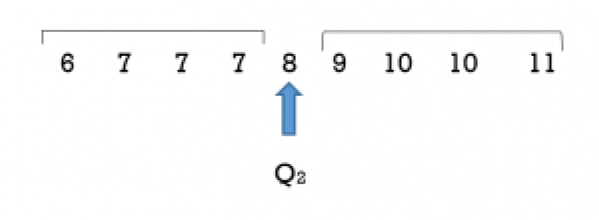
A shoe store wanted to determine the popularity of different shoe sizes for women's tennis shoes. It planned to place its next order using this information. In a five day period it sold nine pairs of women's tennis shoes in the following sizes: 7, 8, 11, 10, 7, 6, 9, 10, 7

Solution

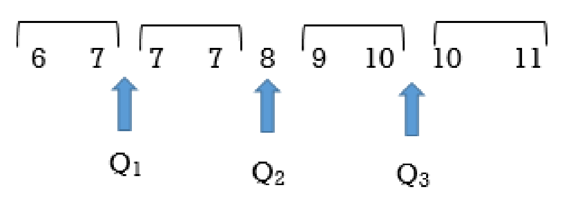
Method 1:

To determine the quartiles:

1. Order the shoe sizes from smallest to largest: 6, 7, 7, 7, 8, 9, 10, 10, 11
2. Count the number of values: n = 9
3. Determine Q_2, the median, which is the middle observation. Since there are nine data observations (shoe sizes) the median, or second quartile, will be the 5th data value. The 5th data value is 8.



4. Determine the first quartile Q_1. It will be the middle observation of the **lower half** of data values. This will be the average of the 2nd and 3rd data values so (7 +7)/2 = 7.



5. Determine the third quartile Q_3. This will be the middle observation of the **upper half**. This will be the average of the 7th and 8th data values so (10+10)/2 = 10

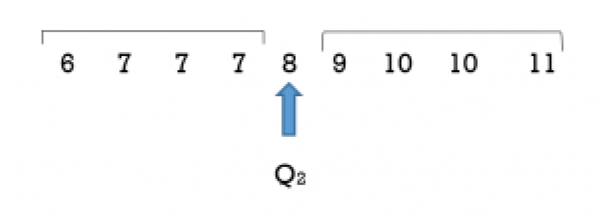
Method 2:

The formulas can be used to determine the quartiles.

1. Order the shoe sizes from smallest to largest: 6, 7, 7, 7, 8, 9, 10, 10, 11 .
2. Determine the number of data values, n. n = 9
3. Use the formula to determine the median. The median , or second quartile, can be determined as follows:

$$Q_2 = \frac{n+1}{2} = \frac{9+1}{2} = 5th\ data\ value$$

Counting from left to right, the 5^{th} data value is 8. The median, or 2nd quartile Q_2, is 8.



4 & 5. The first and third quartiles can be determined as follows:

$$Q_1 = \frac{n+1}{4} = \frac{9+1}{4} = 2.5th\ data\ value \qquad Q_3 = \frac{3(n+1)}{4} = \frac{3(9+1)}{4} = 7.5th\ data\ value$$

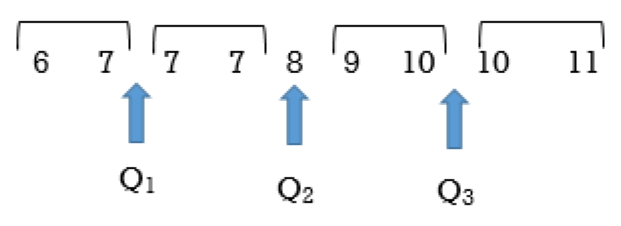
The first quartile is the 2.5th data value. To determine the 2.5th data value we must take the average of the 2nd and 3rd data values. The 2nd data value is 7 and the 3rd data value is 7 so (7+7)/2 = 7.

The first quartile, $Q_1 = 7$

The third quartile is the 7.5th data value. This will be the average of the 7th and 8th data values. The 7th data value is 10 and the 8th data value is also 10 so (10+10)/2 = 10.

The third quartile, $Q_3 = 10$

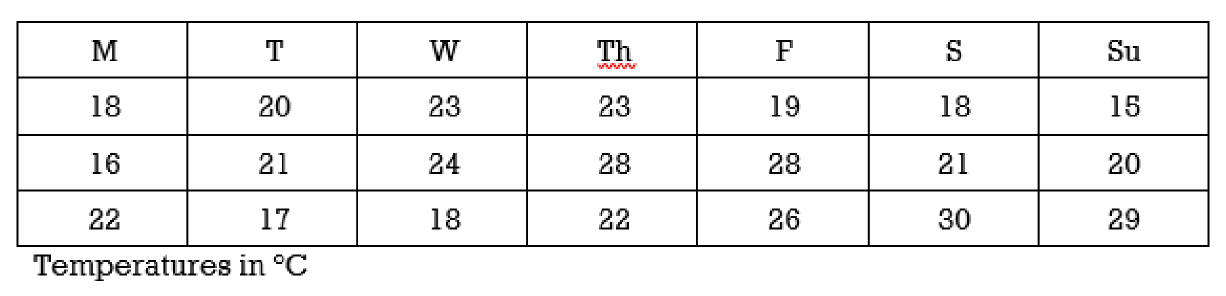
We can see that $Q_2 = 8$ splits the data set into two halves. $Q_1 = 7$ is the middle value of the lower half of the data set and $Q_3 = 10$ is the middle value of the upper half of the data set.



In Example 2 the number of data items was **odd.** When *n* is odd the median or Q_2 will be one of the data observations. When *n* is odd the formula for finding quartiles is straight forward.

TRY IT 2

Determine the quartiles for the following temperature data that was recorded over a 3-week period in May:



M	T	W	Th	F	S	Su
18	20	23	23	19	18	15
16	21	24	28	28	21	20
22	17	18	22	26	30	29

Temperatures in °C

Show answer

$Q_2 = 21$; $Q_1 = 18$; $Q_3 = 25$

It is important to note that a quartile may **not** be a data observation. When the number of data values *n* is **even** the median or Q_2 will **not** be one of the actual data observations. As a result, when *n* is **even** an adjustment must be made to the value of **n** that is to be used in the formula to determine the **first** and **third** quartiles.

Method 1:

Consider the following data set:

1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1

Step 1: To determine the quartiles, order the data values from smallest to largest:

1 1 2 2 4 6 6.8 7.2 8 8.3 9 10 10 11.5

Step 2: The number of data values is 14

Step 3: Determine the **median**, which measures the “centre” of the data. It is the number that separates ordered data into halves. Half the observations are the same number or smaller than the median, and half the observations are the same number or larger.

Since there are 14 observations, the median lies between the seventh observation, 6.8, and the eighth observation, 7.2. To find the median, add the two values together and divide by two. Median = (6.8 + 7.2)/2 = 7

The median, and therefore the 2nd quartile Q_2 , is seven. It is important to note that the median is not actually one of the observed data values.

Step 4: The first quartile, Q_1, is the **middle value of the lower half** of the data.

To determine the **first quartile**, Q_1, consider the lower half of the data observations:

1 1 2 2 4 6 6.8.

Since there are seven observations, the middle observation will be the 4th item. The middle or 4th item of these data observations is 2.

Step 5: The third quartile, Q_3, is the **middle value of the upper half** of the data.

To determine the **third quartile**, Q_3, consider the upper half of the data observations:

7.2 8 8.3 9 10 10 11.5.

Since there are seven observations, the middle observation will be the 4th item in the upper half. The middle item of these data observations is 9.

Figure 4 illustrates the three quartiles, which divide the data set into four equal parts.

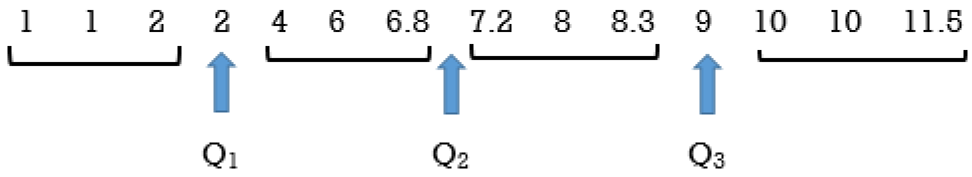


Fig. 4

The number 2 is the first quartile, Q_1. One-fourth of the entire set of observations lie below 2 and three-fourths of the data observations lie above 2.

The third quartile, Q_3, is 9. Three-fourths (75%) of the ordered data set lie below 9. One-fourth (25%) of the ordered data set lie above 9.

Method 2:

Consider the following data set:

1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1

Step 1: To determine the quartiles, order the data values from smallest to largest:

1 1 2 2 4 6 6.8 7.2 8 8.3 9 10 10 11.5

Step 2: The number of data values is 14 so **n is an even number.**

Step 3: Use the formula to determine the position of Q_2, the median. The position will be (14 + 1)/2 = 7.5. This means that the median, or Q_2, will be in the 7.5th observation or halfway between the 7th and 8th position. The observation 6.8 is in the 7th position and the observation 7.2 is in the 8th position therefore the average of these (6.8 + 7.2)/ 2 is the median or Q_2.

Note that the median is **not** an actual observation in the data set. If we use the formula to find Q_1 and Q_3 then we must adjust "n" to include this additional item so in effect "n" will be 15. This is done **only** when determining the positions of Q_1 and Q_3 (and not for determining the position of Q_2)

Step 4: Use the formula to determine the position of Q_1, the first quartile. Remember than **n** will now be 15, not 14. The position will be (15 + 1)/4 = 4 th. This means that Q_1 will be in the 4th position. Counting from the left, the data value 2 is in the 4th position so Q_1= 2.

Step 5: Use the formula to determine the position of Q_3, the third quartile. Remember than **n** will now be 15, not 14. The position will be 3(15 + 1)/4 = 12th. This means that Q_3 will be in the 12th position. Refer to Figure 5. Counting from the left, we include the median value of 7, to determine that the data value in the 12th position. This value is 9 so Q_3 will be 9.

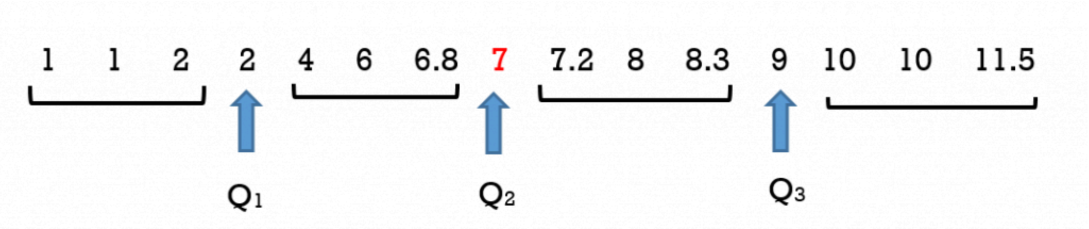


Fig. 5

It is also important to recognize that the median of 7 is not an actual data value in this set. It was included in Figure 5 to illustrate that its **position** must be counted when determing the position of the third quartile. It is not actually part of the data set. The actual data set is illustrated in Figure 6 (and Figure 4).

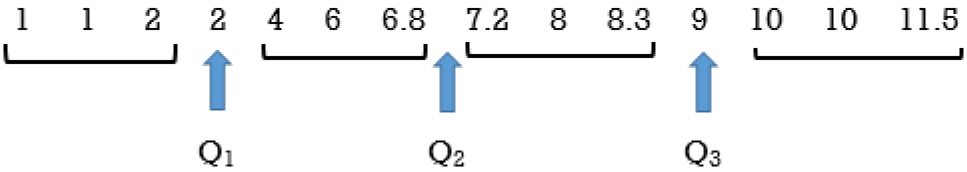


Fig. 6

Consider Figure 7 where the data set that has an even number of data values: 1 2 4 5

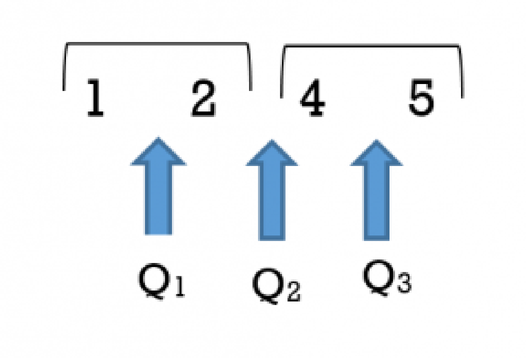


Fig. 7

In this data set $Q_1 = 1.5$, $Q_2 = 3$, and $Q_3 = 4.5$ This illustrates that quartile values need not be actual values in the data set. The second quartile Q_2 is 3 which is the average of the data values 2 and 4. Similarily the first quartile of 1.5 is the average of two data values 1 and 2 and the third quartile of 4.5 is the average of the two data values 4 and 5. Determining the quartile values can become complex as it may require different weightings of the data values but this is beyond the scope of this textbook.

Example 3 illustrates two techniques for determining quartiles when the number of data observations is **even**.

EXAMPLE 3

Consider again the shoe store and a different week. Over a five day period it sold ten pairs of tennis shoes in the following sizes:

6, 8, 11, 10, 7, 6, 9, 10, 8, 9

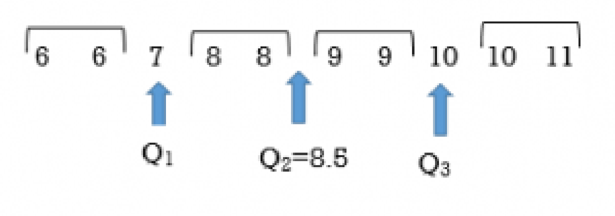
Note that there is an **even** number of data values n = 10

Solution

Method 1:

To determine the quartiles:

1. Rank the sizes from smallest to largest: 6, 6, 7, 8, 8, 9, 9, 10, 10, 11 and divide the data set into four equal quarters.
2. n = 10
3. Start with the median which is the middle observation. The median, or second quartile, will lie between the 5th and 6th data values. The 5th data value is 8 and the 6th data value is 9 so the average of 8 and 9, or 8.5, is the median.
4. Determine the first quartile Q_1. It will be the middle observation of the **lower half** of data values. This is the 3rd data value or the observation of 7.
5. Determine the third quartile Q_3. This will be the middle observation of the **upper half**. This will be the data observation of 10.



Note that each quartile divides the data values such that there are an equal number of data values in each of the four sections.

Method 2:

An alternative is to use the formulas to determine the quartiles.

To determine the quartiles:

1. Rank the sizes from smallest to largest: 6, 6, 7, 8, 8, 9, 9, 10, 10, 11
2. n = 10
3. Determine the position of the median using the formula.

$$Q_2 = \frac{n+1}{2} = \frac{10+1}{2} = 5.5th\ data\ value$$

The 5.5th data value will be the average of the 5th and 6th data values. The 5th data value is 8 and the 6th data value is 9 so

(8 + 9) /2 = 8.5 The median or Q_2 is 8.5.

Note: Q_2 is **not** one of the **actual** data values. In this example Q_2 is the 5.5th data value or 8.5. It is the data value that lies between the 5th and 6th data values but it is not one of the original data values.

4 & 5. Determine the first quartile Q_1 and the third quartile Q_3.

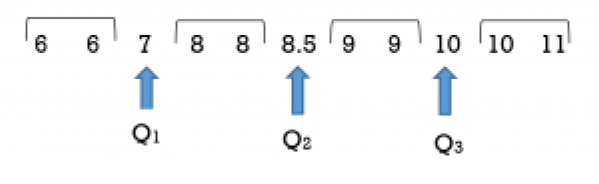
Since the number of data values ***n*** is **even** the median or Q_2 is **not** one of the **actual** data values so when we use the formula to determine Q_1 and Q_3 we must **increase the value of n by 1**. In effect the number of data values has increased by one and therefore the value of **n** in the formula must be increased by 1. This is done only when determining the positions of Q_1 and Q_3 (and not for determining the position of Q_2)

In this example, when determining Q_1 and Q_3 the original value of **n = 10** will now be increased by 1. The new number for **n** to be used in the formula will be **n = 11.** Using the formula, the first and third quartile positions can be determined as follows:

$$Q_1 = \frac{n+1}{4} \qquad Q_3 = \frac{3(n+1)}{4}$$

$$Q_1 = \frac{11+1}{4} = 3 = 3rd\ data\ value \qquad Q_3 = \frac{3(11+1)}{4} = 9 = 9th\ data\ value$$

Using the results from the formula we count to get the 3rd and 9th data values. When determining these values be sure to include and count the **position occupied by the new median value** of 8.5. The 3rd data value is 7 and the 9th data value is 10.

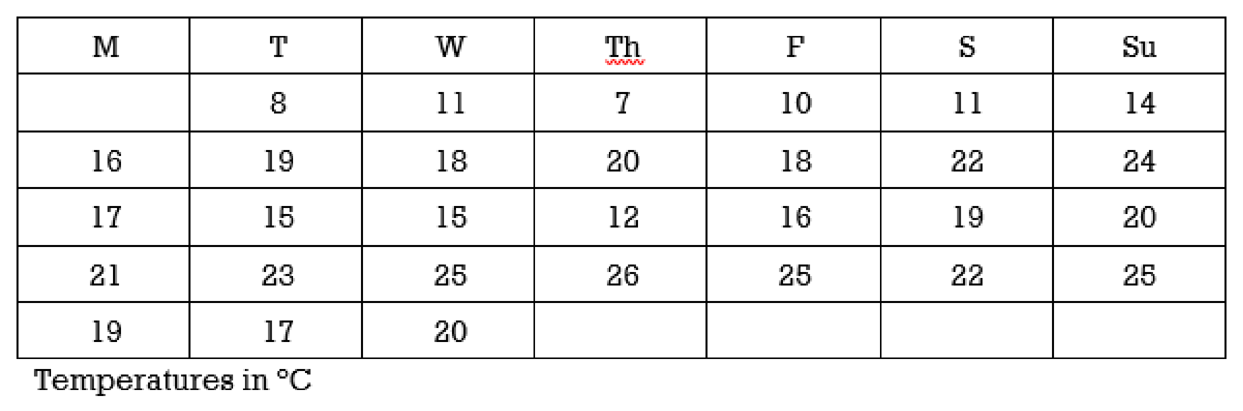


Note that Method 1 and Method 2 yield the same results.

We have seen that either of Method 1 or Method 2 will produce the same quartile values although the formula method can be less intuitive when **n** is **even.**

TRY IT 3

Use either technique to determine the quartiles for the following temperature data that was recorded over the month of April:



M	T	W	Th	F	S	Su
	8	11	7	10	11	14
16	19	18	20	18	22	24
17	15	15	12	16	19	20
21	23	25	26	25	22	25
19	17	20				

Temperatures in °C

Show answer
Q_2 = 18.5 Q_1 = 15; Q_3 = 22

EXAMPLE 4

Consider the data set: 3, 4, 5, 6, 7, 8, 8, 9, 10, 10, 11, 12, 13, 13, 14, 15. Determine the three quartiles using either technique.

Method 1:

Step 1: Order the data values 3, 4, 5, 6, 7, 8, 8, 9, 10, 10, 11, 12, 13, 13, 14, 15

Step 2: n = 16

Step 3: The median will be the average of 9 and 10, so 9.5. This is not one of the observed values.

Step 4: Q_1 is the value that splits the lower half, which will be the average of 6 and 7, so 6.5.

Step 5: Q_3 is the value that splits the upper half, which will be the average of 12 and 13, so 12.5.

Method 2:

Step 1: Order the data values 3, 4, 5, 6, 7, 8, 8, 9, 10, 10, 11, 12, 13, 13, 14, 15

Step 2: n = 16

Step 3: Use the formula (16 + 1) /2 = 8.5. The median will be in the 8.5th position. This is the average of the 8th value of 9 and the 9th value of 10 so the median is 9.5

Step 4 and 5: Since n is **even,** we will use a value of 17, not 16, in the formulas to determine Q_1 and Q_3.

Q_1 will be (17 + 1)/4 = 4.5 th. This means that Q_1 will be in the 4.5th position or the average of the 4th and 5th data values. The 4th value is 6 and the 5th value is 7 so Q_1= 6.5.

Q_3 will be 3(17 + 1)/4 = 13.5 th. This means that Q_3 will be in the 13.5th position or the average of the 13th and 14th data values. Including the median's position when we count, the 13th value is 12 and the 14th value is 13 so $Q_3 = (12 + 13)/2 = 12.5$.

Note that these identical results were obtained without using the formulas. It is also important to recognize that the median of 9.5 is not an actual data value in this set. It serves only to divide the data set into two equal halves and it is not actually part of the data set.

TRY IT 4

An athlete was training for a race and logged the following distances (in km) over a 36 day period. Determine the three quartiles for the distances covered by the athlete.

18	22	34	38	42	14	22	0	18
21	30	41	56	11	18	18	30	0
24	52	11	16	28	36	25	25	11
0	18	24	20	46	38	40	27	10

Show answer

Q_1= 17, Q_2 = 23, Q_3= 35

Key Concepts

- A data set can be divided into one hundred equal parts by ninety-nine percentiles P_1 , P_2 , P_3 , … P_{99} . Percentiles are best used with large sets of data.
- Quartiles divide the data set into **four** equal parts. The first quartile, Q_1, is the same as the 25^{th} percentile, and the third quartile, Q_3, is the same as the 75^{th} percentile. The median can be called both the second quartile, Q_2 , and the 50^{th} percentile.
- Quartiles may or may not be actual observations within a set of data.

Glossary

Percentiles

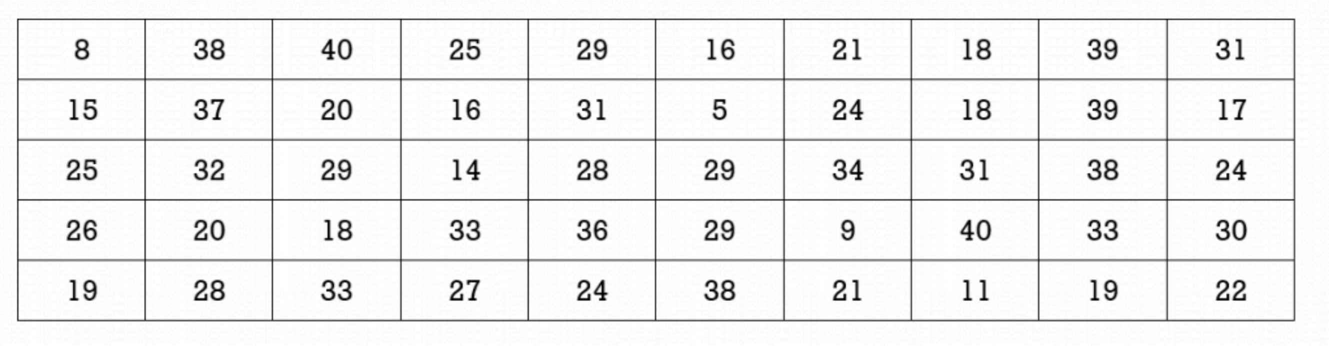
divide ordered data into hundredths.

Quartiles

divide ordered data into four equal parts.

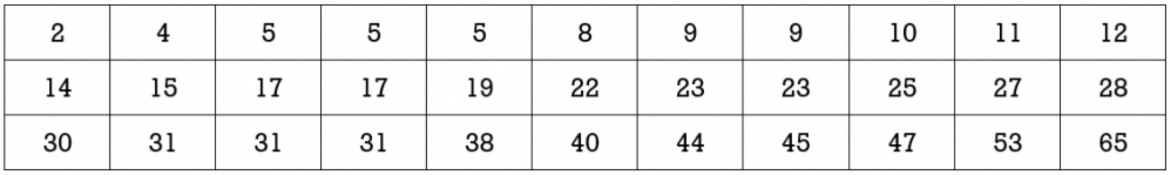
8.1 Exercise Set

1. Your instructor announces to the class that anyone with a midterm exam score of 63% scored in the 80th percentile. If you received a score of 63% how did you do in relation to your classmates?
2. A test consists of 40 marks. Fifty students wrote the test and their scores are in the table below. Determine the percentiles that are associated with scores of:
    a. 15
    b. 25
    c. 37



8	38	40	25	29	16	21	18	39	31
15	37	20	16	31	5	24	18	39	17
25	32	29	14	28	29	34	31	38	24
26	20	18	33	36	29	9	40	33	30
19	28	33	27	24	38	21	11	19	22

3. An employee at a large manufacturing company learns that their salary is in the 45th percentile. If the median salary at the company is $56,000/year can we conclude that this employee's annual salary is more than $56,000?
4. Your instructor announces to the class that the third quartile for the midterm exam was a score of 88%. If you received a score of 88% how did you do in relation to your classmates?
5. A cell phone provider is trying to improve its service by reducing the amount of time that its help desk spends with each customer. It kept track of the average length of time (to the nearest minute) each of its 33 employees spent with customers.

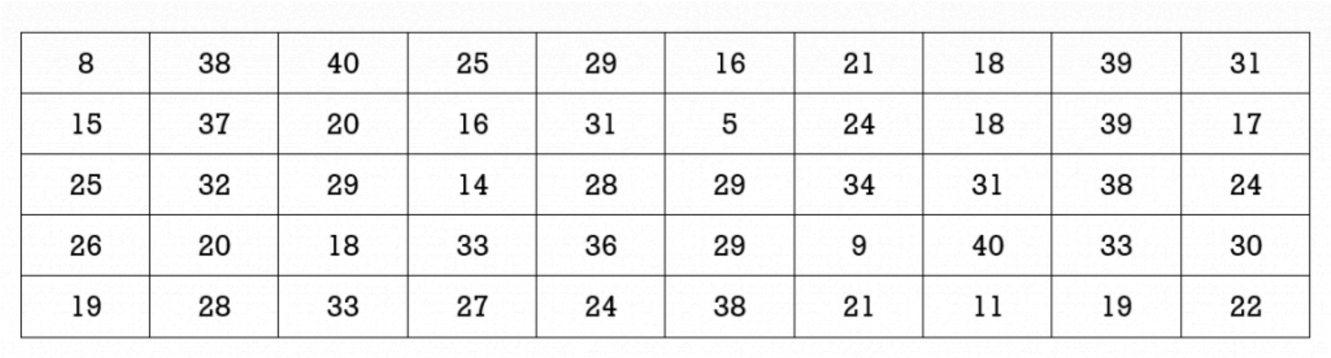


2	4	5	5	5	8	9	9	10	11	12
14	15	17	17	19	22	23	23	25	27	28
30	31	31	31	38	40	44	45	47	53	65

    a. Determine the mean, median and mode for average wait times.
    b. Determine the percentiles that correspond to help times of 14 minutes, 23 minutes and 47 minutes.

   c. Determine the 1st, 2nd and 3rd quartiles for average help times.

6. A test consists of 40 marks. Fifty students wrote the test and their scores are as recorded. Determine the 1st, 2nd and 3rd quartiles.



8	38	40	25	29	16	21	18	39	31
15	37	20	16	31	5	24	18	39	17
25	32	29	14	28	29	34	31	38	24
26	20	18	33	36	29	9	40	33	30
19	28	33	27	24	38	21	11	19	22

7. Which of the following must be an actual data value: mean, median, mode, first quartile, third quartile?
8. At a restaurant one evening the customers were asked to rate the service they received. Scores could range from 1 to 10. The following thirty responses (scores) were provided:1 1
   1 2 2 2 3 3 3 4 4 4 5 5 5 6 6 6 7 7 7 8 8 8 9 9 9 10 10 10
   a. Determine the percentiles that correspond to scores of 2 and 5. Explain what this means.
   b. Determine the first, second and third quartiles.
9. At a restaurant one evening the customers were asked to rate the service they received. Scores could range from 1 to 10. The following twenty-nine responses (scores) were provided:
   1 1 1 2 2 3 3 3 4 4 4 5 5 5 6 6 6 7 7 8 8 8 8 8 9 9 10 10 10
   a. Determine the mean, median and mode.
   b. Determine the first, second and third quartiles.

Answers

1. If your score is 63% and this is in the 80th percentile this means that 80% of your classmates received scores lower than or equal to 63%
   a. A score of 15 is in the10th percentile.
   b. A score of 25 is in the 44th percentile.
   c. A score of 37 is in the 84th percentile.
2. We can conclude that the employee's salary is not more than $56,000/year because the median salary is also the 50th percentile. If the employee's salary is in the 45th percentile they cannot be earning more than the median.
3. You scored better than three quarters of your class mates.

   a. The mean is 765/33 = 23.18; the median is 22; and there are two modes 5 and 31.
   b. A help time of 14 minutes is in the 33rd percentile; 23 minutes is in the 52nd percentile; 47 minutes is in the 91st perecntile
   c. Q_1 = 9.5 min.; Q_2 = 22 min. ; Q_3 = 31 min.

4. Q_1 = 19; Q_2 = 26.5; Q_3 = 33
5. The mode must be an actual data value
   a. A score of 2 is the 10th percentile. This means that 10% of the scores were less than a score of 2. A score of 5 is the 40th percentile. This means that 40% of the scores were less than a score of 5.
   b. The first quartile is 3, the second quartile is 5.5, the third quartile is 8
   a. mean 5.6; median 6; mode 8
   b. Q_1 is 3; Q_2 is 6; Q_3 is 8

Attribution

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8.2 Measures of Dispersion

Learning Objectives

By the end of this section the student should be able to:

- Determine the range for a data set.
- Determine the standard deviation for a data set.
- Determine the standard deviation from a histogram

Measures of Dispersion or Spread

We have seen that **measures of central tendency,** including the **mean** and **median**, are used to identify a central position within a data set. They indicate where the data clusters.

Consider student A's scores on five tests: 32% 95% 89% 74% 55% The **mean,** or average, is (32 + 95 + 89 + 74 + 55)/5 = 69% and the **median** is 74%.

Consider student B’s scores on the same five tests: 68% 69% 72% 74% 62% The **mean,** or average, is (68 + 69 + 72 + 74 + 62)/5 = 69% and the **median** is 69%.

Both student’s have the same test average of 69% but there is a substantial difference in the spread or dispersion of their scores. Student A’s test scores range from a low score of 32% to a high score of 95% so the spread in marks is 63 percentage points. Student B’s test scores range from a low score of 62% to a high score of 74% so the spread in marks is 12 percentage points.

When we analyze data it is important to consider how **dispersed** or spread out the data values are. In this section we will consider two measures of dispersion.

Range

Range is one **measure of dispersion.** A measure of dispersion is used to describe the spread of data.

Range

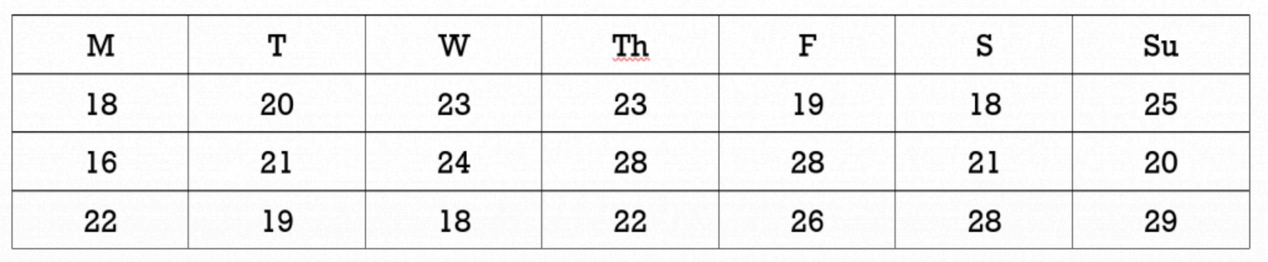
The **range** indicates the total spread in data values. It is the difference between the highest and lowest data values.

Range = highest data value – lowest data value

EXAMPLE 1

The table shows the daily high temperature (°C) over a three week period.

Determine the highest temperature, lowest temperature, and the range in daily high temperatures over the three weeks.



M	T	W	Th	F	S	Su
18	20	23	23	19	18	25
16	21	24	28	28	21	20
22	19	18	22	26	28	29

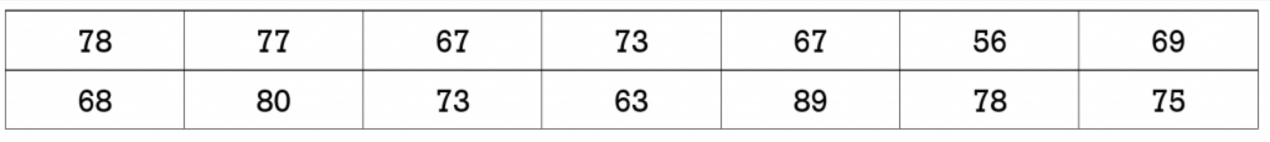
Solution

The highest temperature was 29º, the lowest temperature was 16°, and the range in temperatures was 29 – 16 = 13º.

TRY IT 1

The table shows the test score for a group of fourteen students.

Determine the highest test score, the lowest test score, and the range in test scores for the group of 14 students.



78	77	67	73	67	56	69
68	80	73	63	89	78	75

Show answer

The highest score was 89%, the lowest score was 56%, and the range in scores was 33%.

An advantage of using the range as a measure of dispersion is that it involves a simple calculation. A disadvantage is that the range only provides a measure between the highest and lowest values so it disregards all other data values. If the highest or lowest data value is an **outlier** then the range will not provide a true measure of the spread in the typical values.

EXAMPLE 2

A student wrote five tests and earned the following five scores:

92% 95% 89% 94% 35%

1. Determine the mean, median and the range for these five scores.
2. Which of the five scores is an outlier?
3. Remove the outlier and recalculate the mean, median and the range for the four remaining scores.
4. Comparing the results for the five scores versus four scores, which of the three measures was least impacted by the outlier?
5. Comparing the results for the five scores versus four scores, which of the three measures was most impacted by the outlier?

Solution

1. The student’s mean score: (92 + 95 + 89 + 94 + 35) / 5 = 81%

The median score is 92%.

The range in marks is 95% – 35% = 60%.

2. The score of 35% is an outlier.

3. The student’s mean score: (92 + 95 + 89 + 94) / 4 = 92.5%

The median score is 93%.

The range in marks is 95% – 89% = 6%.

4. The median was least impacted by the removal of the outlier.

5. The range was most impacted by the outlier.

TRY IT 2

The following seven values are salaries at a local computer company.

$62,000 $95, 000 $120,000 $101, 000 $99,000 $98,000 $110,000

1. Determine the mean, median and the range for these seven salaries.
2. Which of the seven salaries is an outlier?
3. Remove the outlier and recalculate the mean, median and the range for the six remaining salaries.
4. Comparing the results for the seven versus six salaries, which of the three measures was least impacted by the outlier?
5. Comparing the results for the seven versus six salaries, which of the three measures was most impacted by the outlier?

Show answer

1. mean salary $97,857; median salary $99,000; range in salaries $58,000
2. $62,000
3. mean salary $103, 833; median salary $100,000; range in salaries $25,000
4. the median
5. the range

Refer back to Example 2 and the measures that were calculated for five test scores.The student’s mean score is 81% and the range in marks is 60%. The range of 60% does not capture the fact that if the outlier is removed then there is a spread of only 6% for the four remaining data values. The range depends on only the **highest** and **lowest** data values. The existence of an outlier can result in a misleading representation of the spread in data values.

An alternative measure of dispersion is called the **standard deviation**. It depends on **all** data values rather than on only the highest and lowest data values.

Standard Deviation

Standard deviation measures the **dispersion of the data values around the mean**. Unlike the range, its value depends on every data value in the data set. The standard deviation is found by determining how much each data value differs from the mean.

What does the standard deviation actually tell us? Consider two sets of test scores:

Set A: 76% 74% 86% 84% 85%

Set B: 53% 95% 62% 99% 96%

Refer to Figure 1. For both sets the mean is 81%. If we plot the scores (indicated by the *****) on a scale of 0% to 100% we see that the scores in Set A are much less spread out around the mean. The scores from Set B are much more dispersed.

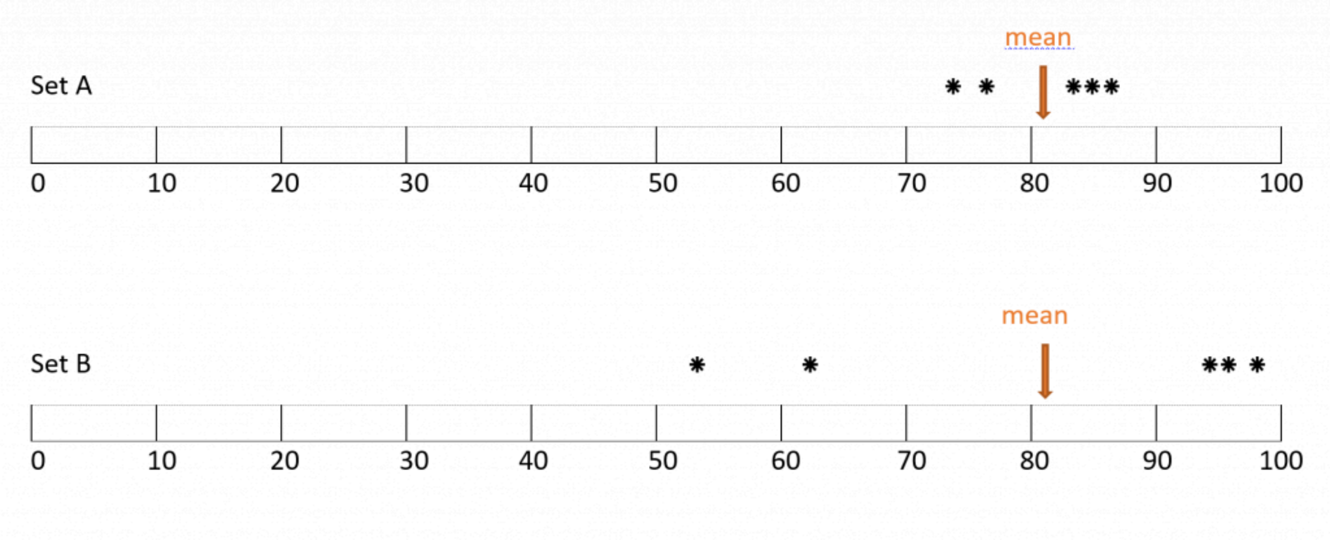


Fig. 1

If we compare the standard deviations for the two data sets we should find that although the mean is the same, the standard deviation for Set B will be greater since the data is more widely spread out from the mean.

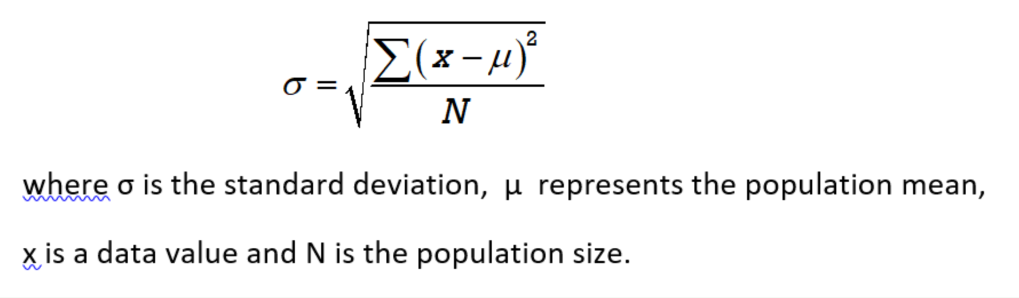
Population versus Sample Standard Deviation

When working with standard deviation it is important to distinguish whether you are working with the entire population or a sample of the population. Statisticians generally survey a **sample** of the population because it is often impossible to survey the entire population. As an example, assume your university wants to determine food preferences for its entire student body. The population would be all students enrolled in the university. Rather than attempt to survey every student, the university will survey only a **sample**.

The symbolic representation of standard deviation is different for a population versus a sample. If you are working with an entire **population,** the symbol for standard deviation is the Greek letter sigma, **σ.** The symbol for the standard deviation of a **sample** is *s*. Similarly, the calculation of standard deviation is different for a population versus a sample. Unless otherwise indicated in this chapter, we will assume that we are working with a sample, rather than an entire population.

Standard Deviation Formula

Standard Deviation Formula for a Population



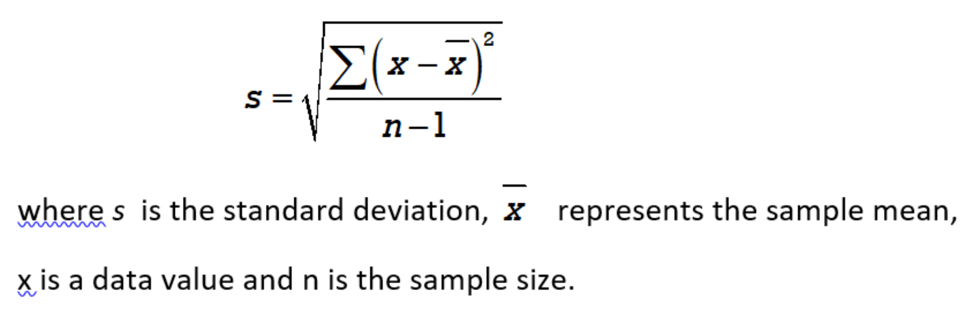
$$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{N}}$$

where σ is the standard deviation, μ represents the population mean,

x is a data value and N is the population size.

The Greek letter sigma ∑ is the summation symbol. It indicates that all of the values $(x - \mu)^2$ must be added.

Standard Deviation Formula for a Sample



$$s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}}$$

where s is the standard deviation, $\bar{x}$ represents the sample mean,

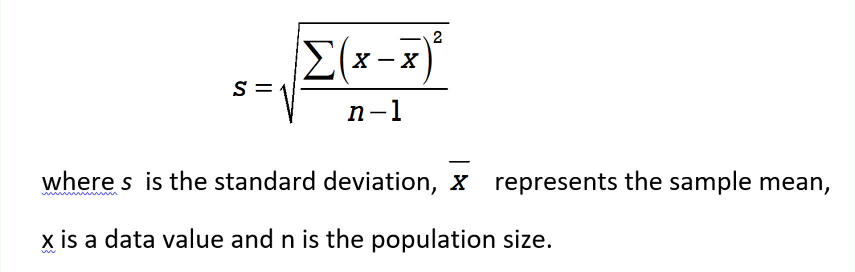
x is a data value and n is the sample size.

Calculating the Standard Deviation

When calculating the standard deviation with the aid of a scientific calculator it is helpful to record the steps using a table.

Calculating Standard Deviation

We will use the formula for finding the standard deviation of a sample:



$$s = \sqrt{\frac{\sum\left(x - \overline{x}\right)^2}{n-1}}$$

where s is the standard deviation, $\overline{x}$ represents the sample mean, x is a data value and n is the population size.

To determine the standard deviation for a sample we will use the following steps :

1. Find the mean
2. Create a table with three columns: data value, data value – mean, (data value – mean)^2
3. Fill in the **data value** column with all values from the sample
4. Subtract the mean from each data value: (data value – mean)
5. Square the results from step 4: (data value – mean)^2
6. Sum the results in column 3 (from step 5)
7. Divide the sum (from step 6) by (n – 1)
8. Find the square root of the result in step 7

EXAMPLE 3

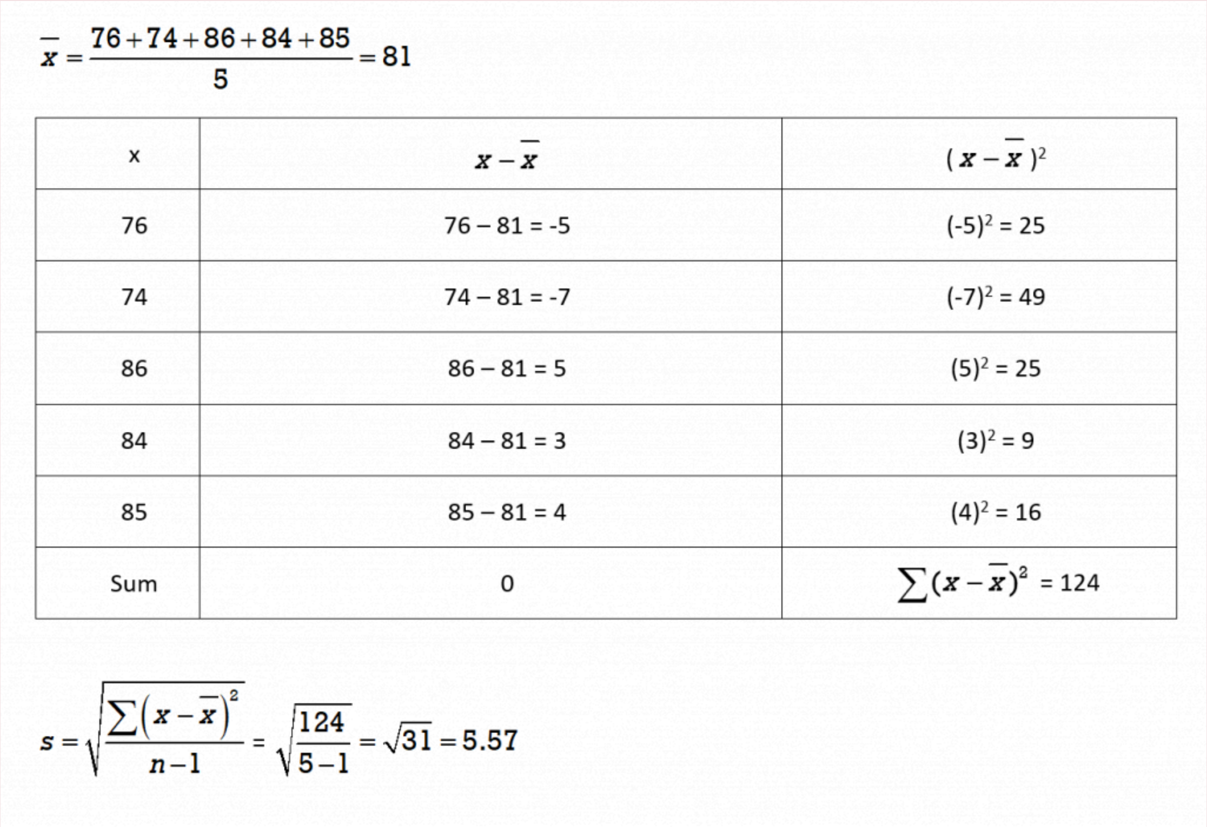
a) Determine the standard deviation for the sample set A test scores: 76% 74% 86% 84% 85%

b) Determine the standard deviation for the sample set B test scores: 53% 95% 62% 99% 96%

c) Compare the means and standard deviations for Set A and Set B. Which set is more spread out (dispersed)?

Solution

a)



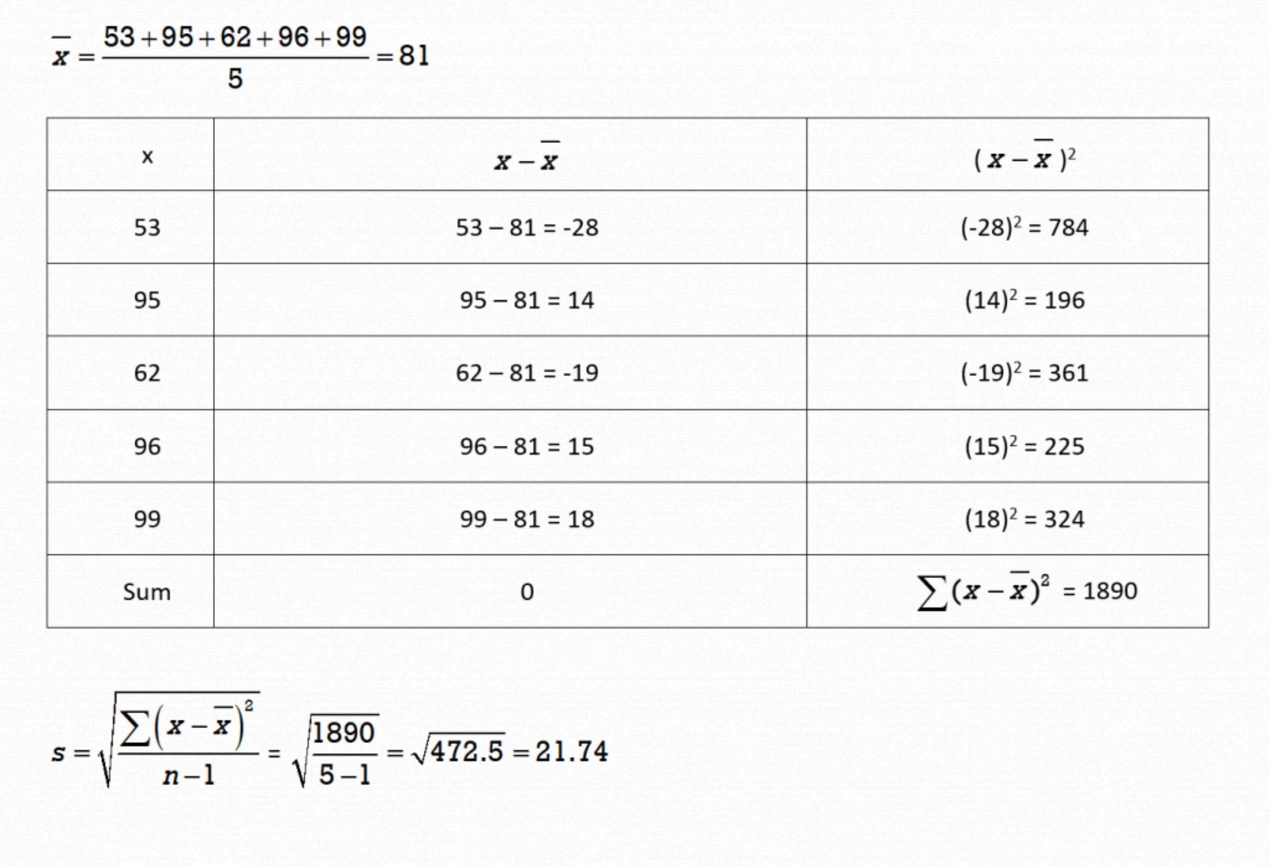
$$\bar{x} = \frac{76+74+86+84+85}{5} = 81$$

x	$x - \bar{x}$	$(x - \bar{x})^2$
76	76 – 81 = -5	$(-5)^2 = 25$
74	74 – 81 = -7	$(-7)^2 = 49$
86	86 – 81 = 5	$(5)^2 = 25$
84	84 – 81 = 3	$(3)^2 = 9$
85	85 – 81 = 4	$(4)^2 = 16$
Sum	0	$\sum (x - \bar{x})^2 = 124$

$$s = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n-1}} = \sqrt{\frac{124}{5-1}} = \sqrt{31} = 5.57$$

The standard deviation for Set A is 5.57 (rounded to 2 decimal places)

b)



$$\overline{x} = \frac{53+95+62+96+99}{5} = 81$$

x	$x - \overline{x}$	$(x - \overline{x})^2$
53	53 – 81 = -28	$(-28)^2 = 784$
95	95 – 81 = 14	$(14)^2 = 196$
62	62 – 81 = -19	$(-19)^2 = 361$
96	96 – 81 = 15	$(15)^2 = 225$
99	99 – 81 = 18	$(18)^2 = 324$
Sum	0	$\sum(x - \overline{x})^2 = 1890$

$$s = \sqrt{\frac{\sum\left(x - \overline{x}\right)^2}{n-1}} = \sqrt{\frac{1890}{5-1}} = \sqrt{472.5} = 21.74$$

The standard deviation for Set B is 21.74 (rounded to 2 decimal places).

c)

In comparing the two data sets A and B:

- The means are the same value of 81.
- The standard deviation for Set A is s = 5.57 and for Set B it is s = 21.74. The much larger standard deviation for Set B indicates that there is a much greater spread in the data values around the mean of 81%.

Note that the sum of the middle column (data value – mean) is 0. This will always be the case. That is why we must square the values before we add them, as is done in column 3.

TRY IT 3

The average high temperatures (in ºC) for one week in April for two different cities in Canada were as follows:

City A:	15	19	22	26	21	19	18
City B:	6	9	15	18	20	19	21

a) Calculate the mean high temperature (if necessary round to 2 decimal places) for each city. Which of the two cities appears to have a wider spread in temperatures around their means?

b) Calculate the standard deviation for each temperature set (if necessary round to 2 decimal places) to see if your observation is correct.

Show answer
The mean for City A is 20 ˚C and the mean for City B was 15.43˚C.

The standard deviation for City A is 3.46 and for City B is 5.80. City B's temperatures are more widely spread out from the mean temperature.

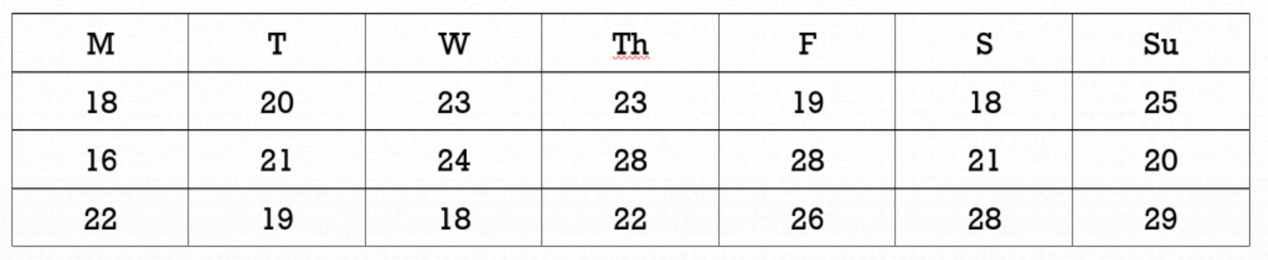
We have seen that two measures of spread or dispersion are the **range** and the **standard deviation**. Although the range is a much simpler calculation, it only takes into consideration the highest and lowest data values. The existence of an outlier can result in a range that is not truly indicative of the spread in data values. The standard deviation is a more complex calculation but takes into consideration all data values. It is important to note that technology is often used to calculate the standard deviation which eliminates the need for tedious calculations.

EXAMPLE 4

In Example 1 the range for the following set of temperature was determined to be 13 ºC.

a) Determine the mean and the standard deviation.

b) Explain why the values for the range and standard deviation are different.



M	T	W	Th	F	S	Su
18	20	23	23	19	18	25
16	21	24	28	28	21	20
22	19	18	22	26	28	29

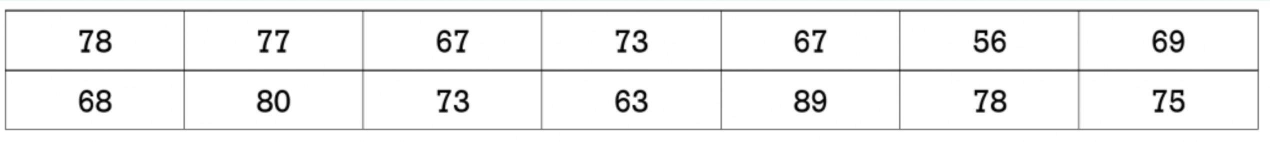
Solution

a) The mean (average) is 22.29 ˚C and the standard deviation is 3.86.
b) The reason for the difference in values is that the range only tells us the difference between the highest and lowest temperature whereas the standard deviation tells us how widespread the temperatures are in relation to the mean temperature of 22.29 ˚C

TRY IT 4

Refer back to TRY IT 1 and the test scores for a group of 14 students. The range was determined to be 33.

Determine the mean (rounded to the nearest whole number) and the standard deviation (rounded to the nearest 2 decimal places) for this set of test scores.



78	77	67	73	67	56	69
68	80	73	63	89	78	75

Show answer
The mean is 72 and the standard deviation is 8.19.

Histograms and the Dispersion of Data Values

We have seen that for two data sets with the same mean, when the standard deviation is larger the data values are more spread out. A **histogram** can be used to illustrate the spread of data values.

Consider a dance competition where teams comprised of seven dancers compete for the prize money in several different dance categories. There were four teams entered in the elite category. The dancers must be between the ages of 18-24 years old. The age breakdown for the members of the four teams in the elite category is:

Team Unity: all seven dancers are age 21

Team Harmony: 2 dancers are 20, 3 dancers are 21 and 2 dancers are 22

Team Mix: 1 dancer is 19, 2 dancers are 20, 1 dancer is 21, 2 dancers are 22 and 1 dancer is 23

Team Extend: 3 dancers are 18, 1 dancer is 21, and 3 dancers are 24

The **mean** age for all four teams is 21 but the standard deviations for each of the four teams are different. Team Unity has a standard deviation of 0, Team Harmony has a standard deviation of 0.82, Team Mix has a standard deviation of 1.412 and Team Extend has a standard deviation of 3.

The histograms for each of the teams appears in Figures 2a through Figures 2d below.

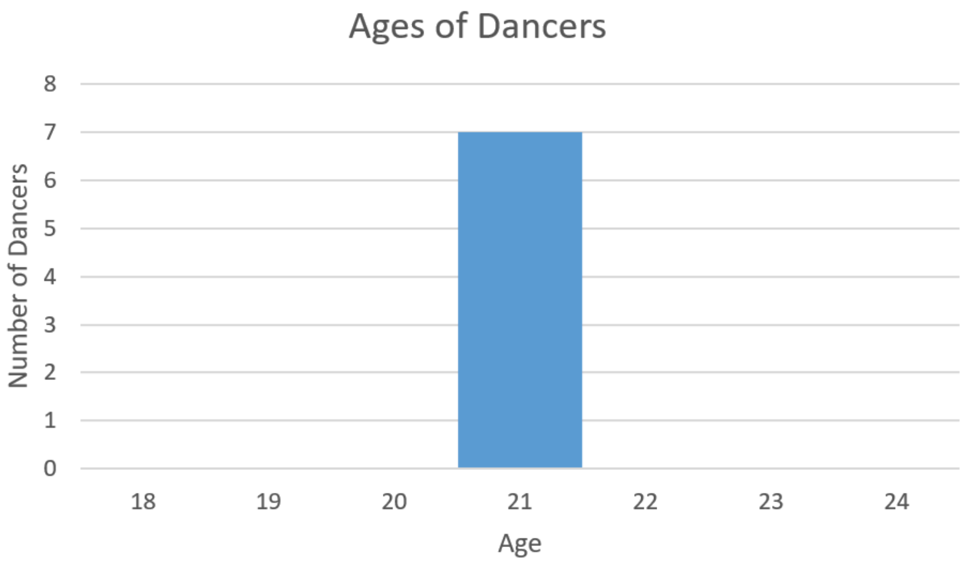


Fig. 2a. Team Unity with s = 0

In Figure 2a Team Unity has a standard deviation of 0 since all ages are the same. None of the ages spread out from the mean of 21.

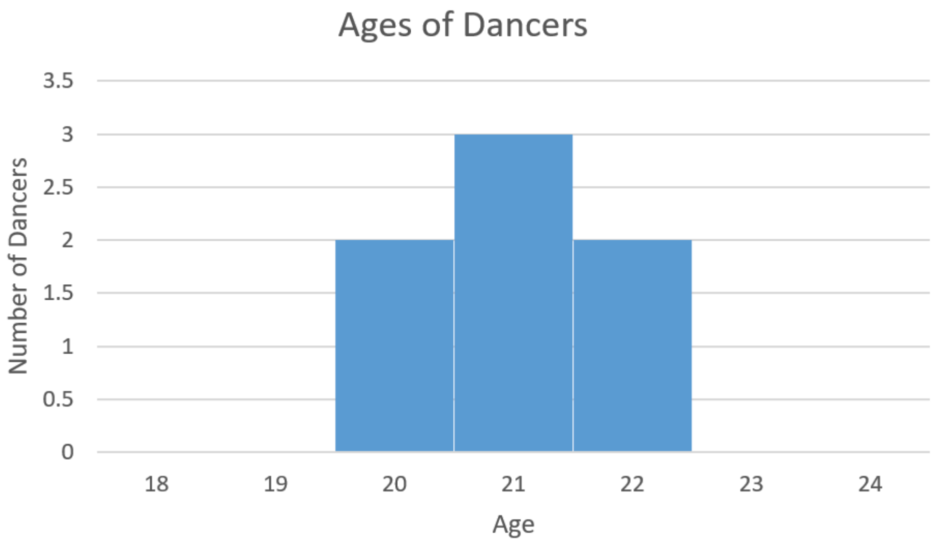


Fig. 2b. Team Harmony with s = 0.82

In Figure 2b Team Harmony has a standard deviation of 0.82 years. The histogram illustrates that the ages are closely clustered around the mean of 21.

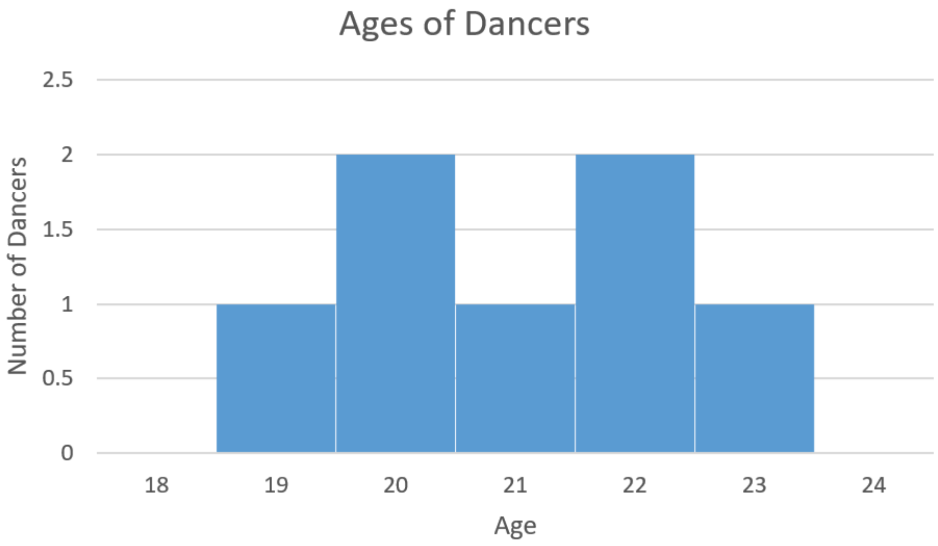


Fig. 2c Team Mix with s = 1.41

In Figure 2c Team Mix has a standard deviation of 1.41. The histogram illustrates that the data (age) spread is greater than for Team Unity and Team Harmony.

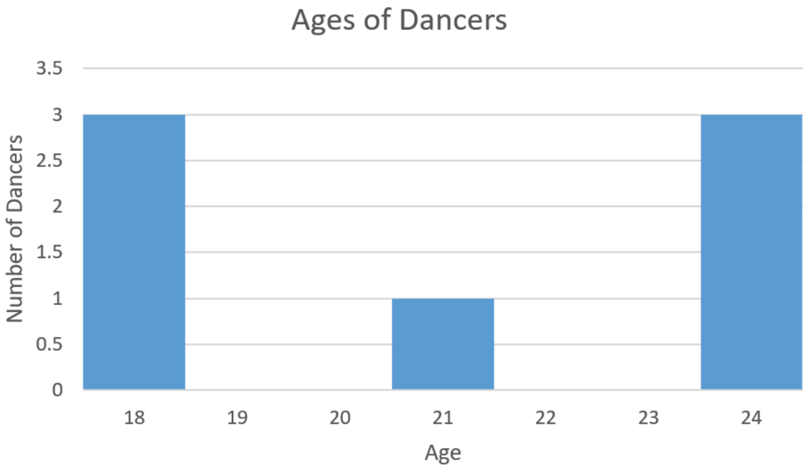
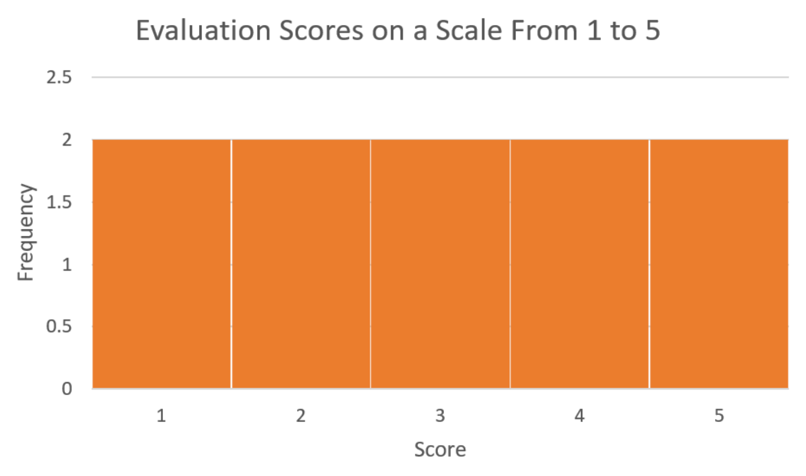


Fig. 2d. Team Extend with s = 3

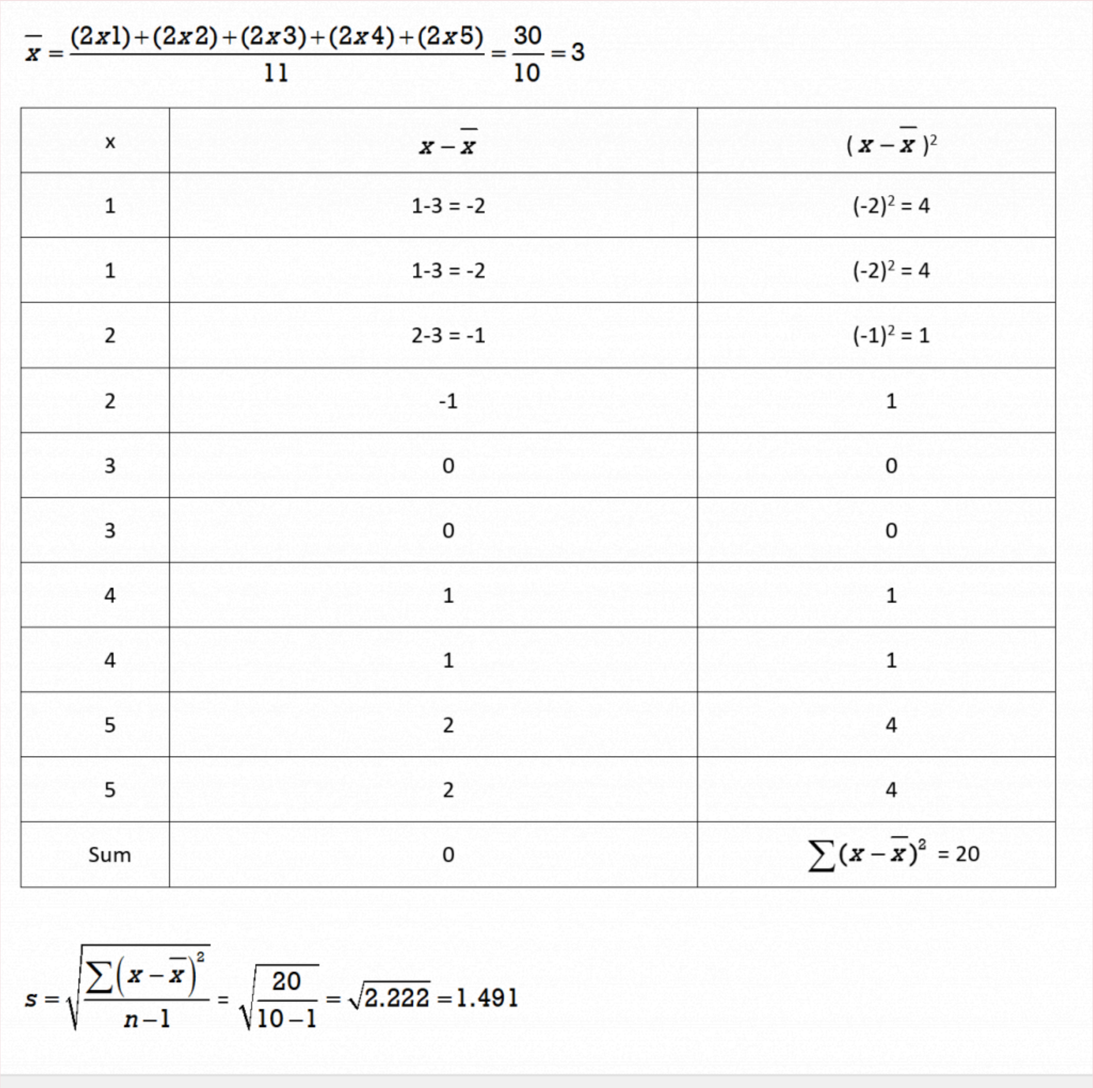
In Figure 2d Team Extend has a standard deviation of 3. The histogram clearly illustrates that the ages for this team are the most spread out from the mean of 21.

EXAMPLE 5

Ten participants in a group fitness class were asked to rank the class on a scale from 1 to 5. Determine the mean and standard deviation for the evaluation scores as depicted in the histogram below.



Solution



$$\bar{x} = \frac{(2x1)+(2x2)+(2x3)+(2x4)+(2x5)}{11} = \frac{30}{10} = 3$$

x	$x-\bar{x}$	$(x-\bar{x})^2$
1	1-3 = -2	$(-2)^2 = 4$
1	1-3 = -2	$(-2)^2 = 4$
2	2-3 = -1	$(-1)^2 = 1$
2	-1	1
3	0	0
3	0	0
4	1	1
4	1	1
5	2	4
5	2	4
Sum	0	$\sum(x-\bar{x})^2 = 20$

$$s = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n-1}} = \sqrt{\frac{20}{10-1}} = \sqrt{2.222} = 1.491$$

The mean score is 3 and the standard deviation is 1.49

TRY IT 5

The hourly wages for ten employees at a small coffee shop are illustrated in the histogram. Determine the mean and standard deviation for the employee hourly wages.

Show answer
The mean is $17.80 and the standard deviation is $1.32

We have seen that standard deviation provides us with a measure of the spread of data values in relation to the mean. We have learned how to calculate the standard deviation for a data set but we have not explored the significance or meaning of these calculated values. In the next section we will explore the significance of the calculated values as we consider the relationship between the standard deviation and the distribution of the data values.

Key Concepts

- Two measures of dispersion or spread in data values are:
  - Range = highest data value – lowest data value
  - Standard deviation. Unlike the range, its value depends on every data value in the

data set. It is found by determining how much each data value differs from the mean.

Glossary

histogram

represents the frequency distribution (number of occurrences) of each data value. The data values are grouped into intervals or “bins”.

outlier

A data observation that is deemed to be unusual based on the pattern of the other data values.

range

indicates the total spread in data values. It is the difference between the highest and lowest data values.

sigma

is the uppercase Greek letter written $\sum$. It is used to indicate the sum of a series of values.

standard deviation

measures the dispersion of the data values around the mean.

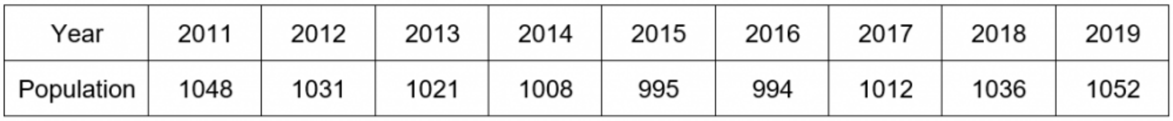
8.2 Exercise Set

1. The daily high temperature (in degrees C) for Calgary AB was recorded over a period of two weeks:

28	25	26	27	27	29	30
30	12	20	22	25	24	25

   a. Determine the mean, median and the range for these fourteen temperatures.
   b. Which of the temperatures appears to be an outlier?
   c. Remove the outlier and recalculate the mean, median and the range for the thirteen remaining values.
   d. Comparing the results for the fourteen temperature values versus thirteen temperature values, which of the three measures was most impacted by the outlier?

2. The population of Cache Creek, B.C. for the years 2011 to 2019 is provided in the table below (*Demographic Analysis Section, BC Stats)*



Year	2011	2012	2013	2014	2015	2016	2017	2018	2019
Population	1048	1031	1021	1008	995	994	1012	1036	1052

a. Determine the mean and median population.

b. Determine the range.

c. Determine the standard deviation.

3. Set A: 6 7 7 10 10 11Set B: 10 11 12 13 14 15Set C: 10 12 16 11 12 13

   a. For each of the three data samples below, calculate the range, mean, and standard deviation.

   b. Based on these calculations, determine any similarities that exist between any of the data samples.

   c. Which of the three measures is different for every set?

   a. Given the two data samples **Set A** and **Set B** below, which appears to have the greatest spread in values?

      - **Set A**: 69 86 74 60 67 65
      - **Set B**: 50 51 86 50 52 51

   b. For each sample calculate the mean, range and standard deviation. If necessary round final answers to 2 decimal places.

   c. Based on these calculations which sample has the larger standard deviation?

   d. If the 86 in each data set is changed to a 56 which set would you predict would have the greatest **change** in standard deviation? Recalculate the standard deviations with a value of 56 instead of 86. Is your prediction correct?

      - **Set A**: 69 56 74 60 67 65
      - **Set B**: 50 51 56 50 52 51

4. The standard deviation for a sample is calculated to be 0. What can you conclude about the data values?

5. The maximum hourly wage (in dollars/hour) for pipefitters and carpenters in the ten Canadian provinces is listed below.(Source: Wages for Steamfitters, pipefitters and sprinkler system installers from the Canadian Job Bank)

   - **Pipefitter**: 48 47 45 45 40 48 42 50 46 43
   - **Carpenter**: 36 33 25 28 39 36 33 35 39 35

   a. Without calculating the average, which **occupation** appears to have the higher average maximum hourly wage?

   b. Calculate the average (mean) maximum hourly wage (to the nearest cent) for each occupation.

   c. Determine the range in maximum hourly wages for each occupation. Which occupation has a greater range in hourly wages?

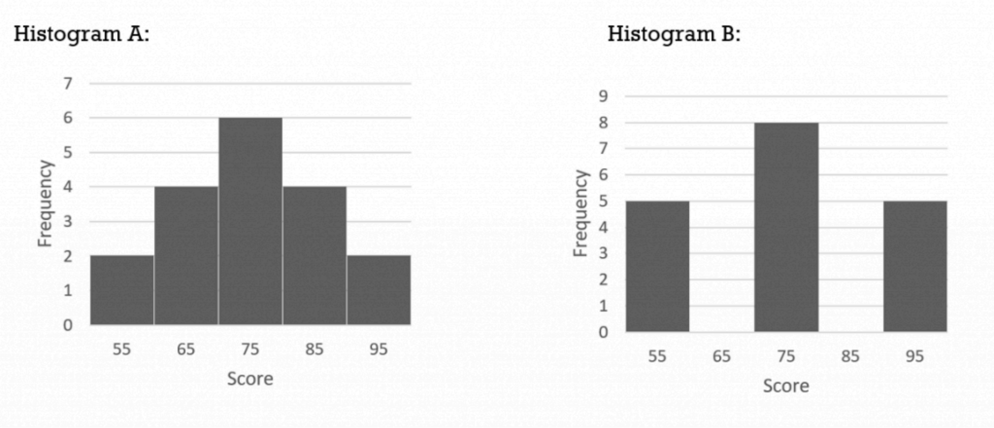
   d. Calculate the standard deviation (to the nearest cent) for each occupation. Which occupation has a maximum hourly wage that is more spread out?

6. Revisit question #1. The daily high temperature (in degrees C) for Calgary AB was recorded over a period of two weeks:

28	25	26	27	27	29	30
30	12	20	22	25	24	25

   a. Calculate the standard deviation for the fourteen temperature values.
   b. Remove the outlier and recalculate the standard deviation for the thirteen temperature values. How has the standard deviation changed with the removal of the outlier?

7. Two histograms are provided below:



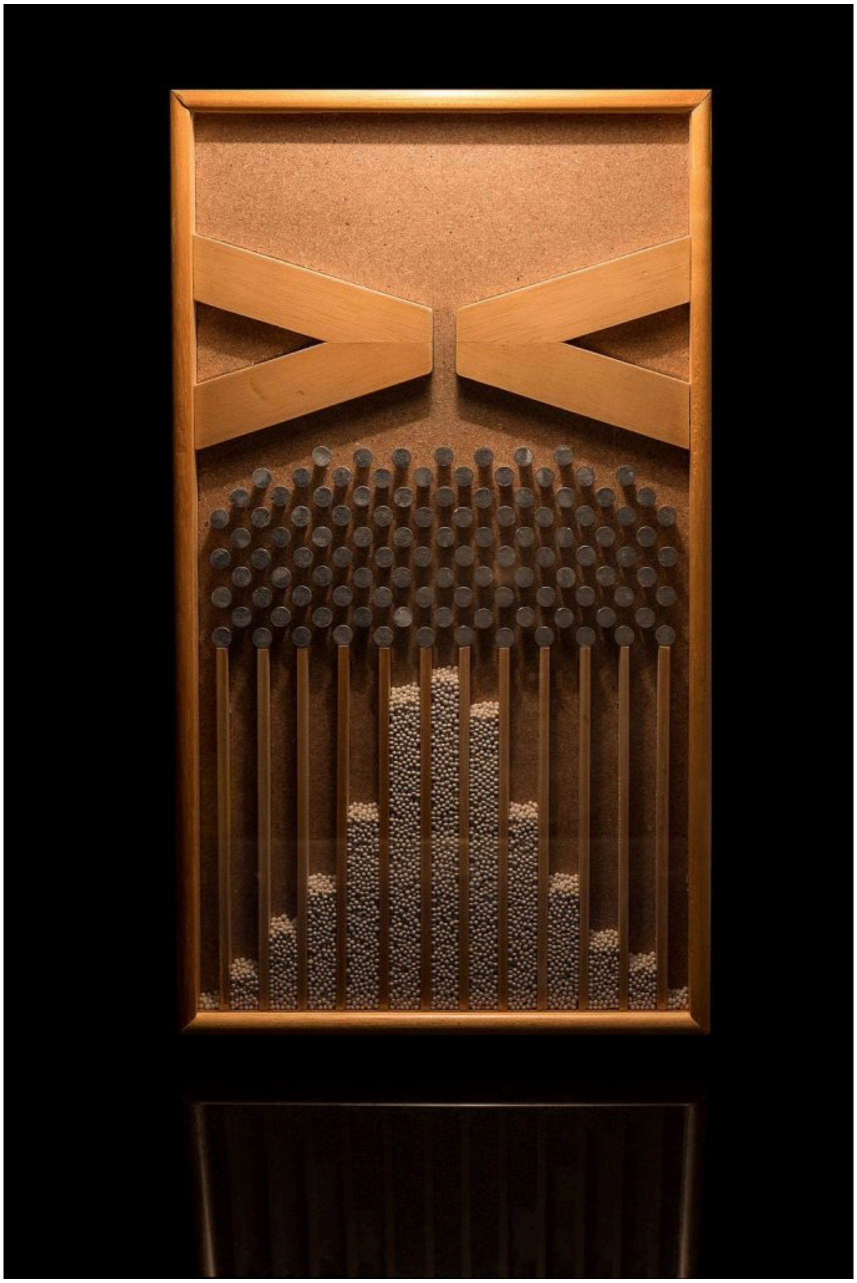
   a. For each histogram determine the range, mean, and standard deviation.
   b. Considering only the means and ranges, what can you conclude about the spread or dispersion of the data values for each set?
   c. Considering only the means and standard deviations, what can you conclude about the spread or dispersion of the data values for each set?

Answers

a. mean 25; median 25.5, range 18
b. outlier is 12
c. mean 26; median 26, range 10
d. The range was most impacted by the removal of the outlier.
a. mean is 1022; median is 1021
b. 58
c. standard deviation = 21.40

   a. Set A: range is 5; mean is 8.5; standard deviation is 2.07Set B: range is 5; mean is 12.5; standard deviation is 1.87Set C: range is 6; mean is 12.33; standard deviation is 2.07
   b. Set A and Set B have identical ranges but different means and standard deviations.Set A and Set C have the same standard deviation but different ranges.
   c. Each set has a different mean.
   a. Answers may vary. Set A scores appear to be more spread out; aside from one score of 86, the scores in Set B are less spread out.
   b. Set A: range is 26; mean is 70.17; standard deviation is 9.02Set B: range is 36; mean is 56.67; standard deviation is 14.39
   c. Set B has a larger standard deviation.
   d. Predictions will vary. For Set A the new standard deviation is 6.43 (compared to 9.02); Set B has a new standard deviation of 2.25 (compared to 14.39) so set B had the greatest change in standard deviation.
2. All data values are equal.
   a. pipefitter
   b. pipefitter ‘s top hourly wage on average is $45.40/hour; carpenter’s top hourly wage on average is $33.90/hour
   c. pipefitter range of $10/hour for the 10 provincescarpenter range of $14/hour for the 10 provincesThe carpenter has a greater range in wages
   d. pipefitter has a standard deviation of $3.06 and the carpenter has a standard deviation of $4.46. The carpenter has a higher standard deviation and therefore the carpenter’s hourly wage is more spread out.
   a. Standard deviation is 4.71
   b. Standard deviation is 2.97; With the removal of the outlier this value has become smaller
   a. Histogram A: range is 40, mean is 75, standard deviation is 11.88Histogram B: range is 40, mean is 75, standard deviation is 15.34
   b. Considering only the means and ranges, these are both identical so it would appear that the data values are equally dispersed.
   c. Histogram B has a greater dispersion of data as indicated by the larger standard deviation.

8.3 The Normal Curve



Learning Objectives

After completing this section the student should be able to:

- Recognize the characteristics of a normal distribution
- Find scores at a designated standard deviation from the mean
- Interpret and use the 68-95-99.7 Rule

The Normal Distribution

The Galton Board, invented by Sir Francis Galton, consists of a vertical board with interleaved rows of pegs. Beads are dropped from the top and, when the device is level, bounce either left or right as they hit the pegs. Eventually they are collected into bins at the bottom, where the height of bead columns accumulated in the bins approximate a **normal distribution.** (https://en.wikipedia.org/wiki/Bean_machine#/)

In this section we will explore the normal distribution and the dispersion of data values around the mean. We have seen that the **standard deviation** provides a measure of the dispersion of the data values around the mean. If the standard deviation is zero then all data vales will equal the mean. The general idea seems to be that as the standard deviation increases the data will be more widely dispersed around the mean. We have also seen that data can be distributed in a variety of ways. Consider the histograms in [Figures 1, 2 & 3](). These histograms represent the evaluation scores (on a scale of 1 to 5) for three instructors. In all three cases a group of 10 students provided feedback for each of the instructors.

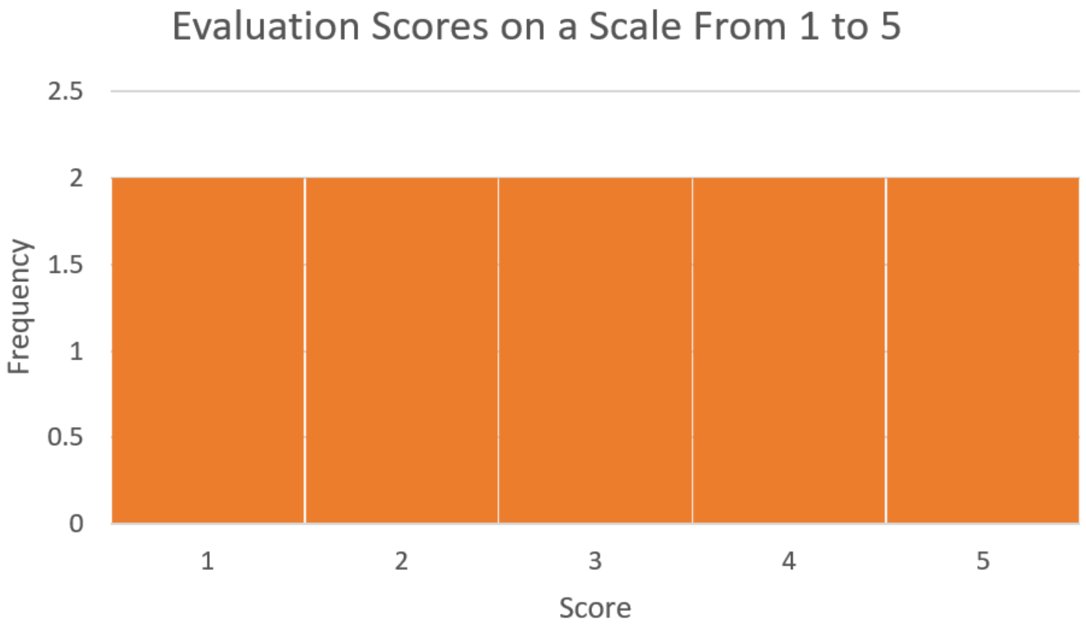


Fig. 1 Instructor A

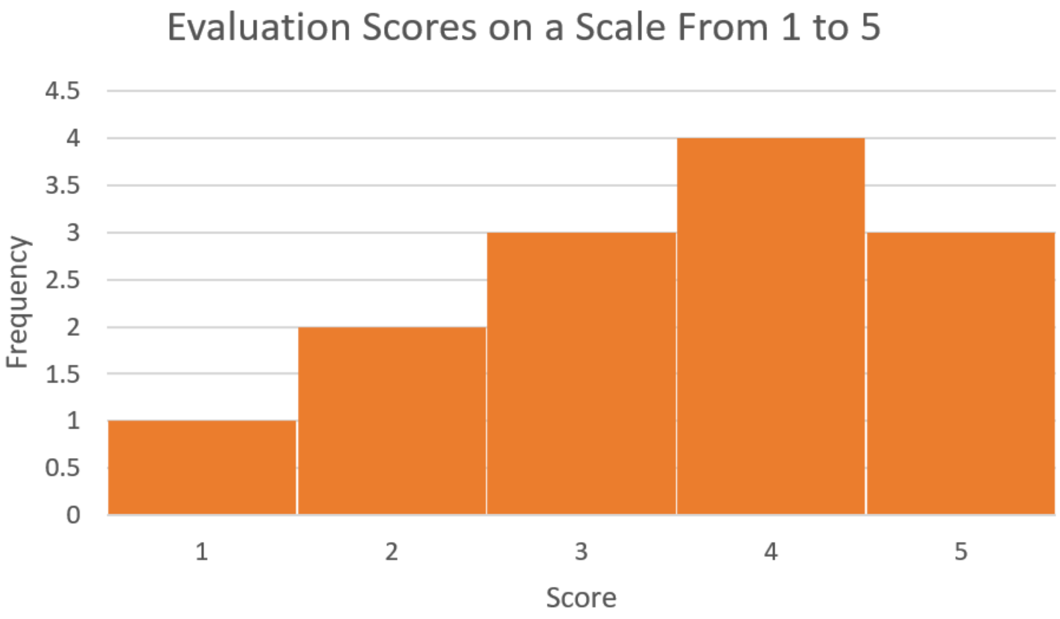


Fig. 2 Instructor B

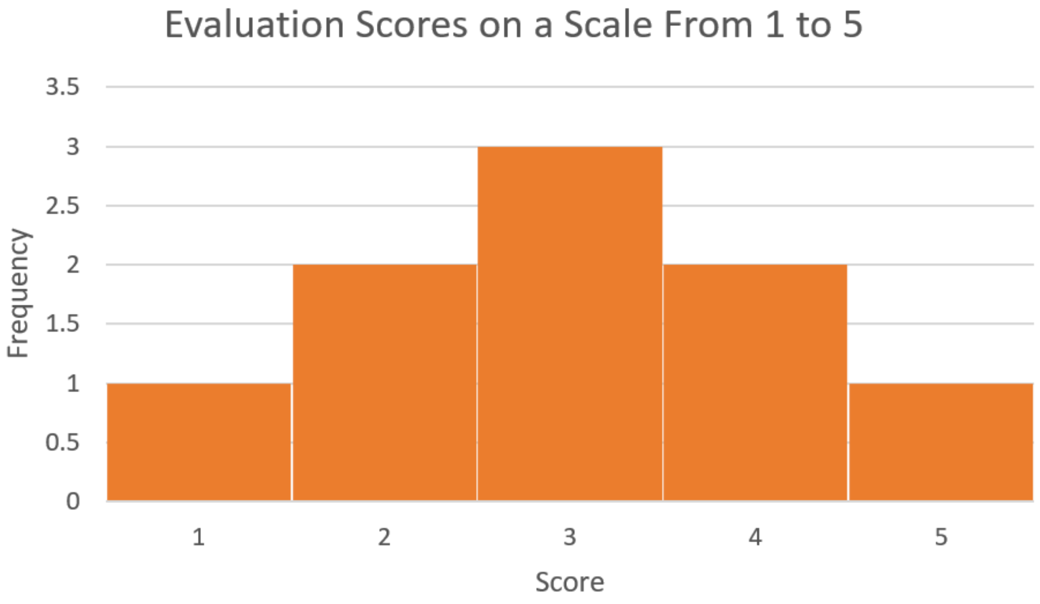


Fig. 3 Instructor C

Referring to Figure 1, Instructor A received each possible score two times. Figure 1 represents a **uniform** distribution since every data value occurs with the same **frequency**.

Referring to Figure 2, Instructor B received a mix of scores. Figure 2 represents a **skewed** distribution where one tail of the distribution is stretched out more than the other.

Referring to Figure 3, Instructor C also received a mix of scores. Figure 3 represents a **symmetrical** distribution. Data values occur most often in the centre of the distribution and spread out equally on either side.

The histogram in Figure 3 is symmetric but because it represents a very small sample size it appears to be a series of rectangles stacked side by side. As the sample size increases a symmetrical distribution will become less boxy as illustrated in Figures 4 & 5.

Fig. 4

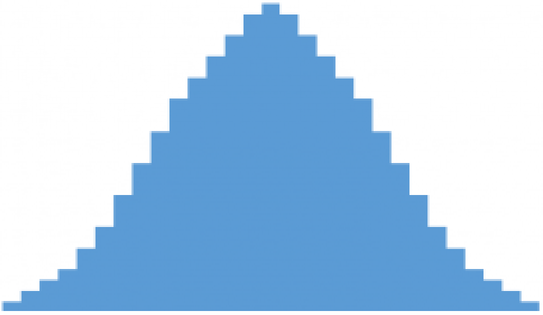


Fig. 5

Eventually if we consider the entire population the distribution will approach what is called the **normal distribution** as in Figure 6. This distribution is also called the **bell curve**.

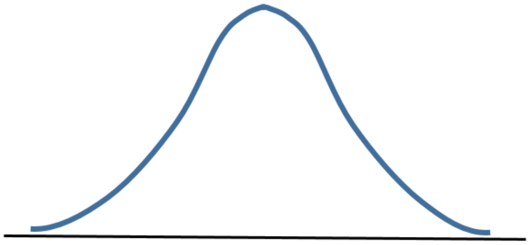


Fig. 6

The normal distribution models many aspects of real life, including height, blood pressure and IQ scores.

Normal Distribution

The **normal distribution** is also called the **bell curve**.

In a normal distribution the data values are **symmetrical** around a vertical line drawn through its centre which is also where the mean is located. Half of the data values lie on either side of the mean.

In a normal distribution the mean, median and mode will all be equal.

Normal Distribution and Standard Deviation

The normal distribution will have symmetry in relation to the mean but it could be flat or high, depending on the standard deviation. In Figure 7 the means are identical but the distribution in A has a smaller standard deviation.

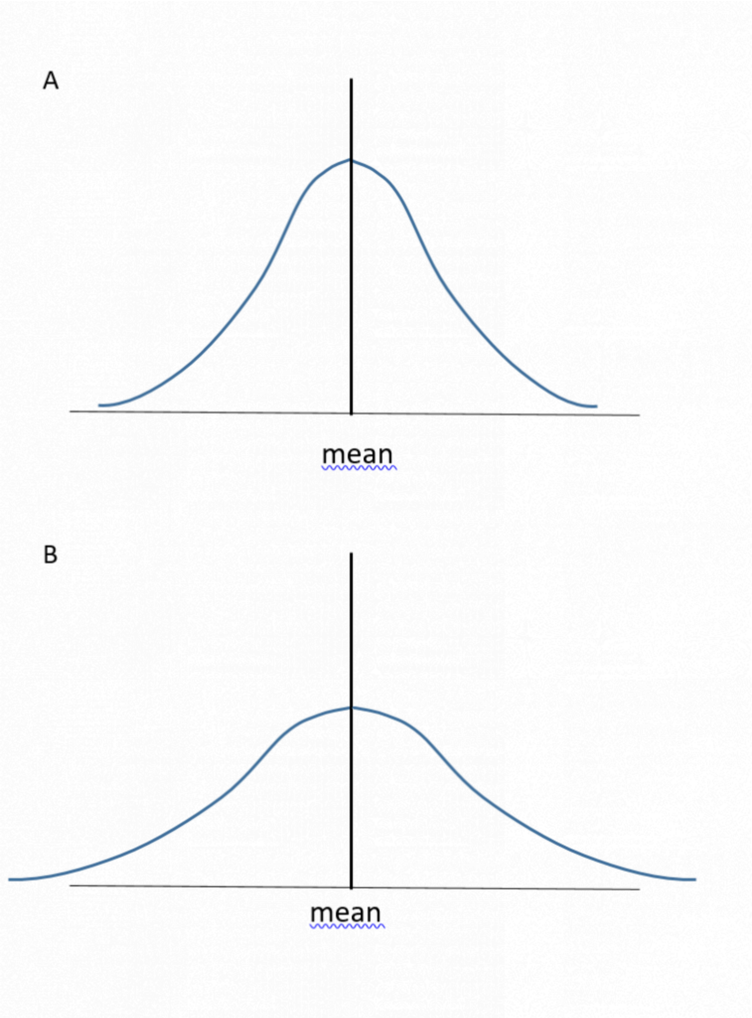


Fig. 7

The standard deviation plays an important role in the normal distribution, as described by the **68-95-99.7 Rule.**

68-95-99.7 Rule

According to the **68-95-99.7 Rule:**

- Approximately 68% (68.26%) of the data items lie within one standard deviation of the mean.
- Approximately 95% (95.44%) of the data items lie within two standard deviations of the mean.
- Approximately 99.7% of the data items lie within three standard deviations of the mean.

Figure 8 depicts the 68-95-99.7 Rule.

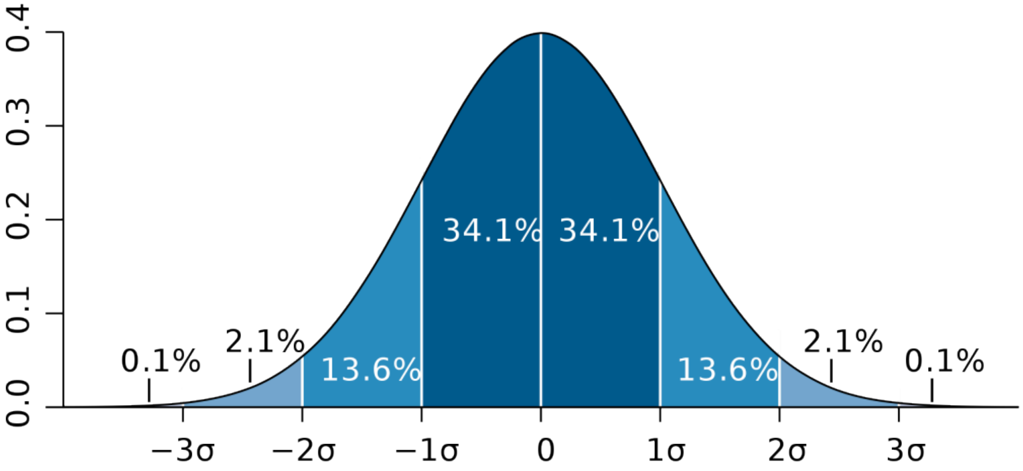


Fig. 8 68-95-99.7 Rule https://commons.wikimedia.org/wiki

In simple language what does this rule tell us? Refer to Figure 8 and consider the percentage of data items that lie within **one standard deviation** (σ) of the mean. This is the region that lies between -1σ and 1σ. If 34.1% lie on either side of the mean, then 34.1% + 34.1% = 68.2% or approximately 68% . For a population with a normal distribution, just over two-thirds of the data (68%) will lie within one standard deviation of the mean.

Consider the percentage of data items that lie within **two standard deviations** of the mean. This is the region that lies between -2σ and 2σ. Between -2σ and 0, we find 13.6% + 34.1% or 47.7% of the data values. So between -2σ and 2σ we will have 47.7% + 47.7% = 95.4% or approximately 95% . For a population with a normal distribution, approximately 95% of the data values will lie within two standard deviations of the mean.

Consider the percentage of data items that lie within **three standard deviations** of the mean. This is the region that lies between -3σ and 3σ. Between -3σ and 0, we find 2.1% + 13.6% + 34.1% or 49.8% of the data values. So between -3σ and 3σ we will have 49.8% + 49.8% = 99.6%. **(Note:** The values in Figure 8 are all rounded to the nearest tenth. The number is actually closer to 49.86% x 2 = 99.72%). For a population with a normal distribution, approximately 99.7% of the data values will lie within three standard deviations of the mean. Another way of stating this, since 99.7% of the data will lie within three standard deviations of the mean, then only 0.3 % of the data will **not** lie within three standard deviations.

Using the 68-95-99.7 Rule

When working with a population that has a normal distribution the **68-95-99.7 Rule** can be used to determine the percentage of the population that will be within one, two or three standard deviations of the mean.

EXAMPLE 1

A certain segment of the economy has a normally distributed salary, with a mean salary of $45,000 and a standard deviation of $4000.

a) Determine the salary that is one standard deviation above the mean.

b) Determine the salary that is three standard deviations below the mean.

c) Determine the salary range for the employees that lie within one standard deviation of the mean. What percent of the employees lie in this salary range?

d) Determine the salary range for the employees that lie within two standard deviations of the mean. What percent of the employees lie in this salary range?

e) What percent of the employees earn a salary less than $33,000?

Solution

a) $45,000 + $4000 = $49,000

b) $45,000 – (3 x $4000) = $33,000

c) $45,000 ± $4000 = $41,000 to $49,000. According to the **68-95-99.7 Rule,** sixty-eight percent of the employees for this segment of the economy lie within this salary range.

d) $45,000 ± (2 x $4000) = $37,000 to $53,000. According to the **68-95-99.7 Rule,** ninety-five percent of the employees for this segment of the economy lie within this salary range.

e) A salary of $33,000 is 3 standard deviations below the mean. According to the **68-95-99.7 Rule,** 100% – 99.7% or 0.3% of the employees lie above or below three standard deviations from the mean. Dividing 0.3 in half, we determine that 0.15% of the employees earn a salary less than $33,000.

TRY IT 1

Birth weights for newborns follow a normal distribution with a mean birth weight of 3.4 kg and a standard deviation of 0.55 kg. (Source O'Cathain et al)

a) Determine the birth weight that is two standard deviations above the mean.

b) Determine the birth weight that is one standard deviation below the mean.

c) Determine the weight range for newborns that lie within two standard deviations of the mean. What percent of the newborns lie in this weight range?

d) Determine the weight range for newborns that lie within three standard deviations of the mean. What percent of the newborns lie in this weight range?

e) What percent of newborns have a mean birth weight greater than 4.5 kg?

O'Cathain A., Walters S.J., Nicholl J.P., Thomas K.J., & Kirkham M. Use of evidence based leaflets to promote informed choice in maternity care: randomised controlled trial in everyday practice. British Medical Journal 2002; 324: 643-646

Show answer
a) 4.5 kg
b) 2.85 kg

c) 2.3 kg to 4.5 kg; 95% of newborns will have birth weights in this range

d) 1.75 kg to 5.05 kg which is 99.7% of the newborns

e) 0.15% of newborns

When we know the total number of data items in the population we are able to extend beyond stating percentages. This is illustrated in the Example 2.

EXAMPLE 2

A final exam was administered to 150 students enrolled in a first year calculus course. The mean score on the exam was 67% with a standard deviation of 8.

a) Determine the number of students who received a score of 67% or greater.

b) Determine the number of students who received a score within one standard deviation of the mean. What was the range in scores for these students?

c) Determine the number of students who received a score ranging between 51% to 83%.

d) What possible scores did the top 0.15% of the students receive? How many students were in this group?

Solution

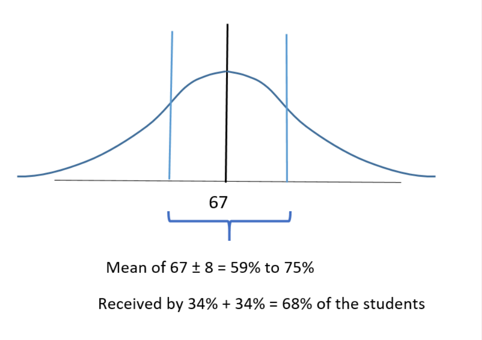
a) Since 67% was the mean or average score, half of the students 0.5 x 150 = 75 students received a score of 67% or greater.

b) According to the Rule, one standard deviation on either side of the mean represents 34% + 34% = 68% of the students so

0.68 x 150 students = 102 students scored within one standard deviation of the mean.

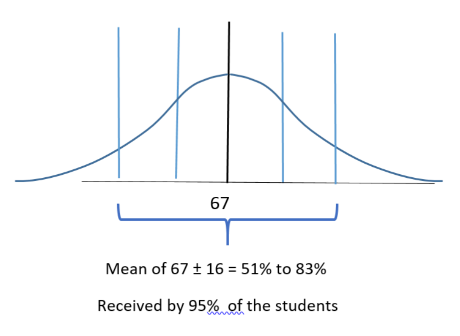
We know that the mean score was 67% and one standard deviation of 8 on either side:

67 – 8 = 59% and 67 + 8 = 75% therefore the 102 students within one standard deviation scored from 59% to 75% on the exam.



c) If we consider the mean of 67% and two standard deviations on either side:

67 – (2×8) = 51% and 67 + (2×8) = 83% This indicates that students who scored from 51% to 83% were two standard deviations on either side of the mean.



According to the Rule, two standard deviations on either side represents 95% of the students therefore 0.95 x 150 = 142.5 or between 142 to 143 students scored between 51% and 83% on the exam.

d) According to the Rule, 99.7% of the exam scores lie within 3 standard deviations of the mean, so 0.15% of the students scored higher than 3 standard deviations above the mean score. The mean score was 67% so:

67% + 3 standard deviations of 8 = 67% + (3 x 8) = 67% + 24% = 91%

Therefore the top 0.15% of the students received exam scores greater than 91%

The number of students receiving this score would be 0.15% x 150 students = 0.0015 x 150 = 0.225 students. This indicates that at most one student received a score greater than 91%.

TRY IT 2

A local run club hosted a recreational race. There were 148 entrants in the men's category and the mean time (rounded to the nearest minute) was 120 minutes with a standard deviation of 15 minutes.

a) Determine the number of runners who had times of 2 hours (120 minutes) or less.

b) Determine the number of runners who clocked a time within one standard deviation of the mean. What were the possible times for these runners?

c) Determine the number of runners who recorded a time between 90 and 150 minutes. (Hint: Consider that one standard deviation is 15 minutes)

d) What possible times did the slowest 2.5% of the runners record? How many runners were in this group?

Show answer
a) 0.5 x 148 = 74 runners

b) 0.68 x 148 = 100.6 runners (100 to 101) runners; 120 min. ± 15 min. = 105 to 135 min.

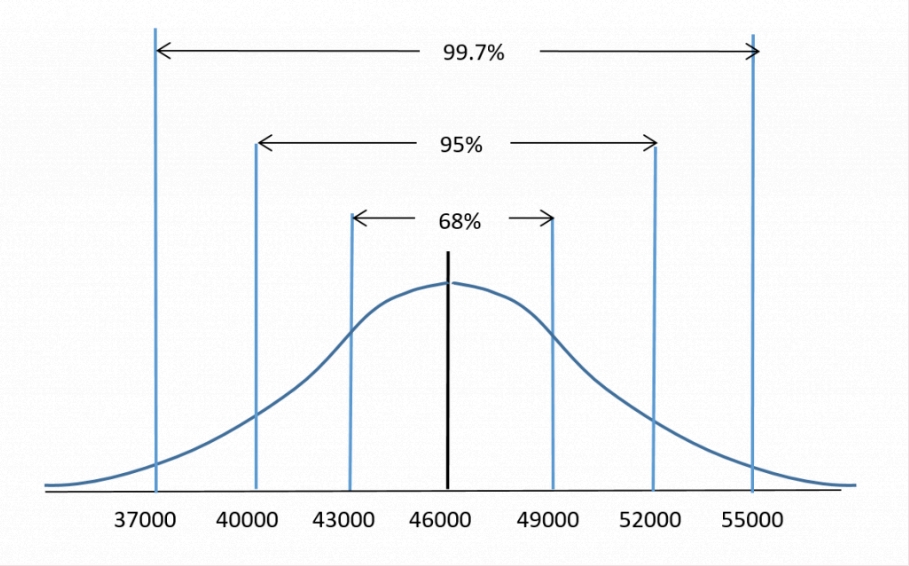
c) mean ± 2 std. deviations = 120 ± 30 min. = 90 to 150 minutes so this is 95% of the runners. 0.95 x 148 = 140.6 (140 to 141 runners)

d) 5% of the runners had times either two standard deviations above or below the mean so 2.5 % had times above the mean (the slowest times). 120 min + (2 x 15min) = 150 min. or greater
For 2.5% of 148 = 3.7 so 3 to 4 runners.

When working with a population that is normally distributed, it can be helpful to sketch the normal curve and calculate values that are one, two and three standard deviations on either side of the mean.

EXAMPLE 3

The average salary for a certain occupation in the trades is determined to be $46,000 (rounded to the nearest thousand) and the standard deviation is $3000. The salaries are normally distributed as indicated in the figure:



Use the 68-95-99.7 rule to determine the percentage of workers in this trade who earn:

a) less than $46,000

b) between $43,000 and $49,000

c) between $37,000 and $55,000

d) less than $55,000

e) between $40,000 and $49,000

Solution:

Note that there is more than one approach for these.

a) Since $46,000 is the mean, 50% of the workers will earn less than $46,000.

b) $43000 is one standard deviation less than the mean and $49,000 is one standard deviation more than the mean. Using the Rule, 68% of the workers will earn between $43000 and $46000.

c) $37000 is three standard deviations less than the mean and $55,000 is three standard deviations more than the mean. Using the Rule, 99.7% of the workers will earn between $37000 and $55000.

d) From the Rule, 99.7% of the data values lie between 3 standard deviations or $37000 to $55000. The remaining 100% – 99.7% = 0.3% of the data values lie equally at either end of the distribution. This means that 0.3% /2 or 0.15% of the data values are greater than $55000 and 0.15% are less than $37,000. So 99.7% + 0.15% = 99.85% of the workers will earn less than $55000.

e) One approach is to work the two halves of the distribution separately and then add the results.

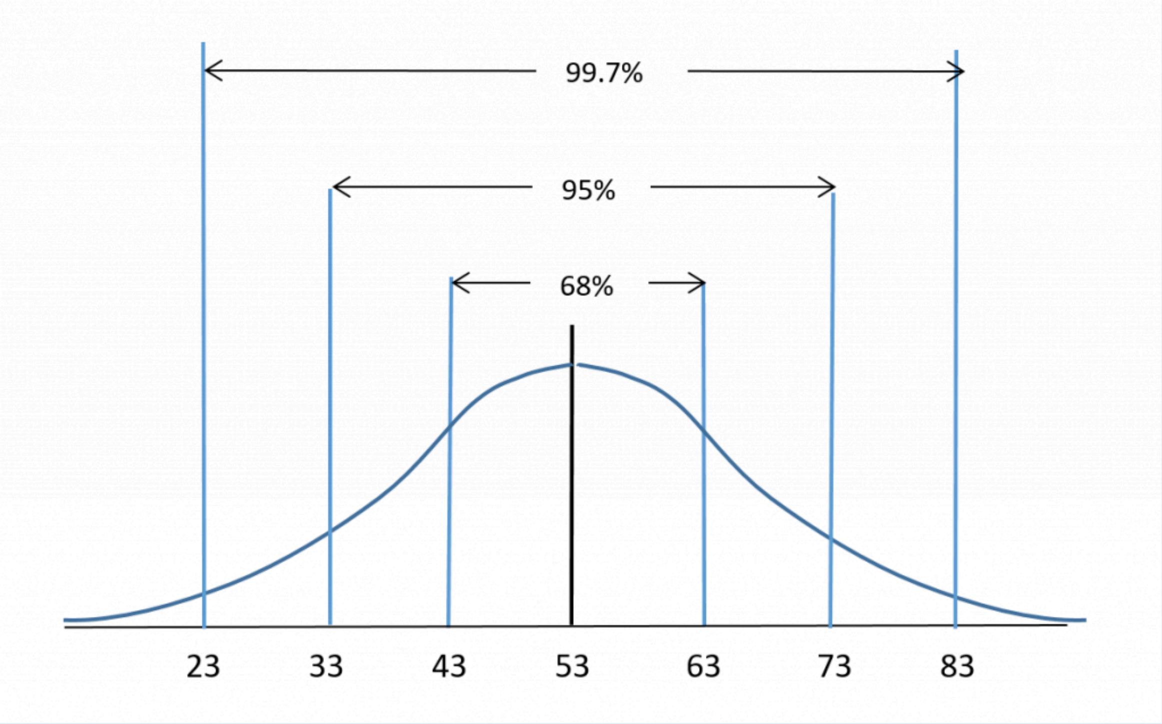
Start with the data values that lie between \$40000 and the mean of \$46000. The salary of \$40000 is is two standard deviations below the mean. If 95% of the values are ± 2 standard deviations from the mean then half this amount 95%/2 = 47.5% of the values are between \$40000 and \$46000.

Now consider \$49000 which is one standard deviation greater than the mean. If 68% of the values are ± 1 standard deviation from the mean then half this amount 68%/2 = 34% of the values are between \$46000 and \$49000.

Now add the two percentages 47.5% + 34% = 81.5%. Therefore 81.5% of the workers earn between \$40000 and \$49000.

TRY IT 3

A physics exam worth 90 points was administered to all first year students. The mean score was 53 points with a standard deviation of 10 points. The scores were normally distributed as indicated in the figure:



Use the 68-95-99.7 rule to determine the percentage of students who scored:

a) less than 63 points

b) between 33 and 53 points

c) more than 73 points

d) between 43 and 83 points

e) less than 43 points

Show answer
a) 50% + 34% = 84%

b) 95%/2 = 47.5%

c) 100% – 95% = 5% split evenly for scores less than 33 and greater than 73 so 5%/2 = 2.5% scored more than 73.

d) 34% + 99.7%/2 = 83.85%

e) 50% – 68%/2 = 16%

Key Concepts

- The **normal distribution** is also called the **bell curve**. The data values have a **symmetrical** distribution around a vertical line drawn through the mean.
- When working with a population that has a normal distribution the **68-95-99.7 Rule** can be used to determine the proportion of the population that will lie within one, two or three standard deviations of the mean.
    - 68% of the data values will lie within 1 standard deviation of the mean
    - 95% of the data values will lie within 2 standard deviations of the mean
    - 99.7% of the data values will lie within 3 standard deviations of the mean

Glossary

Normal Distribution

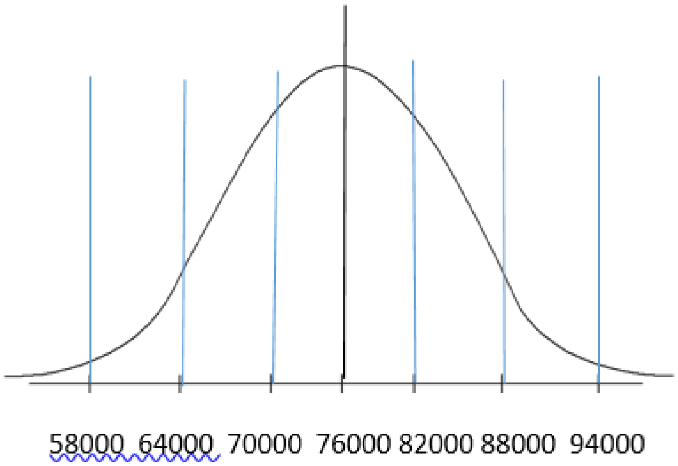
is when the data values lie in a symmetric fashion around the mean. Half of the data values lie on either side of the mean.

Skewed Distribution

is when more of the data values lie at one end of the distribution as compared to the other end.

8.3 Exercise Set

1. A population’s average weight is normally distributed.
    a. What percent of the population will have an average weight that lies within one standard deviation of the mean?
    b. What percent of the population will have an average weight that lies within three standard deviations of the mean?
    c. What percent of the population will have an average weight that lies beyond three standard deviations of the mean?
2. A certain segment of the economy has a normally distributed salary, with a mean salary of \$72,000 and a standard deviation of \$8000.
    a. Determine the salary that is one standard deviation below the mean.
    b. Determine the salary that is two standard deviations above the mean.
    c. Determine the salary range for the employees that lie within one standard deviation of the mean. What percent of the employees lie in this salary range?
    d. Determine the salary range for the employees that lie within three standard deviations of the mean. What percent of the employees lie in this salary range?
    e. What percent of the employees earn a salary more than \$72,000?
3. The average salary for a certain professional occupation is determined to be \$76,000 (rounded to the nearest thousand) and the standard deviation is \$6000. The salaries are normally distributed as indicated in the figure:



Use the 68-95-99.7 rule to determine the percentage of professionals in this occupation who earn:

    a. more than \$76,000
    b. between \$70,000 and \$82,000

   c. between \$64,000 and \$88,000
   d. less than \$58,000
   e. between \$76,000 and \$88,000
   f. between \$58,000 and \$76,000
   g. more than \$82,000

4. A survey of 100 people indicated that the average daily time they spend watching television is 2.5 hours with a standard deviation of 0.75 hours (45 minutes).
   a. Determine the amount of TV time that is one standard deviation above or below the average.
   b. Determine the amount of TV time that is two standard deviations above or below the average.
   c. Determine the amount of TV time that is more than three standard deviations above the average.
5. A survey of 200 people indicated that the average daily time they spend watching television is 2.5 hours with a standard deviation of 0.75 hours (45 minutes).
   a. Sketch a normal distribution and label the TV times (in hours) that represent the mean and the standard deviations from the mean. (Hint: Refer to your answers for question #4)
   b. What percent of those surveyed will watch TV for more than 4.75 hours/day? How many people out of the group watch TV for more than 4.75 hours/day?
   c. What percent of those surveyed will watch TV for less than 2.5 hours/day? How many people out of the group watch TV for less than 2.5 hours/day?
   d. What percent of those surveyed will watch TV for less than 1.75 hours/day? How many people out of the group watch TV for less than 1.75 hours/day?
   e. What percent of those surveyed will watch TV between 1.75 hours/day and 4 hours/day? How many people out of the group watch TV for 1.75 to 4 hours/day?
   f. What percent of those surveyed will watch TV between 0.25 hours/day and 3.25 hours/day? How many people out of the group watch TV for 0.25 to 3.25 hours/day?
6. A local run club hosted a recreational race. There were 230 entrants in the women's category and the mean time (rounded to the nearest minute) was 135 minutes with a standard deviation of 15 minutes.
   a. Determine the number of runners who had times of 135 minutes or more.
   b. Determine the number of runners who recorded a time greater than one standard deviation from the mean. What were the possible times for these runners?
   c. Determine the number of runners who recorded a time between 105 and 135 minutes. (Hint: Consider that one standard deviation is 15 minutes)
   d. What possible times did the fastest 0.15% of the runners record? How many

runners were in this group

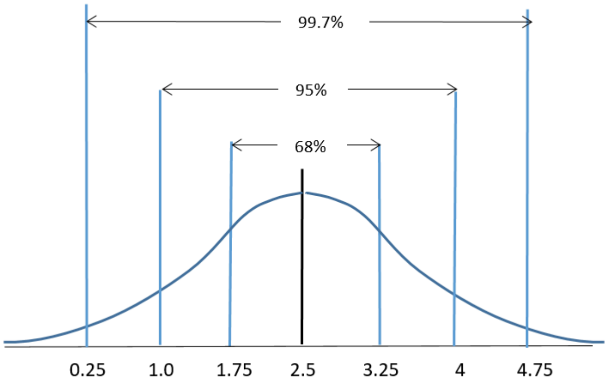
7. A biology exam worth 140 points was administered to all first year students. The mean score was 90 points with a standard deviation of 16 points. The scores were normally distributed. Sketch the normal curve and calculate and label the scores that are one, two and three standard deviations on either side of the mean.
8. A biology exam worth 140 points was administered to all first year students. The mean score was 90 points with a standard deviation of 16 points. The scores were normally distributed. Refer to the sketch in question#7 and use the 68-95-99.7 rule to determine the percentage of students who scored:
    a. more than 106 points
    b. between 74 and 106 points
    c. less than 58 points
    d. between 74 and 122 points
    e. more than 122 points
    f. between 42 and 74 points
9. Your teacher informs you that your exam score was one standard deviation less than the mean. What percentile would this be?
10. Your teacher informs you that your exam score was exactly three standard deviations greater than the mean. What percentile would this be?

Answers

1.
    a. 68%
    b. 99.7%
    c. 100% – 99.7% = 0.3%
2.
    a. $64,000
    b. $88,000
    c. $64,000-$80,000; 68%
    d. $48,00-$96,000; 99.7%
    e. 50%
3.
    a. 50%
    b. 68%
    c. 95%
    d. (100% – 99.7%)/2 = 0.15%
    e. 95%/2 = 47.5% f) 99.7%/2 = 49.85% g) 100% – (34% + 50%) = 16%
4.
    a. 1.75 to 3.25 hours

b. 1 to 4 hours

c. more than 4.75 hours



5. a.

b. (100% – 99.7%)/2 = 0.15%; 30 people

c. 50%; 100 people

d. 50% – (68%/2) = 16%; 32 people

e. 34% + (95% ÷ 2) = 81.5%; 163 people

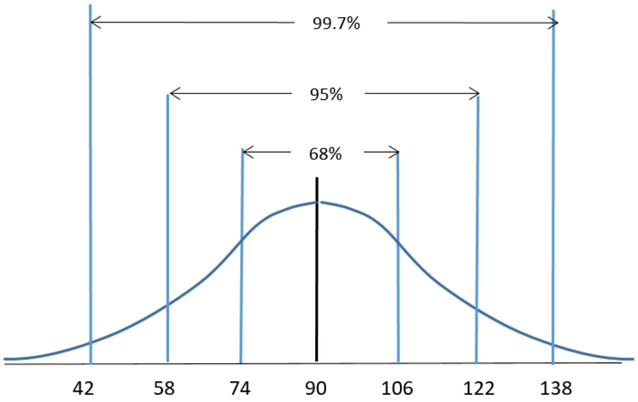
f. (99.7% ÷ 2) + 34% = 83.85%; ≈168 people

6. a. 50% so 115 runners

b. 36.6 so between 36 and 37

c. 109.25 so between 109 and 110

d. less than 90 minutes; at most one runner



7.

8. a. 16%

b. 68%

c. 2.5%

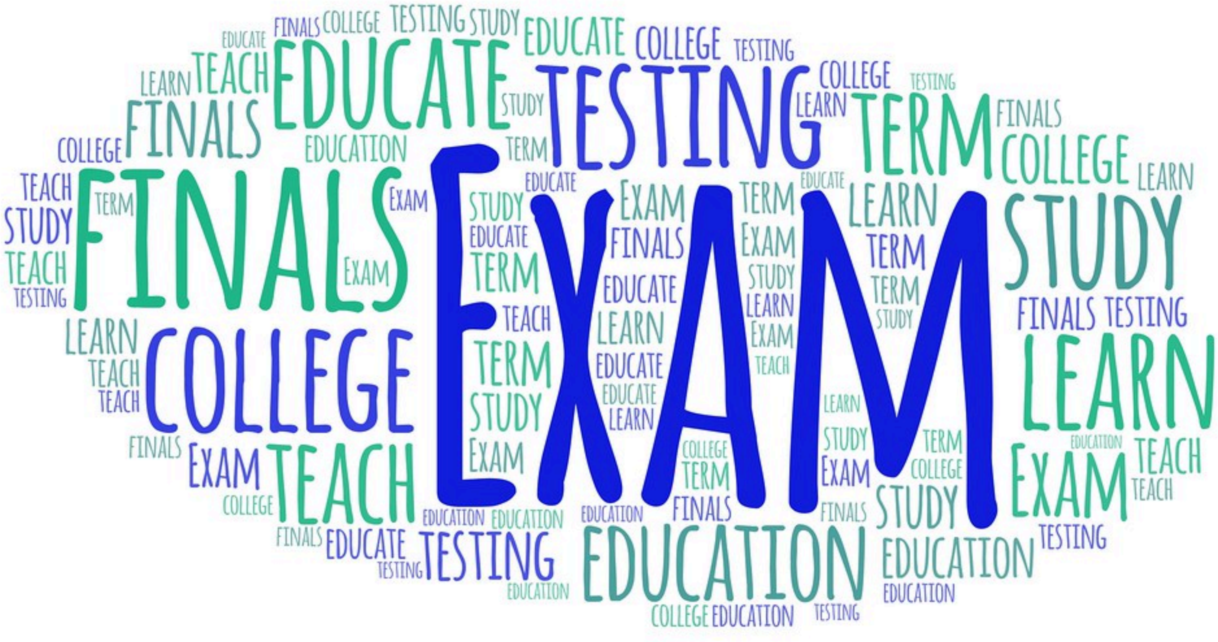
d. 81.5%

e. 2.5%

   f. 15.85%

9. 50% – 68%/2 = 16% of the data values lie below this so this is the 16th percentile
10. 99th percentile

8.4 Z-Scores and the Normal Curve



Learning Objectives

By the end of this section it is expected that you will be able to:

- Convert a data item to a z-score
- Solve applications using z-score tables

The Normal Curve

When a set of data values is normally distributed, the 68-95-99.7 Rule can be used to determine the percentage of values that lie one, two or three standard deviations from the mean. We will shift gears and explore how to determine where a specific data value lies in relation to all other values. As an example, a student who has written a college entrance exam may want to know where they placed in comparison to all other students. This section will explore how to determine this.

Consider the **normal curve** which is an idealized representation of a normally distributed population. The normal curve, also called a bell-shaped curve, is represented in Figure 1. The **area under the curve** represents 100% (or 1.00) of the data (or population) and the mean score is 0.

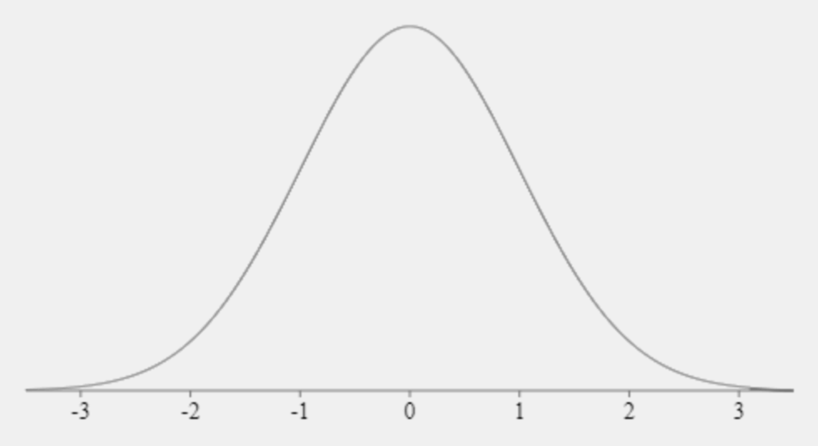


Fig. 1

We have seen that the standard deviation plays an important role in the normal distribution.

Refer to Figure 2 for the visual representation of the 68 – 95 – 99.7 Rule. For a normally distributed set of data:

- Approximately 68% (68.26%) of the data items fall within one standard deviation of the mean.
- Approximately 95% (95.44%) of the data items fall within two standard deviations of the mean.
- Approximately 99.7% of the data items fall within three standard deviations of the mean.

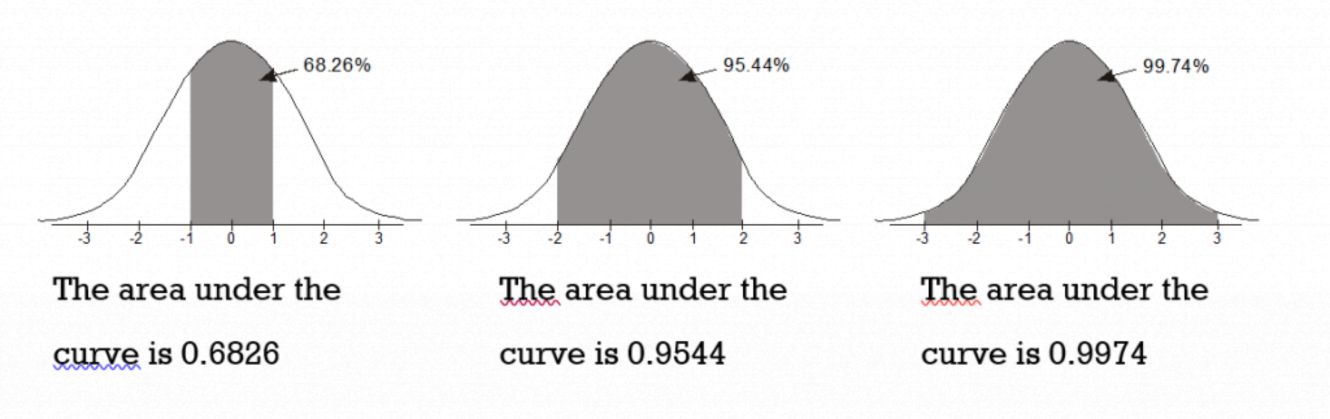


Fig. 2

Z–Scores

When a data set is normally distributed we can use a standardized score, called the **z-score**, to determine the **number of standard deviations** that a data value is from the mean.

Reconsider an example from the previous section. Acertain segment of the economy has a normally distributed salary, with a mean salary of $45,000 and a standard deviation of $4000. Refer to Figure 3.

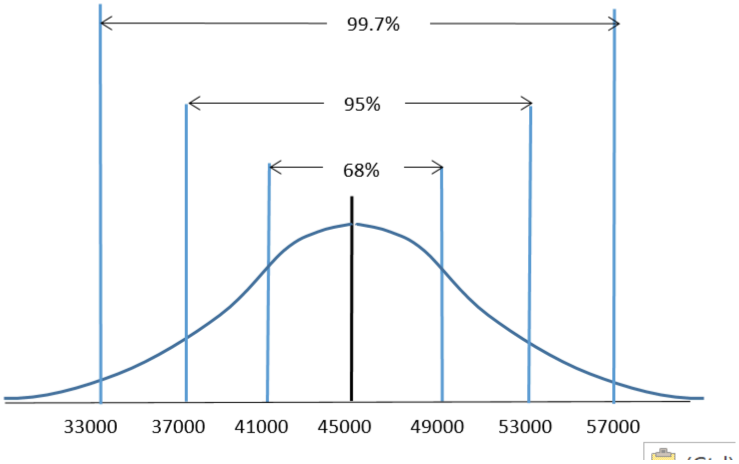


Fig. 3

With this information we are able to determine that a salary of $49,000 lies exactly one standard deviation above the mean since $45,000 + $4000 = $49, 000. In turn, using the 68-95-99.7 Rule we can determine that a salary of $49,000 is higher than 84% of the other salaries for this segment of the economy. The calculation would be 50% + (68%/2) = 84%. This calculation was possible since $49,000 was exactly one standard deviation away from the mean.

Consider a salary which does not lie exactly one, two or three standard deviations from the mean, such as $38,500. The calculation does not appear so straightforward but as it turns out we can use a z-score for situations such as this. A z-score converts a data value and standardizes it so that we are able to determine how many standard deviations a specific data value will lie above or below the mean.

Z-scores can be used in situations with a normal distribution. Consider a chemistry class with a set of test scores that is normally distributed. The average score is 76% and one student receives a score of 55%. Converting the 55% to a z-score will provide the student with a sense of where their score lies with respect to the rest of the class. We can also use z-scores to determine the percent of the data values that will lie between any two data values. Perhaps we wish to determine the percentage of students whose test scores lie between 70% to 85%. This can also be done using z-scores.

We have seen that when calculating standard deviation we must consider whether we are working with the entire population or a sample of the population. We must do the same when calculating a z-score.

Formulas for Finding Z-Scores

The z-score represents the number of standard deviations a data value is from the mean value.

The formula for z is:

$$z = \frac{\text{data value} - \text{mean}}{\text{standard deviation}}$$

For a **population** we calculate the z-score using the population mean μ and standard deviation σ. The data value is represented by x.

$$z = \frac{x - \mu}{\sigma}$$

For a representative **sample** of the population we calculate the z-score using the sample mean and standard deviation.:

$$z = \frac{x - \overline{x}}{s}$$

A z-score is similar to a percentile in that it is a **measure of position**. As a rule, z-scores above 2.0 (or below –2.0) are considered “unusual” values. According to the 68-95-99.7 Rule, in a normal population such scores would occur less than 5% of the time. Z-scores between -2.0 and 2.0 are considered “ordinary” values and these represent 95% of the values.

EXAMPLE 1

IQ scores are normally distributed. The mean IQ is 100 and the standard deviation is 15.

a) If Frank has an IQ of 127, find his z-score. b) Intrepret the meaning of this z-score. c) Using the 68-95-99.7 Rule, how does Frank’s IQ compare to the rest of the population?

Solution

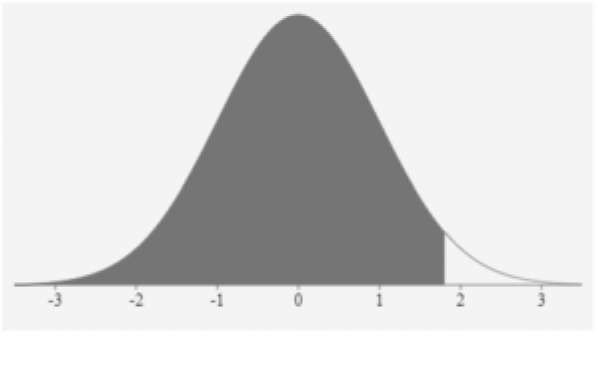
a) We will use the z-score for a population:

Here, $\mu = 100$, $\sigma = 15$, and $x = 127$.

$$z = \frac{127 - 100}{15} = 1.8$$

Frank’s z-score is 1.8.

b) For this z-score, the mean of 100 has been “standardized” to a value of 0 and the score of 127 has been standardized to a value of 1.8. This means that Frank’s IQ score is 1.8 (almost 2 standard deviations) higher than the average.



c) Considering the 68-95-99.7 Rule, Frank's score lies within 2 standard deviations of the mean. His score is certainly better than at least 84% of the population but does not rank in the top 2.5% of the population.

TRY IT 1

Consider a chemistry class and a set of test scores with an average of 76% and a standard deviation of 7%. A student receives a test score of 55%. a) Determine the student's z-score. b) Intrepret the meaning of this z-score. c) Using the 68-95-99.7 Rule, how does the student's test score compare to the rest of the class?

Show answer
a) z-score = -3 b) This z-score is exactly 3 standard deviations less than the mean score of 76%. c) This student scored better than only 0.15% of the class (or 99.85% of the class scored higher than this student).

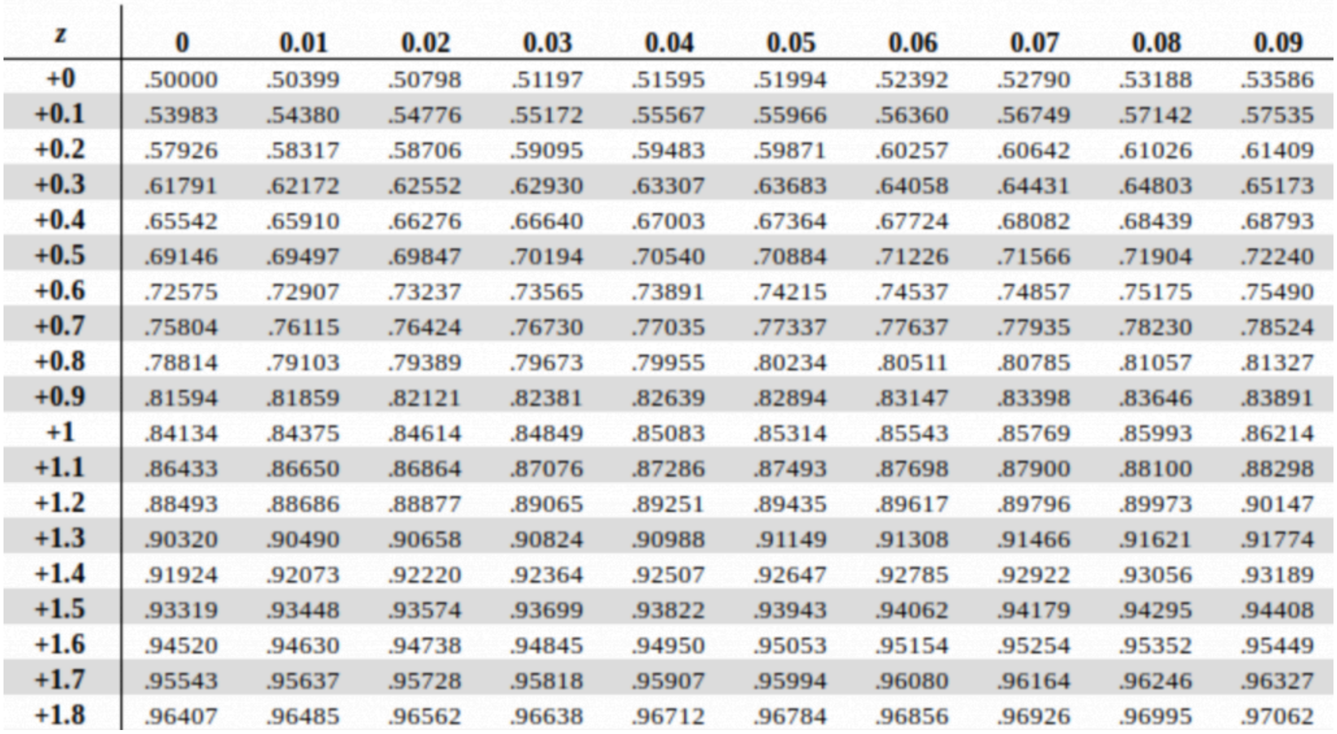
In Example 1 we were able to determine that Frank's score is better than at least 84% of the population but it does not rank in the top 2.5% of the population. This is a fairly broad conclusion. As it turns out we can be more specific if we use z-score tables.

Z-Score Tables

A **z-score table** allows us to determine, for a normal distribution, the percentage of data **values** that lie below (to the left) of a specific **z–score.** This in turn will enable us to determine the percentage of values that lie between or to the right of a given z-score.

We will use two different z-tables, one for **positive** z-scores and one for **negative** z-scores. These tables are available online (or refer to the Tables at the end of this chapter).

A portion of a **positive z-score table** is shown in Figure 4. We use the **positive** z-table when we have z-scores that are greater than 0 or lie to the right of the mean. The number in the z-table represents the area under the bell curve to the **left** of the z-score. The number is stated as a decimal fraction which can then be converted to a percentage by multiplying by 100. If for example the number in the table is 0.62552, then this would be interpreted as 62.552%.



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
+0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
+0.1	.53983	.54380	.54776	.55172	.55567	.55966	.56360	.56749	.57142	.57535
+0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
+0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
+0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
+0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
+0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
+0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
+0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
+0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
+1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
+1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
+1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
+1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91308	.91466	.91621	.91774
+1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
+1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
+1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
+1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
+1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062

Fig. 4

Consider a data value X that is found to have a standardized z-score of 1.42. A data value with a z-score of 1.42 will lie between 1 and 2 standard deviations above the mean (refer to Fig. 5). We can use a z-zcore table to determine the proportion of data vales that are less than (or geater than) this score.

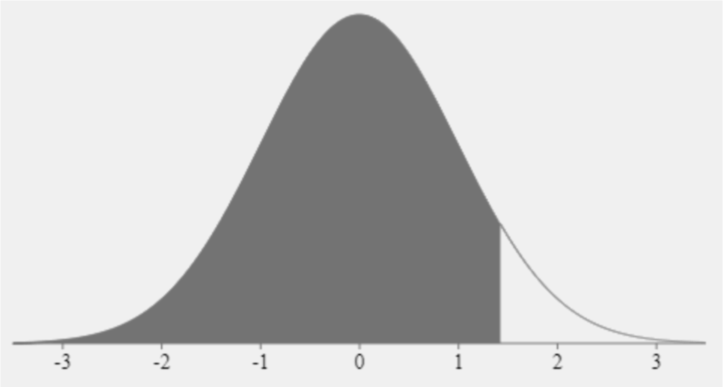


Fig. 5

This z-score value of 1.42 is positive so we refer to the positive z-score table in Figure 4. In the table go down the first column until you reach +1.4. The first column provides the z-score values to the nearest tenth (or one decimal place). To incorporate the digit that is in the second decimal place we must move through the body of the table until the heading of the column matches the digit in the hundredths place. In this example we move across the row for +1.4 until we reach the column headed with a 0.02. The corresponding number in the table is 0.92220 so for a z-score of 1.42 we can state that the area to the left of this score is 0.92220. Alternatively, 92.220% of all data values are less than this data value X. We are also able to conclude that 7.8% of the data values lie above this z-score value since 100% – 92.2% = 7.8%

Using only the z-score of 1.42 we were able to conclude that the data value X will lie between 1 and 2 standard deviations above the mean. By using the z-score table we are able to be more specific in stating that approximately 92% of the data values are less than X.

EXAMPLE 2

Consider a z-score of 0.18. a) Determine the area under the curve for a z-score of 0.18. b) Interpret what this z-score tells us.

Solution

a) This value is positive so we refer to the positive z-score table (a partial table is provided in Figure 4). In the table go down the first column until you reach +0.1. Then move across the row until you reach the column headed with a 0.08. This represents a z-score of 0.18. For a z-score of 0.18 the number in the table is 0.57142.

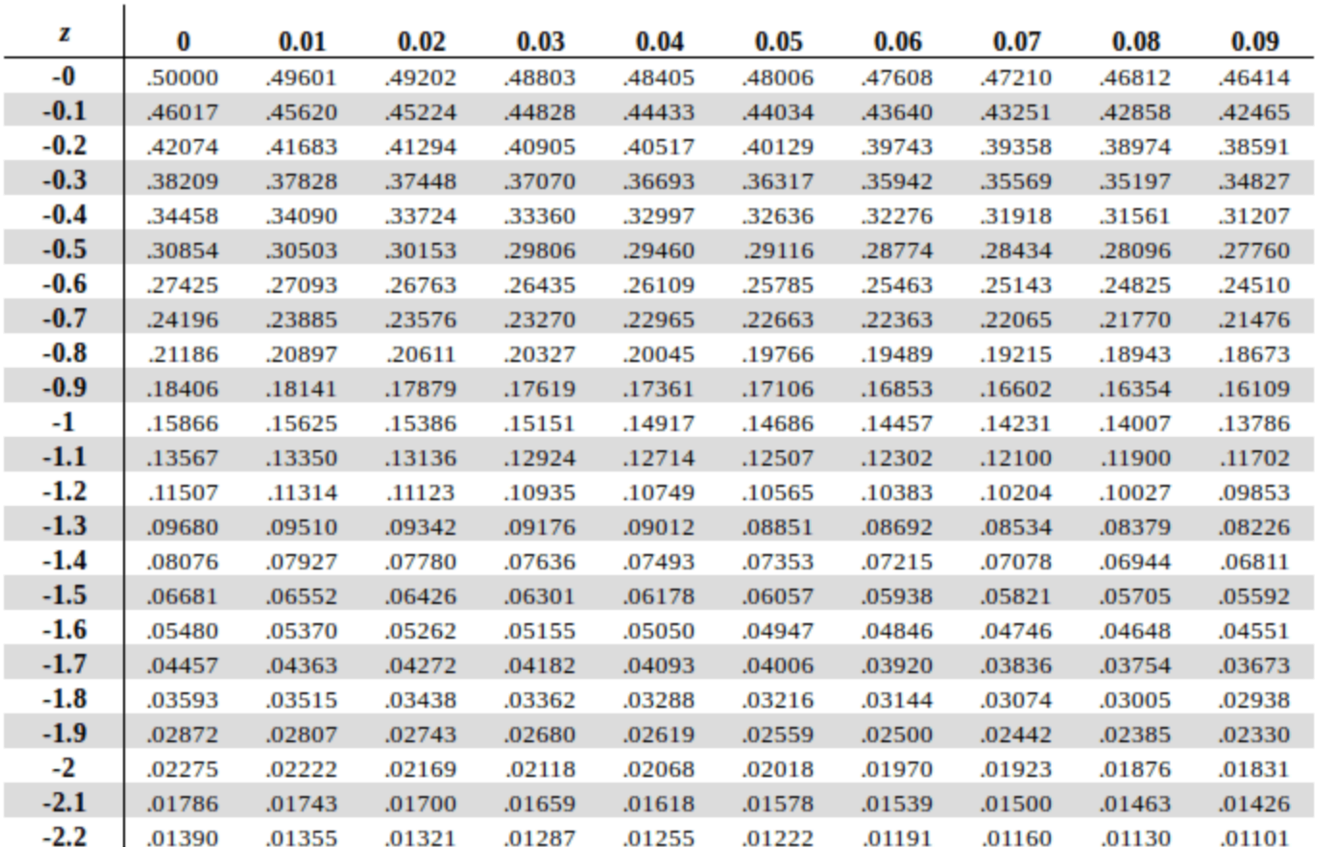
b) For a data value with a z-score of 0.18, approximately 57.1% of the data values will be below this and 42.9% will be above this data value.

TRY IT 2

Consider a z-score of 0.90. a) Determine the area under the curve for a z-score of 0.90. b) Interpret what this z-score tells us.

Show answer
a) area is 0.81594 b) Approximately 81.6% of data values are less than this value (or 18.4% are greater).

If the z-score is negative we use a **negative** z-table, a portion of which is illustrated in Figure 6. This table provides z-scores that are less than 0 or lie to the left of the mean. The number in the z-table represents the area under the bell curve to the **left** of the z-score.



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414
-0.1	.46017	.45620	.45224	.44828	.44433	.44034	.43640	.43251	.42858	.42465
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-1	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08692	.08534	.08379	.08226
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-2	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101

Fig. 6

Consider an observation that has a z-score of -1.17. Refer to the negative z-score table in Figure 6. In the table go down the first column until you reach -1.1. Then move across the row until you reach the column headed with a 0.07. For a z-score of -1.17 the number in the table is 0.12100 or 12.1%. This represents the portion of the data values that lie to the left of the z-score (refer to Figure 7). This means that 12.1% of the data values are less than this data value. Alternatively 87.9% (100% – 12.1%) of the data values are greater than this data observation.

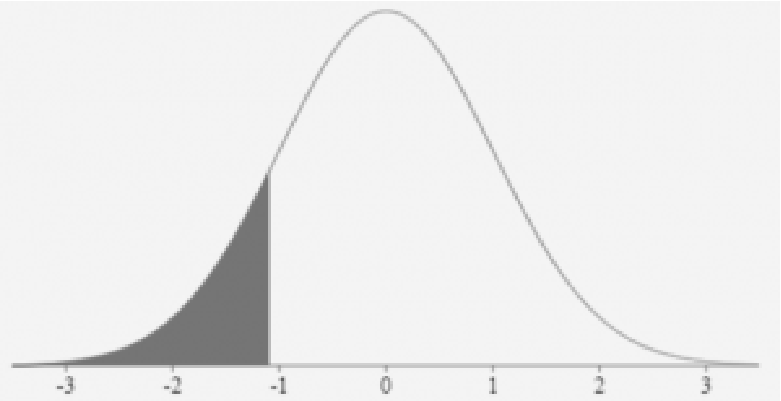


Fig. 7

Z-Scores and the 68-95-99.7 Rule

With a normal distribution half of the data values lie below the mean and half lie above the mean. If we calculate the z-score for a mean of 0 we will find that the z-score will also be 0. From the z-table we can determine that for a z-score of 0 the number is 0.5. This indicates that 50% of the data values lie below the mean and therefore 50% of the data values lie above the mean.

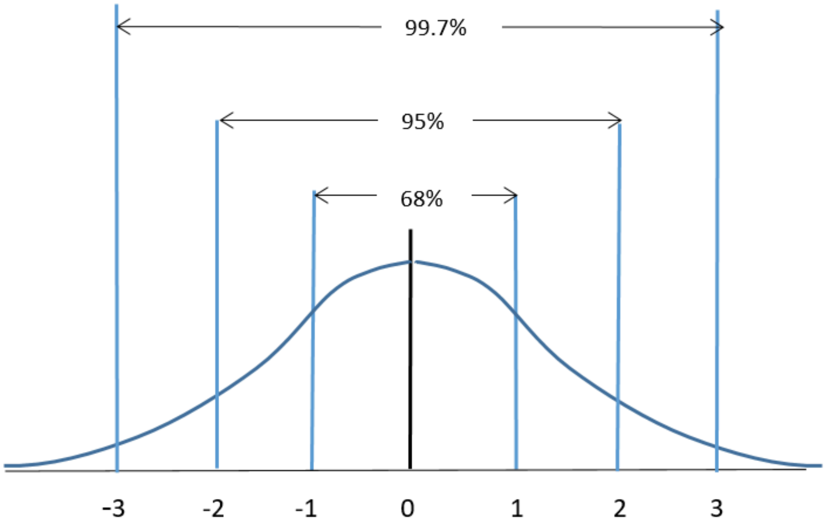


Fig. 8

Refer to the normal distribution that is illustrated in Figure 8. For standard deviations of 1, 2 or 3 we can use the 68-95-99.7 Rule to determine areas under the curve. We can also use z-score tables to do this.

EXAMPLE 3

a) A data value X is found to lie -1 standard deviation from the mean. Use the 68-95-99.7 Rule to determine the percentage of data values that are lower than this data value.

b) For a data value X with a z-score of -1, determine the percentage of data values that are lower than X.

Solution

a) Refer to Figure 8. If 68% of the data values lie between -1 and 1 standard deviations, then 100% – 68% = 32% lie on either side of -1 or 1 standard deviations from the mean. Dividing 32% in half we get 16%. Therefore 16% of all data values are lower than the data value X. (Note: There are other approaches to this calculation).

b) Refer to the negative z-score table in Figure 6. For a z-score of -1 the area will be 0.15866 or 15.866%. Rounded to the nearest whole number we determine that 16% of the data values are lower than this data value.

Note: The z-score table will provide more accurate results than the 68-95-99.7 Rule since the values in the 68-95-99.7 Rule are rounded off approximations.

TRY IT 3

a) A data value X is found to lie 3 standard deviations from the mean. Use the 68-95-99.7 Rule to determine the percentage of data values that are greater than this data value.

b) For a data value X with a z-score of 3, determine the percentage of data values that are greater than X.

Show answer
a) Refer to Figure 8. If 99.7% of the data values lie between -3 and 3 standard deviations, then 100% – 99.7% = 0.3% lie on either side of -3 or 3 standard deviations from the mean. Dividing 0.3% in half we get 0.15%. Therefore 0.15% of all data values are greater than the data value X.

b) Refer to a positive z-score table. For a z-score of 3 the area to the left will be 0.99865 or 99.865% so the area to the right (greater) will be 100% – 99.865% = 0.135% Rounded to the nearest 2 decimal places we determine that 0.14% of the data values are greater than this data value. Note that this slightly different than the answer obtained from using the 68-95-99.7 Rule due to rounding.

Areas Between Z-Scores

We can use z-score tables to determine the area between two z-scores. As an example, we can use the table to determine the area between the mean and a z-score of 1. A z-score of 1 lies one standard deviation to the right of the mean (as in Figure 9). Refer to the positive z-table in Figure 4. For a z-zcore of 1 the value in the table is 0.84134. This indicates that approximately 84% of the data values lie to the left of this z-score.

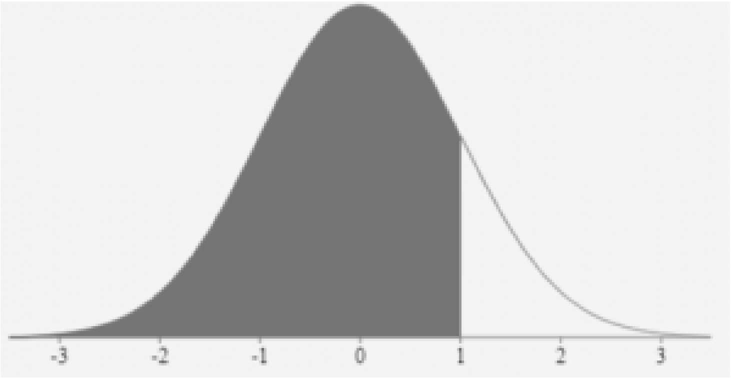


Fig. 9

From the z-score table we determine that for the z-score of 0 the area to the left is 0.5000. The area **between** the mean and one standard deviation would be as follows:

0.84134 – 0.500 = 0.3413 or 34.13%

Figure 10 illustrates the area between the mean and one standard deviation.

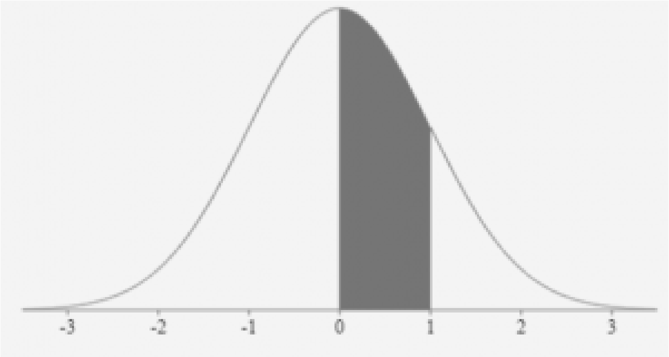


Fig. 10

Note that this is consistent with the 68-95-99.7 Rule. It states that approximately 68% of the data will lie within one standard deviation on either side of the mean. Half this amount, or 34%, will lie between the mean and a standard deviation of one.

The 68-95-99.7 Rule is useful when data values lie exactly 1, 2 or 3 standard deviations from the mean. Z-score tables are useful for data values that have z-scores that are not exactly 1, 2 or 3 standard deviations from the mean.

EXAMPLE 4

Given a normal distribution, use the z-score tables to find the area for each of the following z-scores (rounded to the nearest tenth of a percent):

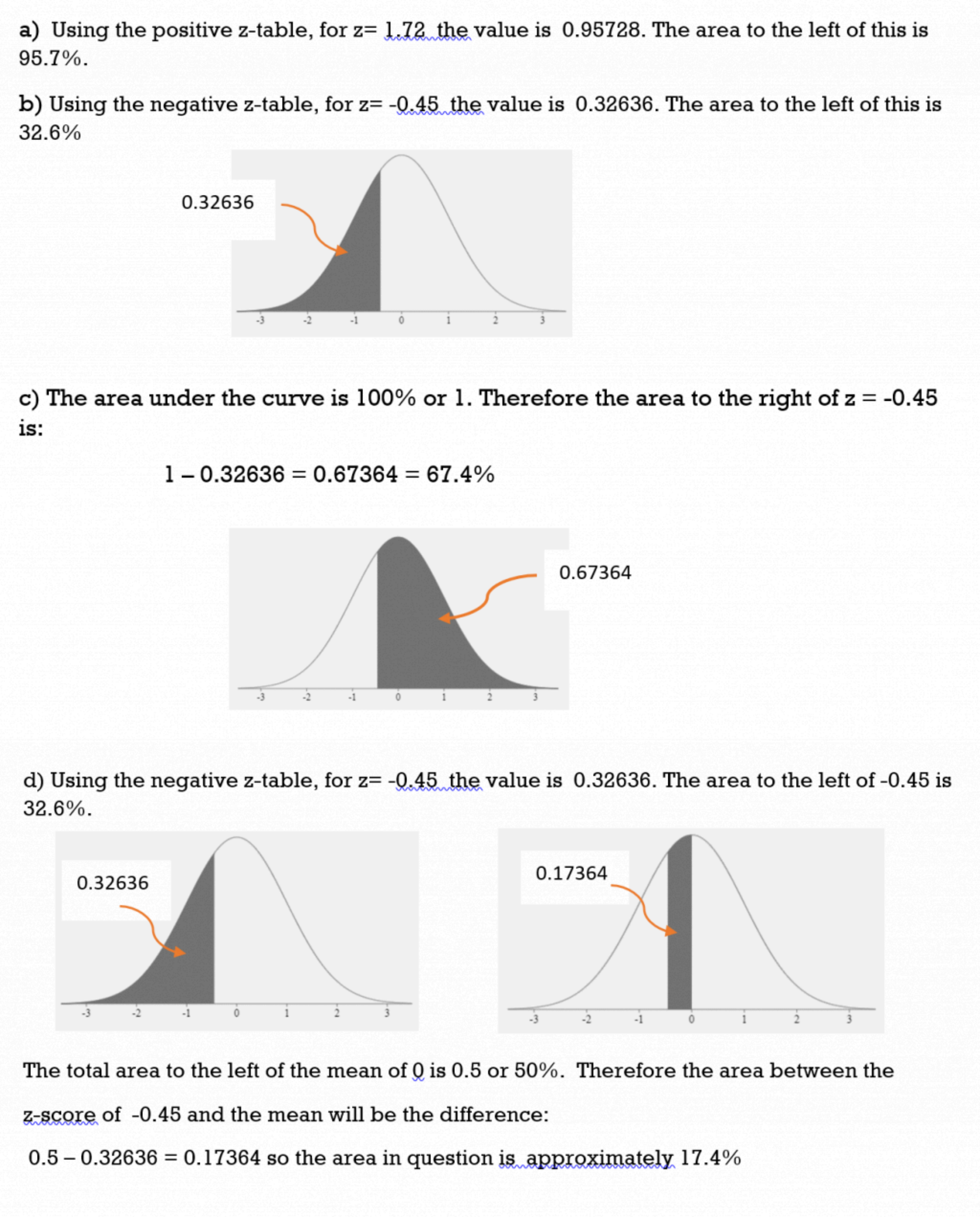
a) to the left of z = 1.72

b) to the left of z = -0.45

c) to the right of z = -0.45

d) between -0.45 and the mean

Solution



a) Using the positive z-table, for z= 1.72 the value is 0.95728. The area to the left of this is 95.7%.

b) Using the negative z-table, for z= -0.45 the value is 0.32636. The area to the left of this is 32.6%

c) The area under the curve is 100% or 1. Therefore the area to the right of z = -0.45 is:

1 – 0.32636 = 0.67364 = 67.4%

d) Using the negative z-table, for z= -0.45 the value is 0.32636. The area to the left of -0.45 is 32.6%.

The total area to the left of the mean of 0 is 0.5 or 50%. Therefore the area between the z-score of -0.45 and the mean will be the difference:

0.5 – 0.32636 = 0.17364 so the area in question is approximately 17.4%

TRY IT 4

Given a normal distribution, find the area for each of the following z-scores (rounded to the nearest tenth of a percent):

a) to the left of z = 0.85

b) to the right of z = 0.85

c) between the mean and z = 0.85

Show answer

a) 0.8023 = 80.2%

b) 0.1977 = 19.8%

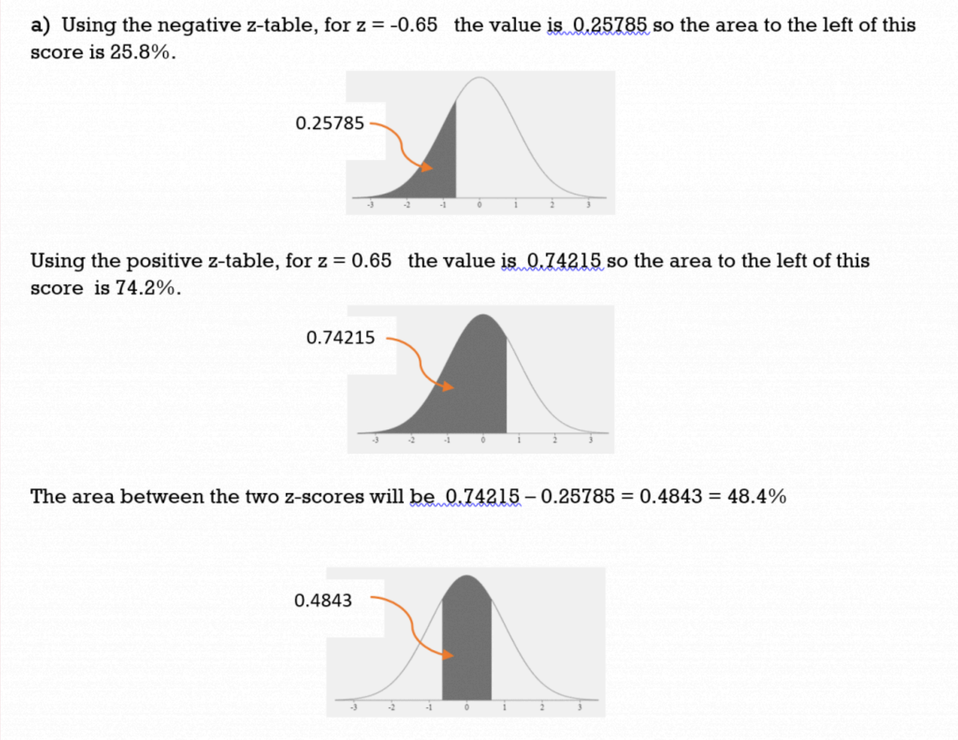
c) 0.8023 – 0.5 = 0.3023 = 30.2%

EXAMPLE 5

Given a normal distribution, find the area for each of the following z-scores (rounded to the nearest tenth of a percent):

a) between z = -0.65 and z = 0.65
b) between z = -0.65 and z = 2.8
c) between z = 1.44 and z = 2.8

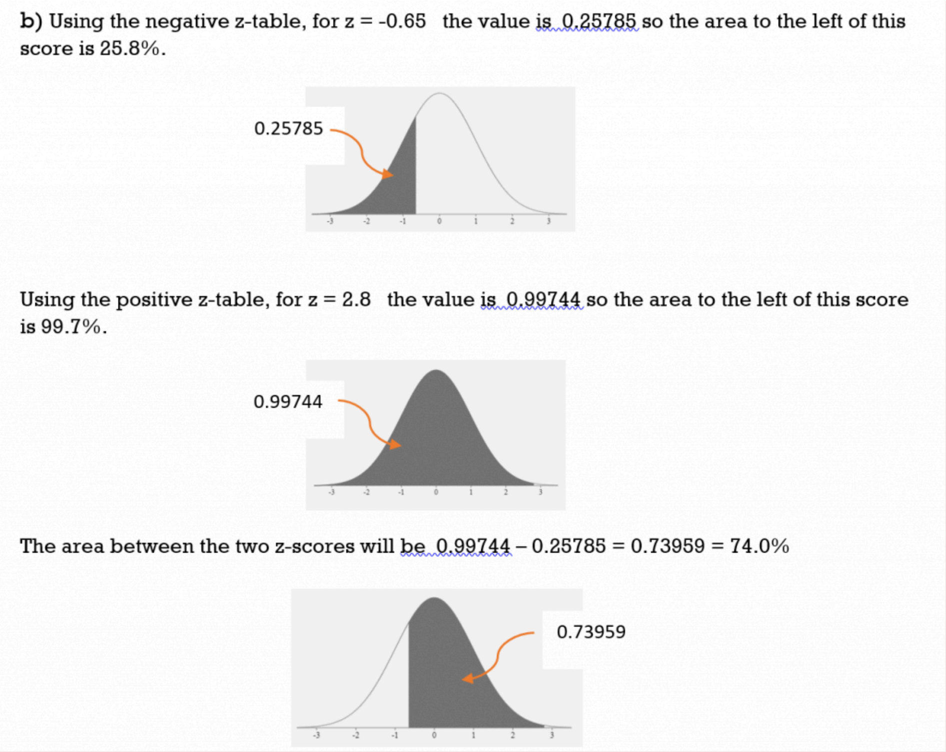
Solution



a) Using the negative z-table, for z = -0.65 the value is 0.25785 so the area to the left of this score is 25.8%.

Using the positive z-table, for z = 0.65 the value is 0.74215 so the area to the left of this score is 74.2%.

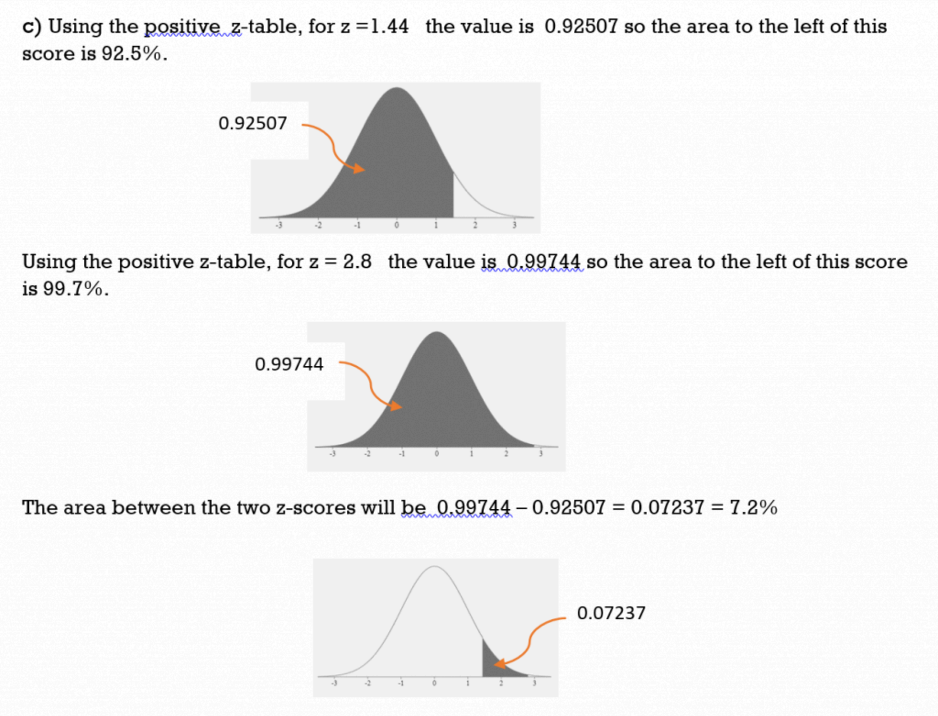
The area between the two z-scores will be 0.74215 – 0.25785 = 0.4843 = 48.4%



b) Using the negative z-table, for z = -0.65 the value is 0.25785 so the area to the left of this score is 25.8%.

Using the positive z-table, for z = 2.8 the value is 0.99744 so the area to the left of this score is 99.7%.

The area between the two z-scores will be 0.99744 – 0.25785 = 0.73959 = 74.0%



c) Using the positive z-table, for z =1.44 the value is 0.92507 so the area to the left of this score is 92.5%.

Using the positive z-table, for z = 2.8 the value is 0.99744 so the area to the left of this score is 99.7%.

The area between the two z-scores will be 0.99744 – 0.92507 = 0.07237 = 7.2%

TRY IT 5

Given a normal distribution, find the area for each of the following z-scores (rounded to the nearest tenth of a percent):

a) between z = 0 and z = -0.73

b) between z = -0.73 and z = 1.95

c) between z = -2.12 and z = -0.73

Show answer

a) 0.2673 = 26.7%

b) 0.7417 = 74.17%

c) 0.2157 = 21.6%

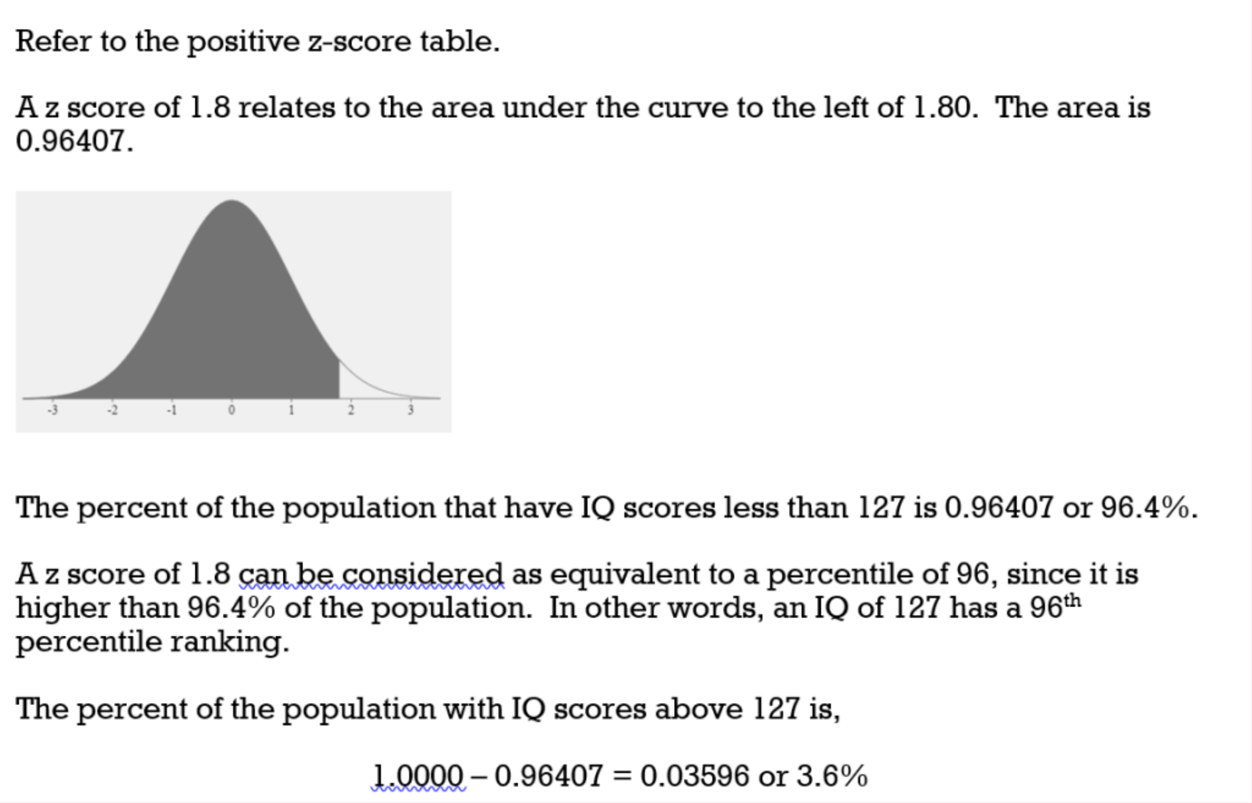
Applications Using Z-Scores

With populations or samples that are normally distributed, z-scores can be used to determine how data values compare (are positioned) with respect to other data values.

EXAMPLE 6

Frank has an IQ of 127, or a z score of 1.8. What percent of the population have IQ scores less than 127 and what percent have IQ scores higher than 127?

Solution



Refer to the positive z-score table.

A z score of 1.8 relates to the area under the curve to the left of 1.80. The area is 0.96407.

The percent of the population that have IQ scores less than 127 is 0.96407 or 96.4%.

A z score of 1.8 can be considered as equivalent to a percentile of 96, since it is higher than 96.4% of the population. In other words, an IQ of 127 has a 96th percentile ranking.

The percent of the population with IQ scores above 127 is,

$$1.0000 - 0.96407 = 0.03596 \text{ or } 3.6\%$$

Refer back to Example 1 where the 68-95-99.7 Rule was used to analyze Frank’s IQ score. From the Rule we were able to conclude that Frank’s IQ was better than at least 84% of the other scores and that his score did not rank in the top 2.5%. Comparing the two methods, the z-score provides us with a much more accurate analysis.

TRY IT 6

Consider a chemistry class and a set of test scores with an average of 76% and a standard deviation of 7%. A student receives a test score of 55% which yields a z-score of -3 (refer to TRY IT 1). a) Use a z-score table to determine what percent of the class had test scores less than 55% and what percent had test scores greater than 55%. b) How does this compare with the answer to TRY IT 1?

Show answer

a) From the table the area to the left of a z-score of -3 is 0.00135 therefore 0.135% of the class had test scores less than 55% and 99.865% of the class scored higher than 55%. b) Using the Rule in TRY IT1 the results were slightly different due to rounding: 0.15% of the students scored less than 55% and 99.85% scored higher than 55%.

For applications involving populations or samples that are normally distributed, we can calculate z-scores if we know the mean and the standard deviation.

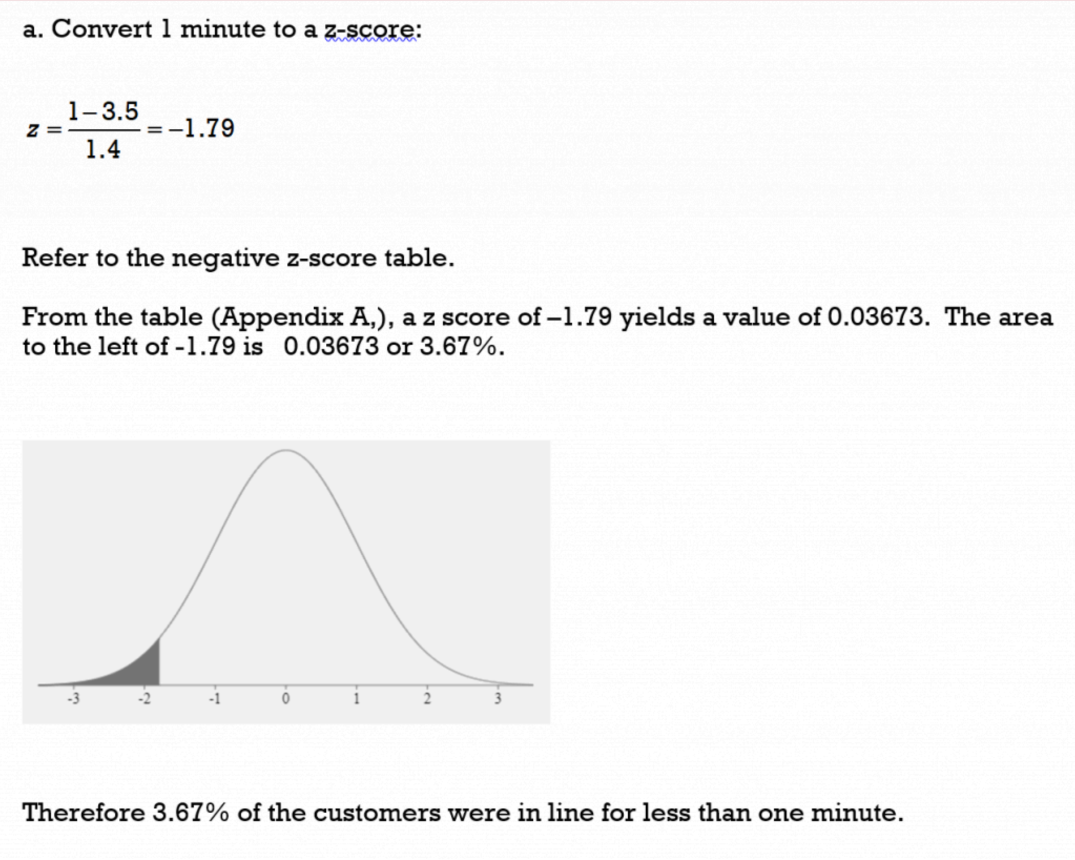
EXAMPLE 7

The waiting-in-line time at a certain grocery store is normally distributed with a mean of 3.5 minutes and a standard deviation of 1.4 minutes.

a) What percent of the customers wait in line less than one minute?

b) What percent of the customers wait in line more than 5 minutes?

Solution



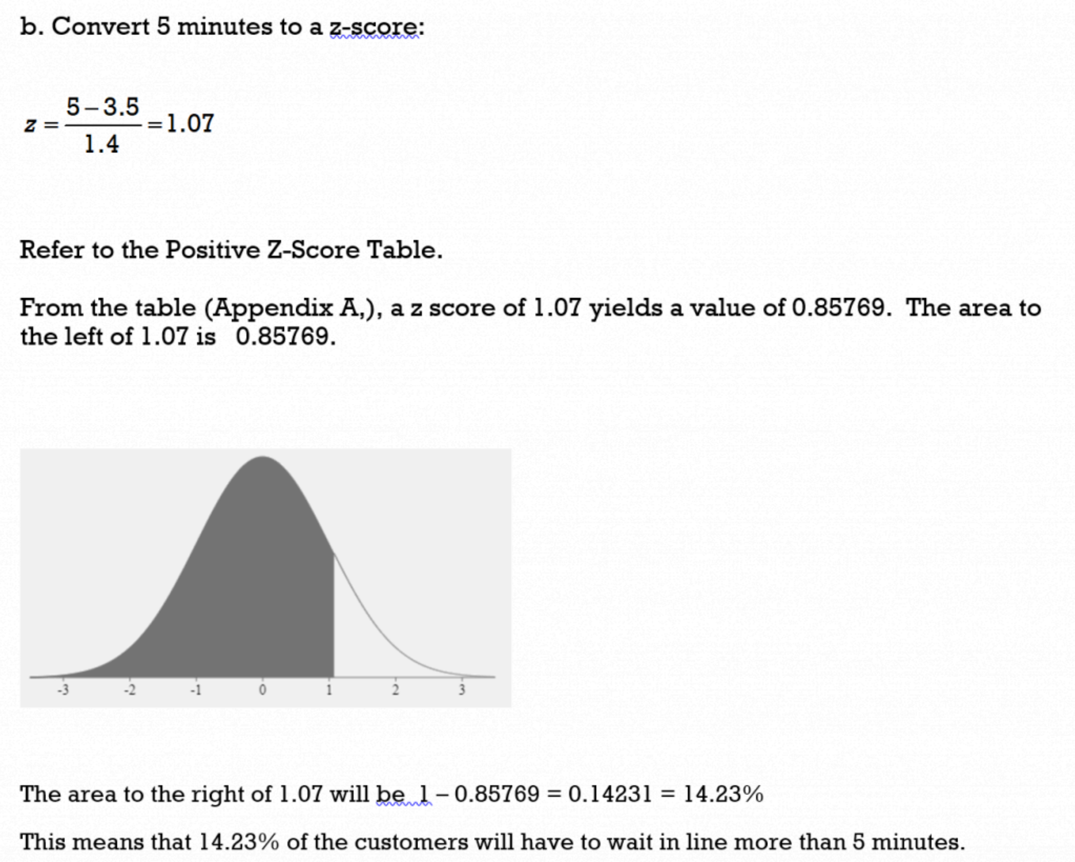
a. Convert 1 minute to a z-score:

$$z = \frac{1-3.5}{1.4} = -1.79$$

Refer to the negative z-score table.

From the table (Appendix A,), a z score of –1.79 yields a value of 0.03673. The area to the left of -1.79 is 0.03673 or 3.67%.

Therefore 3.67% of the customers were in line for less than one minute.



b. Convert 5 minutes to a z-score:

$$z = \frac{5-3.5}{1.4} = 1.07$$

Refer to the Positive Z-Score Table.

From the table (Appendix A,), a z score of 1.07 yields a value of 0.85769. The area to the left of 1.07 is 0.85769.

The area to the right of 1.07 will be 1 – 0.85769 = 0.14231 = 14.23%

This means that 14.23% of the customers will have to wait in line more than 5 minutes.

TRY IT 7

The waiting-in-line time to be seated at a popular retaurant during primetime hours is normally distributed with a mean of 24 minutes and a standard deviation of 11 minutes.

a) What percent of the customers wait in line less than twenty minutes?

b) What percent of the customers wait in line more than forty-five minutes?

Show answer
a) 36% of customers wait less than 20 minutes b) 3% of customers wait more than 45 minutes

EXAMPLE 8

All first year psychology students wrote an exam that had 92 questions. Each question was worth 1 point for a total possible 92 points. The marks were normally distributed with a mean of score of 58 points and a standard deviation of 11 points.

a) Determine the z-score for a mark of 45 points. What percent of the scores were less than 45 points?

b) Determine the z-score for a mark of 84 points. What percent of the scores were greater than 84 points?

c) What percent of the scores were between 76 and 88 points?

Solution:

a) z = -1.1818 and area to the left is 0.11900 so 11.9% of the scores would be less than 45 points

b) z = 2.3636 and the area to the left is 0.99086 so the area to the right is 1- 0.99534 = 0.00914. Therefore 0.9% (almost 1%) would score higher than 84 points

c) for 76 points z = 1.64 and the area to the left is 0.9495. For 88 points z = 2.73 and the area to the left is 0.99683. The difference is 0.99683 – 0.9495 = 0.04733. This means that 4.7% of the students would score between 76 and 88 points.

TRY IT 8

An entrance exam was given to a cohort of students. The mean score was 1000 points with a standard deviation of 150 points.

a) What percent of the students scored less than 820 points?

b) What percent of the students scored more than 1330 points?

c) What percent of the students scored between 950 and 1100 points?

Show answer
a) 11.5% scored less than 820 points b) 1.4% scored more than 1330 points

c) 37.8% scored between 950 and 1100 points

Key Concepts

- For a population we calculate the z-score for a data value x using the population mean μ and standard deviation σ.

$$z = \frac{x - \mu}{\sigma}$$

- For a representative sample of the population we calculate the z-score for a data value x using the sample mean and standard deviation. :

$$z = \frac{x - \overline{x}}{s}$$

- A **z-score table** allows us to determine, for a normal distribution, the percentage of data **values** that lie below (to the left) of a specific **z–score.** This in turn will enable us to determine the percentage of values that lie between or to the right of a given z-score.
- When a z-score is **negative** (lies to the left of the mean) we use a negative z-score table. When a z-score is **positive** (lies to the right of the mean) we use a positive z-score table.

Glossary

Z-score

is a standardized score that has been converted from a data value. The z-score indicates how many standard deviations away from the mean a data value lies.

8.4 Exercise Set

1. Heights of adult males are normally distributed. The mean height of an adult male is 178 cm with a standard deviation of 10 cm.
    a. If Matt is 188 cm tall, find his z-score.
    b. Intrepret the meaning of this z-score.
    c. Using the 68-95-99.7 Rule, how does Matt's height compare to the rest of the population?
2. Heights of adult males are normally distributed. The mean height of an adult male is 178 cm with a standard deviation of 10 cm.
    a. If Keegan is 158 cm tall, find his z-score.
    b. Interpret the meaning of this z-score.
    c. Using the 68-95-99.7 Rule, how does Keegan's height compare to the rest of the population?
3.
    a. A data value X is found to lie -3 standard deviations from the mean. Use the 68-95-99.7 Rule to determine the percentage of data values that are less than this data value.
    b. For a data value X with a z-score of -3, use a negative z-score table to determine the percentage of data values that are lower than X.
4.
    a. A data value X is found to lie 2 standard deviations from the mean. Use the 68-95-99.7 Rule to determine the percentage of data values that are lower than this data value.
    b. For a data value X with a z-score of 2, use a positive z-score table to determine the percentage of data values that are lower than X.

5. Find the area under the normal curve for the followed z scores. Give the answers as both a decimal fraction (as a stated in the z-table) and as a percentage (rounded to the nearest tenth).
    a. less than z = -0.86 ______________________
    b. greater than z = -0.86 ______________________
    c. less than z = 1.34 ______________________
    d. greater than z = 1.34 ______________________
    e. greater than z = -2.88 ______________________
6. Find the area under the normal curve for the following z scores. Give the answers as both a decimal fraction (as stated in the z-table) and as a percentage (rounded to the nearest tenth).
    a. between z = 0 and z = 0.47 ______________________
    b. between z = -0.3 and z = 0 ______________________
    c. between z = -0.3 and z = 0.47 ______________________
    d. between z = -2.24 and z = -0.55 ______________________
    e. between z = 1.46 and z = 2.37 ______________________
    f. between z = -1.5 and z = 1.5 ______________________
7. The average resting heartrate for a normally distributed population of men was found to be 62 beats per minute with a standard deviation of 11 beats per minutes.
    a. What percent of men have resting heartrates under 70 beats per minute?
    b. What percent of men have resting heartrates over 70 beats per minute?
    c. What percent of men have resting heartrates between 40 and 80 beats per minute?
8. In a group of normally distributed women, the average height is 5 feet 4 inches (64 inches) with a standard deviation of 2.8 inches. (1 foot = 12 inches)
    a. What percent of women are shorter than 5 feet?
    b. What percentage of woman are taller than 6 feet ?
    c. What percent of the women are between 5 feet and 6 feet ?
9. A survey of college students enrolled in technology programs indicated that they spend an average of 29 hours a week outside of class time studying for their courses. The data was normally distributed with a standard deviation of 9 hours per week.
    a. What percent of the students spend more than 40 hours per week studying?
    b. What percent spend fewer than 10 hours per week studying?
    c. What percent spend between 20 and 50 hours per week studying?
10. The number of toy cars assembled each day by a worker is normally distributed with a mean of 270 cars and a standard deviation of 16 cars.
    a. What percentage of workers assemble less than 240 cars per day?
    b. What percentage of workers assemble more than 265 cars per day?

   c. Workers are given a bonus every time they assemble more than 310 toy cars in one eight hour day. What percent of the workers receive a bonus each day?

11. A radar unit measures the speed of passing cars on a toll highway where the speed limit is 120/km/hour. The speed of the cars is normally distributed with a mean speed of 114 km/h and a standard deviation of 9.8 km/h.
    a. What percent of the cars are travelling at less than 100 km/h?
    b. What percent of the vehicles are exceeding the speed limit?
    c. What percent of the vehicles are travelling between 115 km/h and 125 km/h?
12. The lengths of cell phone calls in a particular city are normally distributed with a mean time of 8.2 minutes and a standard deviation of 2.6 minutes.
    a. What percent of phone calls are less than 10 minutes?
    b. What percent of phone calls are greater than 5 minutes?
    c. What percent of phone calls are between 7 and 12 minutes?

Answers

1.
    a. z-score = 1
    b. This height is one standard deviation greater than the mean
    c. This height is greater than 84% of the population
2.
    a. z-score = -2
    b. This height is two standard deviations less than the mean
    c. Keegan’s height is greater than approximately 2.5% of the population
3.
    a. From the Rule approximately 1.5%
    b. From the table 1.4%
4.
    a. From the Rule approximately 97.5%
    b. From the table 97.7%
5.
    a. 0.19489 = 19.5%
    b. 0.80511 = 80.5%
    c. 0.90988 = 91.0%
    d. 0.09012 = 9.0%
    e. 0.99801 = 99.8%
6.
    a. 0.18082 = 18.1%
    b. 0.11791 = 11.8%
    c. 0.29873 = 27.9%
    d. 0.06326 = 6.3%

   e. 0.86638 = 86.6%

7.
   a. 76.7%
   b. 23.3%
   c. 92.7%
8.
   a. 7.6%
   b. 0.2%
   c. 92.2%
9.
   a. 11.1%
   b. 1.7%
   c. 83.1%
10.
   a. 3%
   b. 62.2%
   c. 0.6%
11.
   a. 7.7%
   b. 27.1%
   c. 32.9%
12.
   a. z = 0.6923 and area to the left is 0.75490 so 75.5% of the calls would be less than 10 minutes.
   b. z = -1.23 and area to the left is 0.10935 so area to the right is 1- 0.10935 = 0.89065 Therefore 89.1% of the calls would be more than 10 minutes.
   c. 7 minutes has z = -0.4615 and the area to the left is 0.32276 and for 12 minutes z = 1.4615 and the area to the left is 0.92785. The difference is 0.92785 – 0.32276 = 0.60509. This means that 60.5% of the calls would be between 7 and 12 minutes.

Attribution

1. Figure 6 is from https://www.ztable.net/.
2. Some of the content for this chapter is from “Unit 9: Mortgages”, “Unit 10: Interest rates on loans”, and “Review Questions” in Financial Mathematics by Paul Grinder, Velma McKay, Kim Moshenko, and Ada Sarsiat, which is under a CC BY 4.0 Licence.. Adapted by Kim Moshenko. See the Copyright page for more information.

9 Financial Mathematics

9.1 Simple Interest

Learning Objectives

By the end of this section it is expected that you will be able to:

- Determine the simple interest earned on an investment or charged on a loan
- Determine the principal amount, the interest rate, or the time for applications involving simple interest
- Determine the maturity value of a loan that involves simple interest

Interest

Some people keep money at home in an easily accessible location, perhaps a piggy bank, a safe or locked box, or perhaps even a mattress. Although this provides instant access to funds it does not provide any return or earnings on this money. For that reason, most people hold their money in accounts or investments that provide some form of return or earning power.

Interest is the price paid for the use of money. If you borrow money from another person or a lending institution, eventually you must pay back this amount plus the interest owing. When you deposit money in a bank, you are lending them money and after some time they will pay you interest on the money you lent them.

The amount of interest you will owe or receive is determined by the **principal,** the **interest rate,** and the **time** (the length of the loan). The amount of money that you lend or borrow is called the **principal**. The **length** of the loan can range between a few days to several years. The **interest rate** is stated as an annual percentage. It may be **simple** interest or **compound** interest. With **simple interest** the interest is calculated only **once** during the entire time period of the loan or deposit. Simple interest is calculated solely on the principal investment or loan. With **compound** interest the interest is calculated **more than once** during the time period of the loan. It will be calculated on the principal as well as the accumulated interest This section will focus on simple interest and in the next section we will consider compound interest.

Simple Interest

Simple interest is calculated by finding the product of the principal (*P*), the rate (*r*), and the time (*t*).

Simple Interest

The **simple interest formula** is $I = Prt$ where

I = interest earned

r = annual interest rate (stated as a decimal)

P = principal

t = time (in **years**)

Interest rates are quoted for periods of **one** year and when used in a formula must be converted to a decimal fraction. The time must be expressed in the same unit of time as the interest rate so time must be stated in years or portions of a year. If you deposit money in a savings account earning 3% interest then the annual interest rate is 3% per year.

EXAMPLE 1

Jo borrows $2000 at an interest rate of 5% per year. How much interest will Jo owe after one year?

Solution

Identify the *P*, *r*, and *t*. $P = \$2000$ $r = 5\% = 0.05$ $t = 1$ year

Here,	$P = \$2000$	$I = Prt$	
	$r = 5\%\, or\, 0.05$	$I = 2000\,(5\%)\,(1)$	Replace *P*, *r* and *t* with their values
	$t = 1\, year$	$I = 2000\,(0.05)\,(1)$	Change 5% to its decimal equivalent, 0.05
		$I = 100$	

Start with the formula I = *Prt*

I = 2000 (5%) (1) Replace *P*, *r*, and *t* with their values

I = 2000 (0.05) (1) Change 5% to its decimal equivalent, 0.05

I = 100

Jo will pay **$100** in interest.

TRY IT 1

Terri borrowed $3200 at an interest rate of 4.75%. How much interest will Terri owe on the loan at the end of one year?

Show answer

$$I = Prt = \$3200\,(0.0475)\,(1) = \$55$$

Terri will owe $152 after one year.

Note that the time *t* is expressed in terms of years. When the time period is not exactly one year, the value for *t* will be the fraction of the year during which interest is earned.

If the investment is made for 3 months, then *t* = 3 months/12 months = 0.25 years.

If the investment is made for 35 days then t = 35 days/365 days = 7/73 years.

EXAMPLE 2

a) If an investment is made for a period of 145 days, what portion of the year does this represent?

b) If an investment is made for a period of 48 weeks, what portion of the year does this represent?

c) If an investment is made for a period of 10 months, what portion of the year does this represent?

Solution:

a) 145 ÷ 365= 29/73 years

b) 48 ÷ 52 = 12/13 years

c) 10 ÷ 12 = 5/6 years

TRY IT 2

a) If an investment is made for a period of 220 days, what portion of the year does this represent?

b) If an investment is made for a period of 32 weeks, what portion of the year does this represent?

c) If an investment is made for a period of 2 months, what portion of the year does this represent?

Show answer

a) 220 ÷ 365= 44/73 year

b) 32 ÷ 52 = 8/13 year

c) 2 ÷ 12 = 1/6 year

EXAMPLE 3

Determine the interest that will be earned on a deposit of $1350 at 2.8% over:

a) 7 months

b) 25 days

Solution:

a)

$I = Prt$	$P = \$1350$ $r = 2.8\% = 0.028$ $t = 7/12 years$
$I = (1350)\,(0.028)\,(7/12)$	
$= \$22.05$	

Interest of $22.05 over 7 months

b)

$I = Prt$	$P = \$1350$ $r = 0.028$ $t = \frac{25}{365}$
$I = (1350)\,(0.028)\left(\frac{25}{365}\right)$	
$=2.59$	

Interest of $2.59 over 25 days. Note that the answer is rounded to the nearest two decimal places or to the nearest cent.

TRY IT 3

Determine the interest that will be earned on a deposit of $2200 at 4.52% over:

Determine the interest earned after a) 1 month b) 300 days

Show answer

a) $8.29 b) $81.73

EXAMPLE 4

Determine the interest that will be earned on a deposit of $4200 at 4.65% over:

a) $1\frac{1}{2}$ years

b) 5 weeks

Solution

a)

$I = Prt$	$P = \$4200$ $r = 4.65\% = 0.0465$ $t = 1.5 years$
$I = (4200)\,(0.0465)\,(1.5)$	
$= \$292.95$	

Interest of $292.95 over $1\frac{1}{2}$years

b)

$I = Prt$	$P = \$4200$ $r = 0.0465$ $t = \frac{5}{52}$
$I = (4200)\,(0.0465)\left(\frac{5}{52}\right)$	
$= 18.78$	

Interest of $18.78 over 5 weeks. Note that the answer is rounded to the nearest two decimal places or to the nearest cent.

TRY IT 4

Max deposited $1500 in a savings account at an interest rate of 3.28%.

Determine the interest earned after i) 3 months ii) 65 days iii) two years.

Show answer

$$I = Prt = \$1500\,(0.0328)\left(\frac{3}{12}\right) = \$12.30$$
$$I = Prt = \$1500\,(0.0328)\left(\frac{65}{365}\right) = \$8.76$$
$$I = Prt = \$1500\,(0.0328)\,(2) = \$98.40$$

Maturity Value

The total amount of money due at the end of a loan period is called the **maturity value** of the loan. It is the amount to be paid on the due date of a loan or the amount to be paid to an investor at the end of the period for which an investment has been made.

Maturity Value

The **Maturity Value** (MV) of a loan is the sum of the principal P plus the interest I.

MV = P + I

In Example 1, Jo borrowed $2000 at an interest rate of 5%. At the end of one year Jo owed $100 in interest.

The maturity value of the loan is MV = P + I where P = $2000 and I = $100.

MV = \$2000 + \$100 = \$2100

The maturity value of the loan is \$2100. At the end of the year Jo will be expected to pay back \$2100.

EXAMPLE 5

Linda lends Ed \$500. Ed says he will pay her back in 60 days at 9% simple interest. How much interest should Linda receive? How much must Ed pay Linda altogether?

Solution

$P = \$500$	$I = Prt$	
$r = 9\% \; or \; 0.09$	$I = 500\,(0.09)\left(\frac{60}{365}\right)$	Replace *P*, *r* and *t* with their values
$t = 60\, days = \frac{60}{365} years$	$I = 7.39726$	Multiply
	$I = \$7.40$	Round to the nearest cent

MV = P + I = \$500 + \$7.40 = \$507.40

Linda should receive \$7.40 in interest. At the end of 60 days Ed will owe Linda \$507.40.

TRY IT 5

In order to purchase equipment, a barbershop takes out a short term loan of \$3000 at a rate of 4.35%. The loan is due in 80 days.

Determine the interest that will be owed at the end of 80 days and find the maturity value of the loan.

Show answer
Interest owed is \$28.60, MV is \$3028.60

Variations On Simple Interest

The amount of interest earned on an investment or due on a loan is calculated using I = Prt.

This formula can also be used to determine:

- the amount of **principal** (P) that needs to be invested in order to earn a certain amount of interest over a certain period of time.
- the **interest rate** (r) that is needed in order to earn a certain amount of interest over a given time period.

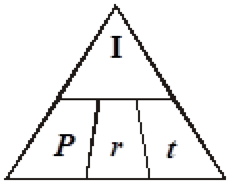
- the amount of **time** (t) it will take in order to earn a certain amount of interest at a stated interest rate.

These amounts can be determined by solving the **simple interest formula** for any of r, P or t.

Finding the Principal, Interest Rate, or Time

		where
To determine the **principal** use:	$P = \frac{I}{rt}$	I = interest earned
To determine the **interest rate** use:	$r = \frac{I}{Pt}$	*r* = annual interest rate
To determine the **time** use:	$t = \frac{I}{Pr}$	*P* = principal
		t = time (in **years**)

The following memory aid is often called the “Magic Triangle”, because if you cover the variable you are trying to find, the formula will magically appear!



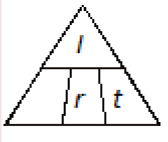
Determining the Principal

EXAMPLE 6

A six month investment will earn 5.25%. How much would you need to invest if you want to earn $100 in interest?

Solution

The principal is unknown. Cover *P* in the Magic Triangle.

P = ? or  appears. Use the formula: $P = \frac{I}{rt}$

$I = \$100$

$r = 5.25$

$t = 6\,months = \frac{6}{12}$ or $0.5\,years$	$P = \frac{100}{0.0525(0.5)}$	Replace *I*, *r* and *t* with their respective values
	$P = \frac{100}{0.02625}$	Multiply 0.0525 by 0.5
	$P = 3809.52$	Divide 100 by 0.02625 and round answer to nearest cent

You would need to invest **$3809.52**

TRY IT 6

A student borrowed money from his best friend at the very low interest rate of 1.5% for a period of 9 months. At the end of 9 months the friend had earned $22.50 in interest. Determine the original amount of the loan.

Show answer

$$P = \frac{22.5}{(0.015)(0.75)} = \$2000$$

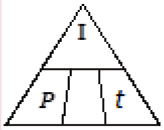
Determining the Interest Rate

EXAMPLE 7

Mariko had $240 in the bank for the month of April. At the end of the month she had earned $0.90 in interest. What interest rate was the bank paying?

Solution

The interest rate is unknown. Cover *r* in the Magic Triangle.

r = ? or 

appears. Use the formula: $r = \frac{I}{Pt}$

$P = \$240$ $I = \$0.90$ *t* = 1 month = $\frac{1}{12}\,year$	$r = \frac{0.90}{240\left(\frac{1}{12}\right)}$
	$r = \frac{0.90}{20} = 0.045$

Since r = 0.045, the interest rate as a percentage is **4.5%**

TRY IT 7

Kris deposited \$800 in an account. At the end of 6 months he had earned \$10.40. Determine the interest rate.

Show answer

$r = \frac{10.4}{(800)(0.5)} = 0.026 = 2.6$

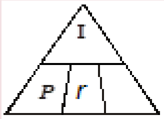
Determining the Time

EXAMPLE 8

Carol invested \$500 at an interest rate of 6%. How long will it take her to earn \$250 in interest?

Solution

The time is unknown. Cover *t* in the Magic Triangle.

t = ? or 

appears. Use $t = \frac{I}{Pr}$

$I = \$250$ $P = \$500$ *r* = 6% = 0.06	$t = \frac{250}{500(0.06)}$
	$t = \frac{250}{30} = 8.33$ or $8\frac{1}{3}$ years

It will take **8.33 years**.

TRY IT 8

An account earns 4% interest. How long will it take for a deposit of \$4000 to earn \$240 in interest?

Show answer

$t = \frac{240}{(4000)(0.04)} = 1.5 years$

When calculating time “t” using the simple interest formula, the answer will be in terms of years. Sometimes it is more reasonable to express the answer in terms of days or months.

Time Conversions

When converting time (in years) to months or days:

To express the time in **months** (m):

Multiply the time “t” in years x 12 months/year.

If time t = 0.25 years then the number of months m = 0.25 years x 12 months/year = 3 months

To express the time in days (d):

Multiply the time “t” in years x 365 days/year.

If time t = 0.25 years then the number of days d = 0.25 years x 365 days/year = 91.25 days

EXAMPLE 9

Troy invested $4000 in an account offering 3.8%. How long will it take him, in days, to earn $30 in interest?

Solution

$$t = \frac{I}{Pr} = \frac{30}{4000(0.038)} = 0.1974 years$$

Time in days = 0.1974 years x 365 days/year = 72 days

It will take 72 days to earn $30 in interest

TRY IT 9

Tam invested $1875 in an account offering 4%. How long will it take her, in months, to earn $62.50 in interest?

Show answer

$t = \frac{62.50}{(1875)(0.04)} = 0.833\ldots$ years so 0.833… years $\times 12$ = 10 months

Key Concepts

- to calculate the simple interest earned on an investment or charged on a loan we use the formula $I = Prt$ where:

I = interest earned r = annual interest rate (stated as a decimal)

P = principal amount t = time (in **years**)

- to determine the principal amount (P) for simple interest applications:
  - $P = \frac{I}{rt}$
- to determine the time in years (t) for simple interest applications:
  - $t = \frac{I}{Pr}$
- to determine the interest rate (r) for simple interest applications:
  - $r = \frac{I}{Pt}$
- The **Maturity Value** (MV) of a loan is the sum of the principal P plus the interest I:

MV = P + I

Glossary

maturity value

is the amount to be paid on the due date of a loan or the amount to be paid to an investor at the end of the period for which an investment has been made.

principal

is the amount of money that has been invested or borrowed.

simple interest

is interest that is calculated only **once** during the entire time period of the loan or deposit. Simple interest is calculated solely on the principal investment or loan.

9.1 Exercise Set

1.
   a. How many days are in 1 year? If an investment is made for a period of 20 days, what portion of the year does this represent?
   b. How many weeks are in one year? If an investment is made for a period of 16 weeks, what portion of the year does this represent?
   c. How many months are in one year? If an investment is made for a period of 5 months, what portion of the year does this represent?
2. Calculate the simple interest earned for each of the following.

a. $1000 at 10% for 1 year	b. $150 at 5% for 1 year
c. $500 at 4.5% for 0.5 years	d. $200 at 11% for 3 months
e. $100 at 7.25% for 6 months	f. $480 at 3.6% for 5 months
g. $2500 at $6\frac{1}{2}\%$ for 100 days	h. $1800 at 5.25% for 30 weeks

3.

   a. Mike borrowed $1500 from his mother. He agreed to pay her back in 9 months at 5%.

   b. How much in total will he owe her?

4. Mark won $10,000 and invested it for 32 weeks at 7.25% interest. a) How much interest did his investment earn? b) How much will he have in total at the end of the 32 weeks?
5. Barb invested $100. At the end of one year the investment had earned 16%. She then invested the whole amount (principal plus interest) and earned 12% in the second year.
   a. How much interest did Barb earn at the end of the first year?
   b. How much did she invest at the beginning of the second year?
   c. How much interest did she earn in the second year?
   d. How much did Barb have at the end of the two years?
6. Larry loaned Mary $2500 at 7%. Mary said she would pay Larry the $2500 plus interest in 90 days. What is the total amount of money that Mary should pay Larry in 90 days?

Find the principal needed to earn the following interest amounts:

7.

a. $100 at 5% in 1 year	b. $15 at 2.5% in 18 weeks
c. $60 at 9.5% in 90 days	d. $1000 at 2.75% in 9 months

8. Find the interest rate (if necessary round final answers to the nearest hundredth) when:

a. $1000 earns $25 in 1 year	b. $100 earns $3.60 in 5 months
c. $4000 earns $10.60 in 13 weeks	d. $550 earns $4.80 in 73 days

. Find the time (if necessary round final answers to the nearest hundredth) needed to earn:

9.

a. $5 interest on $100 at 10% (in months)	b. $1 interest on $1,000 at 12.5% (in days)
c. $4 interest on $100 at 7.5% (in days)	d. $3 interest on $100 at 10% (in months)

10. Fill in the missing values.

I	*P*	*r*	*t*
	$100.00	3%	1 year
$50.00		5%	6 months
$3.41	$630.00		1 month
$9.50	$800.00	4.75%	months

11. At the beginning of the year, Bill invested $500 in a special account. At the end of the year the account was worth a total of $523.25. What interest rate did he earn on the $500 investment?
12. Velma invests $1200 at 6.5%. How long (to the nearest day) will it take to earn $10 in interest on the investment?
13. A short term lender charged $3.45 interest on a $230 purchase over a 30 day period. What interest rate did the lender charge?

Answers

1.
   a. 365; 4/73
   b. 52; 2/13
   c. 12; 5/12
2.
   a. $100
   b. $7.50
   c. $11.25
   d. $5.50
   e. $3.63
   f. $7.20
   g. $44.52
   h. $54.52
3.
   a. interest $56.25
   b. $1500.00 + 56.25 = $1556.25
4.
   a. I = $446.15
   b. $10446.15
5.
   a. $16
   b. $116

   c. I = 116(0.12)(1) = $13.92
   d. $16.00 + $13.92 = $29.92

6. Mary owes Larry $2500 + $43.15 = $2543.15
7.
   a. $2000.00
   b. $1733.33
   c. $2561.40
   d. $48 484.85
8.
   a. 2.5%
   b. 8.64%
   c. 1.06%
   d. 4.36%
9.
   a. 0.5 years = 6 month
   b. 0.008 years = 2.92 days
   c. 0.533 years = 194.67 days
   d. 0.3 years = 3.6 months
10.

I	*P*	*r*	*t*
$3.00	$100.00	3%	1 year
$50.00	**$2000.00**	5%	6 months
$3.41	$630.00	**6.5%**	1 month
$38.00	$800.00	4.75%	**3 months**

11. 4.65%
12. 0.13 years or 47 days
13. 18.25%

Attribution

Some of the content for this chapter is from “Unit 1: Simple interest” and “Unit 2: Variations on simple

9.2 Compound Interest

Learning Objectives

By the end of this section it is expected that you will be able to:

- Determine the compound amount (future value) of an investment or loan
- Determine the interest component of an investment or loan that involves compound interest
- Determine the present value of a compound amount

Compound Interest

We have seen that with simple interest an investment will earn interest on the original amount. For an investment of $100 earning 10% simple interest, the interest earned after one year will be $10 since 10% of $100 = $10. An investment will grow more quickly if the interest is calculated more often than once a year. Interest will not only be calculated on the principal amount but also on the previously earned interest. This process is referred to **compounding.**

Figure 1 illustrates the process of compounding or earning interest on interest. Consider an investment of $100 that earns 10%/year with interest being compounded semiannually. With semiannual compounding the interest on the investment will be calculated twice during the year.

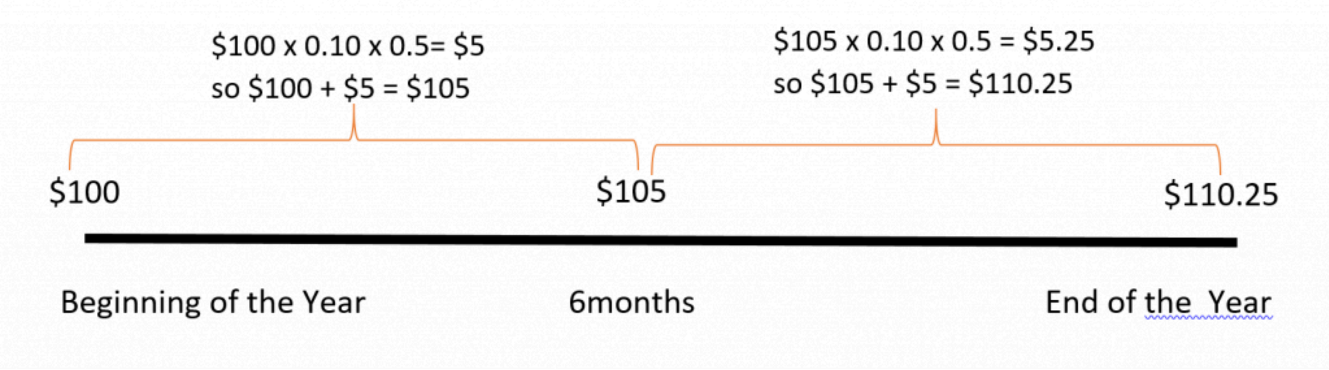


Fig. 1

Using the simple interest formula I = Prt, at the end of six months (half a year) interest will be calculated as follows:

I = $100 x 10% x 1/2 year = $5.

Adding this $5 to the principal of $100 you will have $105 at the end of the first six months. At the end of the year interest will be calculated again on the $105:

I = $105 x 10% x 1/2 year = $5.25.

Adding this $5.25 to $105 you will have $110.25 at the end of the year. In this case you would be earning interest not only on the original principal of $100, but also on the previously earned interest of $5. When interest is earned on interest, we say the interest is **compounded.** The total amount of principal and accumulated interest at the end of a loan or investment is called the **compound amount.**

Consider a $100 investment that earns 10%/year compounded annually. The table in Figure 2 shows how the value of the $100 investment will grow over a 6-year period.

Year	**Amount at the beginning of the year**	**Earned Interest**	**Year End Total**
1	$100	$10	$110
2	$110	$11	$121
3	$121	$12.10	$133.10
4	$133.10	$13.31	$146.41
5	$146.41	$14.64	$161.05
6	$161.05	$16.11	$177.16

Fig. 2

At the beginning of Year 1, $100 is invested, so the interest earned in the first year will be:

I = Prt = $100 × 0.10 × 1 = $10. This is added to the original $100 to result in $110 at the end of Year 1.

At the beginning of Year 2 the process will repeat but the principal P is now $110.

I = Prt = $110 × 0.10 ×1 = $11 in interest so at the end of Year 2 there will be:

$110 + $11 = $121 in the account.

Notice that the **compound amount** at the end of the six year period is $177.16. The investment has earned an accumulated $77.16 in interest. If the investment had earned **simple** interest as opposed to **compound** interest it would have only earned:

I = Prt = 100 × 0.10 × 6 = $60 in interest.

The above method of calculating the compound amount is very time consuming. Fortunately, there is a mathematical formula that we can use when working with compound interest.

Compound Interest Formula

The **compound interest formula** is:

$A = P\left(1+\frac{r}{n}\right)^{nt}$	where,	*A* = total compound amount(includes principal and interest)
		P = principal
		r = annual interest rate
		n = number of times in one year that interest is calculated
		t = time (in years)

Since A includes both the principal and interest, to find the interest amount I calculate:
$I = A - P$

EXAMPLE 1

Find the compound amount and the interest earned on $100 compounded annually at 10% for 6 years.

Solution

P = $100 r = 10% = 0.1 n = 1 (since the interest is calculated once a year) t = 6 years	$A = P\left(1+\frac{r}{n}\right)^{nt}$ $A = 100\left(1+\frac{0.1}{1}\right)^{1\times 6}$	Replace the variables with their values
	$A = 100\left(1+0.1\right)^{6}$	$\frac{0.1}{1} = 0.1$ and $1 \times 6 = 6$
	$A = 100\left(1.1\right)^{6}$	Raise $\left(1.1\right)^{6} = 177.1561$
	$A = 100\left(1.771561\right) = 177.1561$	

The interest earned is $A - P = \$177.16 - \$100 =$ **$77.16**

The compound amount is **$177.16**

TRY IT 1

Kyle won $10,000 in a lottery and deposited the full amount in a 3 year investment at 3.8% compounded annually. Find the compound amount and the interest earned over the three years.

Show answer
Compound Amount = $11,183.87; Interest = $1183.87

Interest can be compounded using a variety of **compounding periods**. The compounding period is the span of time between when interest is calculated and when it will be calculated again. If there is one month between every interest calculation then the compounding period is monthly. With monthly compounding there will be 12 compounding period in one year since there are twelve months in a year . The variable ***n*** in the compound interest formula reflects the number of times in one year that interest is calculated.

Compounding Periods

If interest is **compounded:**

annually (once per year) ⇒ $n = 1$

semi-annually (twice a year) ⇒ $n = 2$

quarterly (four times per year) ⇒ $n = 4$

monthly (twelve times per year) ⇒ $n = 12$

weekly (fifty-two times per year) ⇒ $n = 52$

daily (three hundred sixty-five times per year) ⇒ $n = 365$

EXAMPLE 2

Find the compound amount and the interest earned on $500 compounded semiannually at 6% for 3 years.

Solution

$P = \$500$	$A = P\left(1+\frac{r}{n}\right)^{nt}$
$r = 6$	$A = 500\left(1+\frac{0.06}{2}\right)^{2\times3}$
$n = 2$ (since the interest is calculated semiannually or 2 times a year)	$A = 500\left(1+\frac{0.06}{2}\right)^{6}$
$t = 3$	$A = 500\left(1.03\right)^{6}$
	$A = 500\left(1.19405\right)$

The compound amount is **$597.03** and the interest earned is **$597.03 – $500 = $97.03**

TRY IT 2

Kam won $10,000 in a lottery and deposited the full amount in a 3 year investment at 3.8% compounded monthly. Find the compound amount and the interest earned over the three years.

Show answer
Compound Amount = $11,205.50; Interest = $1205.50

The greater the number of compounding periods in a year, the greater the total interest earned will be.

EXAMPLE 3

Find the compound amount and the interest earned on $500 compounded daily at 6% for 3 years.

Solution

$P = \$500$	$A = P\left(1+\frac{r}{n}\right)^{nt}$
$r = 6$	$A = 500\left(1+\frac{0.06}{365}\right)^{365\times 3}$
$n = 365$ (since the interest is calculated daily)	$A = 500\left(1+\frac{0.06}{365}\right)^{1095}$
$t = 3$	$A = 500\left(1.000164\right)^{1095}$
	$A = 500\left(1.1972\right)$

The compound amount is **$598.60** and the interest earned is **$598.60 – $500 = $98.60**

TRY IT 3

Kam won $10,000 in a lottery and deposited the full amount in a 3 year investment at 3.8% compounded daily. Find the compound amount and the interest earned over the three years.

Show answer
Compound Amount = $11,207.45; Interest = $1207.45

Loan recipients must repay the principal amount borrowerd plus any interest charged. They will pay a greater price (in terms of total interest) when interest is compounded.

EXAMPLE 4

Pat borrows \$3200 an interest rate of 6.5% compounded semiannually. The original loan amount plus interest must be paid back in 3 years. Calculate the total amount that must be paid back in three years and determine the interest amount.

Solution

$P = \$3200$	$A = P\left(1+\frac{r}{n}\right)^{nt}$
$r = 6.5$	$A = 3200\left(1+\frac{0.065}{2}\right)^{2\times3}$
$n = 2$ (since the interest is calculated semiannually or 2 times a year)	$A = 3200\left(1+\frac{0.065}{2}\right)^{6}$
$t = 3$	$A = 3200\,(1.0325)^{6}$
	$A = 3200\,(1.21155)$

The compound amount is **\$3876.95** and the interest owing is **\$3876.95 – \$3200 = \$676.95**

TRY IT 4

Determine the compound interest on a 2 year loan of \$5000 at an interest rate of 4.8% compounded quarterly.

Show answer
Compound Amount = \$5500.65; Interest = \$500.65

EXAMPLE 5

Pat borrows \$3200 an interest rate of 6.5% compounded monthly. The original loan amount plus interest must be paid back in 3 years. Calculate the total amount that must be paid back in three years and determine the interest amount.

Solution

$P = \$3200$	$A = P\left(1+\frac{r}{n}\right)^{nt}$
$r = 6.5$	$A = 3200\left(1+\frac{0.065}{12}\right)^{12\times3}$
$n = 12$ (since the interest is calculated monthly or 12 times a year)	$A = 3200\left(1+\frac{0.065}{12}\right)^{36}$
$t = 3$	$A = 3200\left(1.00542\right)^{36}$
	$A = 3200\left(1.21467\right)$

The compound amount is **$3886.95** and the interest owing is **$3886.95 – $3200 = $686.95**

TRY IT 5

Determine the compound interest on a 2 year loan of $5000 at an interest rate of 4.8% compounded daily.

Show answer

Compound Amount = $5503.76; Interest = $503.76

Variation on Compound Interest – Present Value

We might want to know how much money we should invest now in order to make a purchase in the future. Say for example that you want to know how much principal you needed to invest now in order to have $2000 in two years. The amount you need to invest now is called the **present value** of $2000. It is the amount of money that if invested now will accumulate to $2000 in two years. Assuming that your investment earns interest, the amount required now will be less than the future amount. Assuming annual compounding at an interest rate of 5% you will need to invest $1814.06 now to have $2000 in two years. Refer to Figure 3 below.

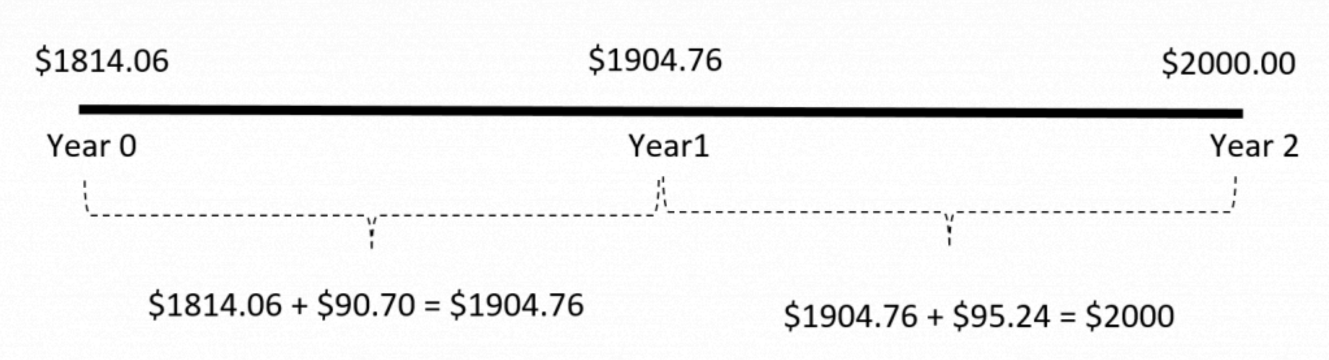


Fig. 3

In Figure 3 we can see that at the beginning of the year $1814.06 is invested.

At the end of the first year the interest earned on $1814.06 is 5% × 1814.06 = $90.70.

At the end of the first year there will be $1904.76 in the account. In the second year the interest earned on $1904.76 is $1904.76 × 0.05 = $95.24.

At th eend of the second year there will be $2000.00 in the account. The $2000 that is needed in two years is said to have a **present value** of $1814.06.

The present value can be calculated by solving the compound interest formula for ***P***.

Formula for Present Value

The **present value formula** is:

$$P = \frac{A}{\left(1+\frac{r}{n}\right)^{nt}}$$

where	*P* = present value
	A = desired future amount
	r = interest rate (as a decimal fraction)
	n = number of times interest is calculated in one year
	t = times (in years)

EXAMPLE 6

A house painting company is planning to expand its operations in three years time. It will require $24,000 in order to expand. How much must it invest now, at 4.6% interest compounded annually?

Solution

P = ? A = $24000 r = 4.6% = 0.046 n = 1	$P=\frac{A}{\left(1+\frac{r}{n}\right)^{nt}}$	
t = 3 years	$P=\frac{24000}{\left(1+\frac{0.046}{1}\right)^{1\times3}}$	Replace the variables with their values
	$P=\frac{24000}{(1.046)^3}$	Add $1+0.046=1.046$
	$P=\frac{24000}{1.144445}$	Raise $(1.046)^3=1.144445$
	$P=20970.86$	

The present value is **$20,970.86** so the company must invest that amount now to have $24,000 in three years.

TRY IT 6

Mae is planning on buying a vehicle when she turns 18 in five years. How much money must she invest now in an account earning 7% interest, compounded annually, in order to afford a used vehicle at a cost of $5000?

Show answer
An investment of $3564.93 is required

EXAMPLE 7

Pat and her friends are planning a reunion in five years. She estimates that the cost of the trip plus expenses will be approximately $2000. How much should she invest right now in order to have $2000 five years from now, if she knows her money will earn 6% compounded quarterly?

Solution

P = ? A = \$2000 r = 6% = 0.06 n = 4	$P=\frac{A}{\left(1+\frac{r}{n}\right)^{nt}}$	
t = 5 years	$P=\frac{2000}{\left(1+\frac{0.06}{4}\right)^{4\times5}}$	Replace the variables with their values
	$P=\frac{2000}{\left(1+\frac{0.06}{4}\right)^{20}}$	Multiply $4\times5=20$
	$P=\frac{2000}{(1+0.015)^{20}}$	Divide $\frac{0.06}{4}=0.015$
	$P=\frac{2000}{(1.015)^{20}}$	Add $1+0.015=1.015$
	$P=\frac{2000}{1.346855}$	Raise $(1.015)^{20}=1.346855$
	1484.94	

The present value is **\$1484.94**

In other words, if Pat invested \$1484.94 now at 6% compounded quarterly, then in 5 years the compound amount would be \$2000.

TRY IT 7

You are planning on attending college in four years and your parents plan to help out with \$10,000 in assistance. How much money must they invest now in an account earning 5.6% compounded monthly if they plan to have \$10000 in the account in four years?

Show answer
\$7997.31

Key Concepts

- to determine the compound amount (A) of an investment or loan:
  - $A=P\left(1+\frac{r}{n}\right)^{nt}$
- to determine the interest component (I) of a principal or original amount (P) that has grown to a compound amount (A):
  - Interest = Compound Amount – Principal Amount I = A – P

- to determine the present value **P** of a compound amount the formula is:
  - $P = \frac{A}{\left(1+\frac{r}{n}\right)^{nt}}$
- **Compounding Periods**
  - annually (once per year) ⇒ *n* = 1
  - semi-annually (twice a year) ⇒ *n* = 2
  - quarterly (four times per year) ⇒ *n* = 4
  - monthly (twelve times per year) ⇒ *n* = 12
  - weekly (fifty-two times per year) ⇒ *n* = 52
  - daily (three hundred sixty-five times per year) ⇒ *n* = 365

Glossary

compound amount

is the total amount of principal and accumulated interest at the end of a loan or investment period.

compound interest

is when the interest on a loan or deposit is calculated based on both the initial principal and any accumulated interest from previous periods.

compounding period

is the span of time between when interest is calculated and when it will be calculated again.

present value

is the current value of a sum of money that has been invested and has grown to a larger compound amount.

9.2 Exercise Set

1. Determine the value of n in each of the following:
   a. weekly, then *n* = ___________
   b. semi-annually, then *n* = ________
   c. quarterly, then *n* = _______
   d. daily, then *n* = _______
2. Ada invested $1000 at 5% compounded annually.
   a. Complete the table below to determine the compound amount of Ada's investment at the end of 5 years.

Year	**Principal Amount**	**Earned Interest**	**Year End Total**
1	$1000	$50	$1050
2	$1050	$52.50	
3			
4			
5			

   b. Use the compound interest formula to determine the compound amount Ada will earn in 5 years.

3. Find the compound amount and the earned interest when $1000 is invested under the following conditions:
   a. $1000 compounded annually at 9% for 5 years.
   b. $1000 compounded semi-annually at 9% for 5 years.
   c. $1000 compounded quarterly at 9% for 5 years.
   d. $1000 compounded monthly at 9% for 5 years.
   e. $1000 compounded daily at 9% for 5 years.
4. When Penny was born her parents put $5000 in a special fund paying 4.4% compounded quarterly.
   a. How much will the fund be worth when Penny turns 10 years old?
   b. Penny's parents take the money from the fund when Penny turns 10 and reinvest it at 7.2% componded monthly. How much will the investment be worth when Penny turns 18 years old?
5. Anne's parents invested $8400 at 5% with daily compounding. How much money will they have when Anne starts college in 5 years? How much interest did their investment accumulate over the 5 years?
6. Theresa is considering two options for investing $10 000 : a savings account offering 8% simple interest or a savings certificate that earns 7.75% compounded monthly.
   a. How much will the savings account earn (in interest) in one year?
   b. How much interest will the savings certificate earn after one year?
   c. Which option yields more interest and by how much?
7. You have $2500 to invest, compounded monthly, over a period of 4 years. Calculate the compound amount and the interest earned when interest rates are as follows:
   a. 3%
   b. 6%
   c. 9%

   d. Notice that the interest rate of 9% is triple that of the 3% rate. How many times higher is the total interest earned at 9% than at the 3% rate?

8. L. Shark says that he will lend you the $5000 you need but he wants 50% compounded daily on the loan. (Note that time t = number of days/365)
   How much will you owe L. Shark if you pay the loan back in:
   a. 30 days
   b. 60 days
   c. 90 days
   d. How much more interest will you owe if you wait 90 days to pay back the loan rather than paying it back in 30 days? Does it make sense to pay off a loan quickly?
9. For each of the following pairs of investment options, determine which option results in a higher compound amount.
   a. Option A: $8000 invested for 3 years at 2.6% compounded quarterlyOption B: $8000 invested for 2 years at 6.8% compounded monthly
   b. Option A: $20,000 invested for 7 years at 8.6% compounded annuallyOption B: $20,000 invested for 8 years at 7.4% compounded semiannually
10. Find the present value for each of the following:
    a. $1800 due in 5 years at 4.6% compounded semi-annually.
    b. $2500 due in 2 years at 3.6% compounded monthly.
    c. $4000 due in 10 years at 8.4% compounded yearly.
    d. $650 due in one and one-half years at 4% compounded quarterly.
    e. $1000 due in 6 months at 2.8% compounded monthly.
11. In 6 years, Sylvia’s son will be going to college. Sylvia estimates that her son will need about $28000 to get started in the first year of his education. How much should she invest now if she can earn 7% compounded semi-annually?
12. The Smiths inherited $20,000. They would like to spend some of the money now, but still have $20,000 ten years from now when they retire. They have found an investment that will earn them 8.4% compounded annually over this time.a. How much of the $20,000 should they invest now to guarantee that they will have $20,000 when they retire in 10 years?b. How much of the $20,000 can they spend now?
13. A certain savings certificate will pay the owner $5000 in two years. If the interest rate is 3.8% compounded weekly, how much will be invested now to accumulate to $5000 in two years?

Answers

1.
   a. 52
   b. 2

c. 4

d. 265

2\. a.

Year	Principal Amount	Earned Interest	Year End Total
1	$1000	$50	$1050
2	$1050	$52.50	$1102.50
3	$1102.50	$55.13	$1157.63
4	$1157.63	$57.88	$1215.51
5	$1215.51	$60.78	$1276.29

b. $1276.28

3\. a. $1538.62; $538.62

b. $1552.97; $552.97

c. $1560.51; $560.51

d. $1565.68; $565.68

e. $1568.23; $568.23

4\. a. $7744.91

b. $13,753.79

5\. Compound Amount = $10, 785.63 Interest = $2385.63

6\. a. $800

b. $803.13

c. savings certificate pays $3.13 more interest

7\. a. Compound Amount = $2818.32 so interest = $318.32

b. Compound Amount = $3176.22 so interest = $676.22

c. Compound Amount = $3578.51 so interest = $1078.51

d. $1078.51/$318.32 = 3.4 times as great

8\. a. $5209.61

b. $5428.00

c. $5655.55

d. $445.96 so yes

9\. a. Option A $8646.80; Option B $9161.94 so Option B is better

b. Option A $35,631.88; Option B $35,767.62 so Option B is better

10\. a. $1433.89

- b. $2326.58
- c. 1785.53
- d. $612.33
- e. $986.11

11. $18,529.93
12. invest $8927.65; spend $11072.35
13. $4634.21

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9.3 Interest Rates

Learning Objectives

By the end of this section it is expected that you will be able to:

- State the difference between nominal and effective interest rates
- Determine the effective annual interest rate
- Determine the best option when comparing nominal interest rates under different situations

Nominal and Effective Rates of Interest

As consumers and investors we are inundated with all kinds of offers for both investments and financing. When we require a loan we seek the lowest possible interest rate but when we invest we want the highest rate of return. As we search for the best offer it is important to recognize that the advertised interest rate is not necessarily the true interest rate.

An advertised rate of 10% with **annual** compounding works out to be equivalent to a rate of 10% annually. We refer to this stated rate of 10% as the **nominal** interest rate. An advertised rate of 10%

with **daily** compounding works out to be equivalent to a rate of 10.52% since interest is calculated on the interest 365 times in one year. We refer to the 10.52% as the **effective** interest rate.

The stated interest rate is referred to as the **nominal interest rate**, as it describes the named or numerical value. It is the rate that is usually stated in advertisements. The actual interest rate, or **effective interest rate**, reflects the real rate of return as it takes the compounding periods into account.

With simple interest calculations (where there is no compounding), the stated annual interest rate indicates the true rate of return. If $1000 is borrowed at 6% for one year, then the interest owined will be I = Prt = $1000 × 0.06/yr × 1yr = $60.

With compound interest calculations, the stated annual interest rate does not indicate the true interest cost. If $1000 is invested at 6% for one year compounded semiannually, then the interest owed will be $60.90 rather than $60.

One way to determine the **effective interest rat**e is to divide the total compound interest for the first year by the principal amount. If in the first year $60.90 is the interest charged on a principal of $1000 then the effective interest rate is $60.90/$1000 = 0.0609 = 6.09%. Although the nominal interest rate is 6%, the effective rate is 6.09%.

It is also possible to use a formula to calculate the effective interest rate.

Effective Interest Rate Formula

The **effective interest rate formula** is:

$f=\left(1+\frac{r}{n}\right)^n-1$	where	*f* = effective interest rate
		r = nominal interest rate (annual interest rate)
		n = number of times in one year that interest is calculated

EXAMPLE 1

Use the formula to determine the effective interest rate for 6% compounded annually.

Solution

f = ? r = 6% = 0.06 n = 1	$f=\left(1+\frac{r}{n}\right)^{n}-1$	
	$f=\left(1+\frac{0.06}{1}\right)^{1}-1$	Replace the variables with their values
	$f=(1.06)^{1}-1$	Add $1+0.06$
	$f=1.06-1$	
	$f=0.06=6$	

The effective interest rate is 6%. Note that the nominal and effective rate are the same since the number of compounding period is one (n = 1).

TRY IT 1

What is the effective rate of 4% compounded yearly?

Show answer
4%

EXAMPLE 2

Use the formula to determine the effective interest rate for 6% compounded monthly.

Solution

f = ? r = 6% = 0.06 n = 12	$f = \left(1 + \frac{r}{n}\right)^n - 1$	
	$f = \left(1 + \frac{0.06}{12}\right)^{12} - 1$	Replace the variables with their values
	$f = (1 + 0.005)^{12} - 1$	Divide $\frac{0.06}{12}$
	$f = (1.005)^{12} - 1$	Add $1 + 0.005$
	$f = 1.0617 - 1$	Raise $(1.005)^{12}$
	$f = 0.0617 = 6.17\%$	

The effective interest rate is 6.17%

TRY IT 2

What is the effective rate of 4% compounded monthly?

Show answer
4.07%

EXAMPLE 3

What is the effective rate for a nominal rate of 9.8% compounded weekly?

Solution

f = ? r = 9.8% = 0.098 n = 52	$f=\left(1+\frac{r}{n}\right)^n-1$	
	$f=\left(1+\frac{0.098}{52}\right)^{52}-1$	Replace the variables with their values
	$f=(1+0.0018846)^{52}-1$	Divide $\frac{0.098}{52}$
	$f=(1.0018846)^{52}-1$	Add $1+0.0018846$
	$f=1.102861-1$	Raise $(1.0018846)^{52}$
	$f=0.102861=10.29\%$	

The effective interest rate is 10.29%

TRY IT 3

Determine the effective rate of interest on a loan that is advertised at a rate of 7.8% compounded daily.

Show answer
8.11%

It is important to consider the effective interest rate, rather than the nominal rate, when deciding on investments or loans.

Consider Bank A which offers a savings plan at 6.25% compounded monthly and Bank B which offers 6.5% compounded semi-annually. Which of the two banks offers the better rate of return? Although both banks offer the same nominal interest rate, their effective rates differ. The effective rate will reflect the actual rate of return in one year. Example 4 will illustrate this.

EXAMPLE 4

Bank A offers 6.25% compounded monthly while Bank B offers 6.5% compounded semi-annually. Which bank offers the better effective rate of return?

Solution

For Bank A:

$f = ?$ $r = 6.25\% = 0.0625$ $n = 12$	$f = \left(1 + \frac{0.0625}{12}\right)^{12} - 1$
	$f = (1 + 0.0052083)^{12} - 1$
	$f = (1.005208)^{12} - 1$
	$f = 1.064322 - 1$
	$f = 0.064322 = 6.43\%$

For Bank B:

$f = ?$ $r = 6.5\% = 0.065$ $n = 2$	$f = \left(1 + \frac{0.065}{2}\right)^{2} - 1$
	$f = (1 + 0.0325)^{2} - 1$
	$f = (1.0325)^{2} - 1$
	$f = 1.066056 - 1$
	$f = 0.066056 = 6.61\%$

Note that Bank A’s effective rate, 6.43%, is less than both Bank B’s nominal rate of 6.5% and Bank B’s effective rate of 6.61%. Bank B offers the better effective rate of return.

TRY IT 4

Sam plans to invest a lottery win of $15,000. He is considering two different options. Option A offers 3.56% compounded weekly and Option B offers 3.48% componded monthly. Which option offers a better rate of return?

Show answer
Option A 3.62%; Option B 3.54%; Option A offers a better rate of return.

EXAMPLE 5

Consider two options for a 2 year loan. Bank A will charge 7.2% compounded monthly while Bank B will charge 7.4% compounded semi-annually. Which bank offers the less expensive loan (charges the lower effective rate)?

Solution

For Bank A:

$f = ?$ $r = 7.2\% = 0.072$ $n = 12$	$f = \left(1 + \frac{0.072}{12}\right)^{12} - 1$
	$f = (1 + 0.006)^{12} - 1$
	$f = (1.006)^{12} - 1$
	$f = 1.074424 - 1$
	$f = 0.074424 = 7.44\%$

For Bank B:

$f = ?$ $r = 7.4\% = 0.074$ $n = 2$	$f = \left(1 + \frac{0.074}{2}\right)^{2} - 1$
	$f = (1 + 0.037)^{2} - 1$
	$f = (1.037)^{2} - 1$
	$f = 1.075369 - 1$
	$f = 0.075369 = 7.54\%$

Bank A's effective rate, 7.44%, is less than Bank B's effective rate of 7.54. By a slight margin, Bank A offers the less expensive loan.

TRY IT 5

Sam needs to borrow $5500. He is offered two different loans. One loan is at a bank for 6.8% compounded quarterly and the other is at a credit union for 6.9% compounded semiannually.

Which is the better option for Sam?

Show answer
Bank 6.98%; Credit Union 7.02%; the Bank is a slightly better option for a loan.

Key Concepts

- to determine the effective annual interest rate (f):
    - $f=\left(1+\frac{r}{n}\right)^n-1$
- when investing money you want the higher effective interest rate; when borrowing money you want the lower effective interest rate.

Glossary

effective interest rate

takes the compounding periods into effect so it is a better reflection of the actual interest charges.

nominal interest rate

is normally the stated rate. It does not take the compounding periods into effect.

9.3 Exercise Set

1. Determine the effective interest rates (rounded to two decimal places) for the following nominal interest rates when there is monthly compounding.
    a. 8%
    b. 3.7%
    c. 2.64%
    d. 5%
2. Determine the effective interest rates (rounded to two decimal places) for the following nominal interest rates when there is daily compounding.
    a. 8%
    b. 3.7%
    c. 2.64%
    d. 5%
3. Determine the effectiveinterest rate (rounded to two decimal places) when 10% is compounded

   a. Yearly
   b. Semi-annually
   c. Quarterly
   d. Monthly
   e. Weekly
   f. Daily

4. You have a choice between purchasing a savings certificate offering 3.6% simple interest or putting your money in a savings account at 3.6% compounded monthly. What is the difference between the effective rates?
5. What simple interest rate would give you the same return as
   a. 6% compounded daily?
   b. 5% componded semiannually?
   c. 4.2% componded weekly?
6. a. What is the effective interest rate?
   b. What total amount do you owe in four years?
   c. What amount of this will be the interest charged?You borrow $4800 to be paid back in 4 years at a rate of 4.4% componded quarterly.
7. You are needing to borrow $10,000 to be paid back over a 3 year period and you are consider two options. With Option A the interest rate is 3.5% compounded daily and with Option B the interest rate is 3.52% componded semiannually. Which option offers the less expensive loan (charges the lower effective rate)?
8. You invest $5600 for two years at a rate of 5.2% componded monthly.
   a. What is the effective interest rate?
   b. What total amount will be in your account after two years?
   c. What amount of this will be the interest earned?
9. You can invest $2000 for one year under the following two options: Option A 6.2 % simple interest or Option B 6.15% compounded weekly. For each of these
   a. Determine the effective interest rate.
   b. Determine the compound amount at the end of one year.
   c. Determine the interest that is earned.
10. L. Shark offers to lend you $1000 for one year at 50% interest compounded daily.
   a. What is the effective rate of interest on this loan (rounded to the nearest hundred
   b. What total amount do you owe at the end of one year?
   c. What is the interest component?
   d. What would the interest component be if instead you were charged 50% simple

interest?

Answers

1.
   a. 8.30%
   b. 3.76%
   c. 2.67%
   d. 5.12%
2.
   a. 8.33%
   b. 3.77%
   c. 2.68%
   d. 5.13%
3.
   a. 10%
   b. 10.25%
   c. 10.38%
   d. 10.47%
   e. 10.51%
   f. 10.52%
4. The effective rate for 3.6% simple interest is 3.6%. The effective rate for 3.6% compounded monthly is 3.66% so a difference of 0.06%.
5.
   a. 6.18%
   b. 5.06%
   c. 4.28%
6.
   a. 4.47%
   b. $5718.21
   c. $918.21
7. Option A the effective rate is 3.56%; Option B the effective rate is 3.55%. Option B is less expensive by 0.01%
8.
   a. 5.33%
   b. $6212.37
   c. $612.37
9.
   a. Option A 6.2% and Option B 6.34%
   b. Option A $2124 and Option B $2126.78
   c. Option A $124 and Option B $126.78
10.
   a. 64.82%
   b. $1648.16
   c. $648.16
   d. $500

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9.4 Annuities

Learning Objectives

By the end of this section it is expected that you will be able to:

- Distinguish between an ordinary annuity and an annuity due
- Determine the future value of an ordinary annuity
- Determine the payment, given the future value for an ordinary annuity

Annuities

A common financial goal is to be comfortable in retirement. During our working lives we contribute to a retirement fund so that upon retirement we receive a financial payment at regular intervals. Financial transactions that involve a series of equal payments at equal intervals are called annuities. Other examples of annuities include payments on a loan, rental payments, and insurance premiums.

The **term of the annuity** is the time from the beginning of the first payment interval to the end of the last payment interval. A **payment interval** is the time between successive payments. If, for example, a vehicle is purchased with monthly payments on a four-year loan then the **term** of the loan is 4 years and the **payment interval** is monthly.

In some cases, as with salaries or a senior's pension, the payments are made at the end of a payment interval. This is referred to as an **ordinary annuity**. When payments are required at the beginning of a payment interval, as with many loans and mortgages, this is referred to as an **annuity due.**

It is important to note that the **term** of the annuity does not necessarily coincide with the first and last payment. Consider a one year loan where 12 equal payments are made on the first of each month. This is an example of an **annuity due**. The **term of the annuity** is one year and the **payment interval** is one month. Refer to Figure 1. There are twelve payments, each occurring on the first day of the month. The first payment is made on Jan. 1 and the last payment is made on Dec. 1. Note that the final payment on Dec. 1 does not occur on the last day of the term of the annuity which is Dec. 31.

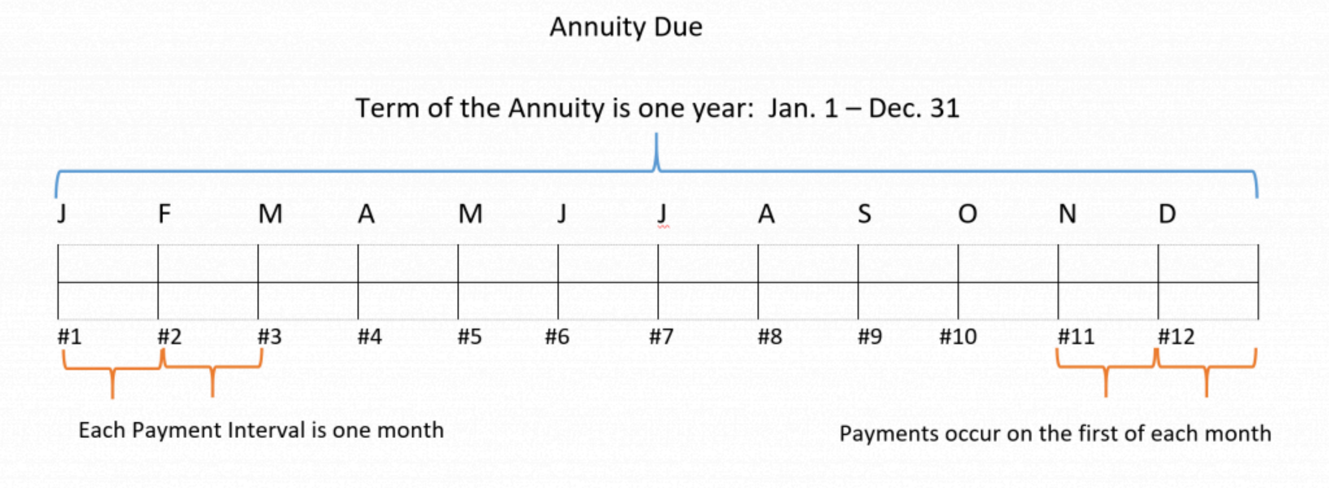

Fig. 1

Consider a one year loan where 12 equal payments are made on the last day of each month. This is an example of an **ordinary annuity**. The **term** of the annuity is one year and the **payment interval** is one month. Refer to Figure 2. There are twelve payments, each occurring on the last day of the month. The first payment is made on Jan. 31 and the last payment is made on Dec. 31. Note that the first payment on Jan. 31 does not occur on the first day of the term of the annuity which is Jan. 1.

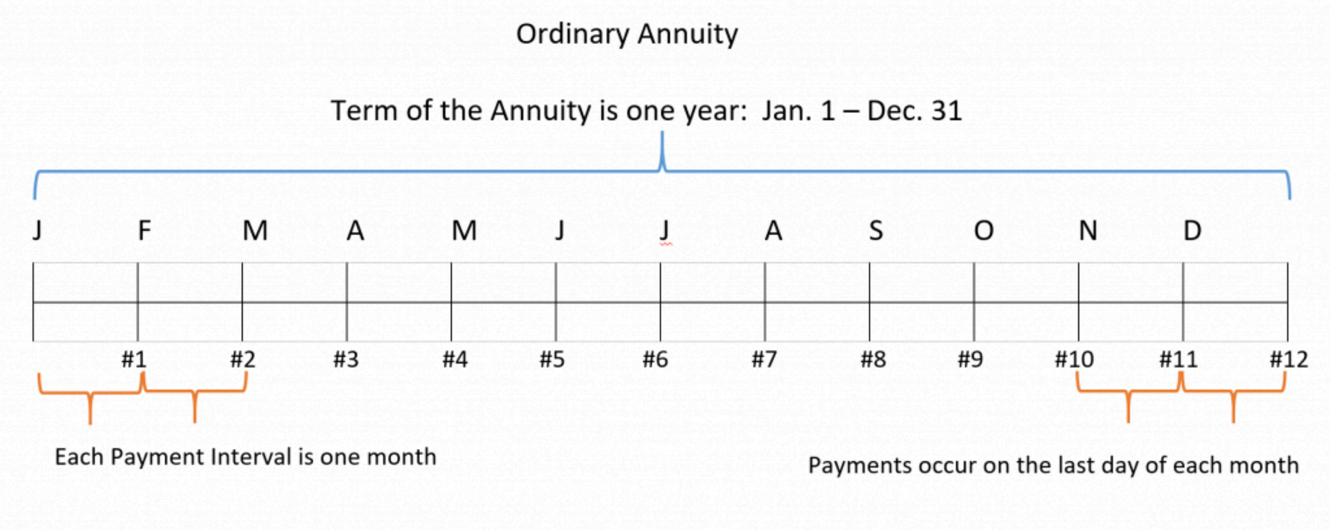

Fig. 2

In this section we will only be concerned with **ordinary simple annuities**. As with all **ordinary annuities** the payments are made at the **end** of each payment interval. It is also the case that the **compounding interval equals the payment interval**. This means that if the payment interval is monthly then interest will also be compounded monthly.

Future Value of an Ordinary Annuity

Consider an investment that is in the form of an ordinary simple annuity. This means that a deposit is made at the end of regular intervals and interest is compounded at each of these intervals. The value of the annuity can grow substantially. The final amount of the annuity is called the **future value** of the annuity. It is the total of all annuity payments and the accumulated compound interest as illustrated in Figure 3.

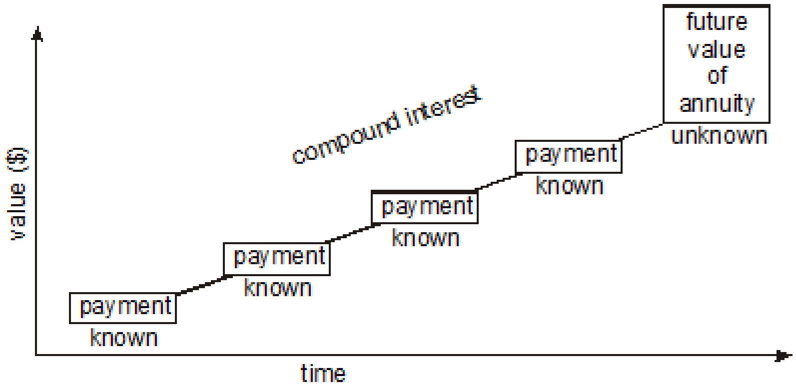

Fig. 3

To see how the annuity process works, consider the table in Figure 4 below. This table depicts an ordinary 6-month annuity commencing on Jan. 1 and ending on June 30. The monthly payments are $1000 and the annual interest rate is 6% compounded monthly.

Month	**Balance on the first day of the month**	**Interest earned during the interval**	**Deposit at end of the month**	**Balance at end of the month**
Jan	$0	$0	$1000	$1000
February	$1000	$5.00	$1000	$2005
March	$2005	$10.025 = $10.03	$1000	$3015.03
April	$3015.03	$15.075 = $15.08	$1000	$4030.11
May	$4030.11	$20.15	$1000	$5050.26
June	$5050.26	$25.25	$1000	$6075.51

Fig. 4

Since this is an ordinary annuity the payments are made at the **end** of the month. Interest is calculated as simple interest $I = Prt$ where r = 0.06 and t = 1/12 year and P = the balance at the beginning of the month.

The first payment of $1000 is deposited at the end of January and therefore no interest is earned for the month of January. At the beginning of February there is $1000 in the account. Interest for the month of February is I = $1000(0.06)(1/12) = $5. At the end of February there will be a deposit of $1000 so the balance at the end of February is $1000 + $5 + $1000 = $2005.

At the beginning of March there is $2005 in the account. Interest for the month of March will be I = $2005(0.06)(1/12) = $10.025 or $10.03 (rounded off). At the end of March $1000 is deposited so the balance at the end of March is $2005 + $$10.03 + $1000 = $3015.03.

At the beginning of April there is $3015.03 in the account. Interest for the month of April will be I = $3015.03(0.06)(1/12) = $15.075 or $15.08 (rounded off). At the end of April $1000 is deposited so the balance at the end of April is $3015.03 + $$15.08 + $1000 = $4030.11.

The process is continued for the remaining two months. At the end of June, the balance will be $6075.51. This is the **future value** of the annuity, which is the total of all annuity payments and the accumulated compound interest.

Notice that with an **ordinary annuity** the interest calculation is based on the balance at the beginning of the interval. Since the first payment does not occur until the end of the first payment interval there will not be any interest in the first payment interval. Although the term of the annuity is six months there will only be five intervals where interest is calculated.

With an **annuity due** the payment would be made at the beginning of each payment interval so for a six month term there would be six interest calculations. Since many loans are set up as an annuity due it is advantageous to the lending institution (but not to the loan recipient).

The calculation of the future value of an annuity can be very time consuming. Fortunately there is a formula for this.

Ordinary Annuity Formula

The **ordinary annuity formula** is:

$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$	where	A = amount of annuity (Future Value) P = periodic payment amount r = annual interest rate n = number of compounding periods per year t = time (in years)

It is important to note that there are variations on how the ordinary annuity formula is written. This is due to the way in which the variables are defined. The formula that is provided in this section defines *r* as the **annual interest rate**, *n* as the number of compounding periods **per year**, and t as the **time in years** (term of the annuity in years).

EXAMPLE 1

Assume that the formula will be used to calculate the future value of a two year ordinary annuity that offers an annual interest rate of 6%, monthly payments of $1000, and monthly compounding.

Define each of the variables but do not calculate the future value.

Solution

$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$	where	A = amount of annuity (Future Value) = unknown P = periodic payment amount = $1000 r = annual interest rate = 6% = 0.06 n = number of compounding periods per year = 12 t = time (in years) = 2 years

Note that the term of the annuity is 2 years. The interest calculation involves monthly compounding so n = 12 since there are 12 compounding periods in a year.

TRY IT 1

Assume that the formula will be used to calculate the future value of a 1 year ordinary annuity that offers an annual interest rate of 3%, semiannual payments of $500, and semiannual compounding.

Define each of the variables but do not calculate the future value.

Show answer

Amount (A) = unknown

Payment (P) = $500

Annual interest rate (r) = 3% = 0.03

Number of compounding period (n) = 2

Time (t) = 1 year

EXAMPLE 2

Assume that the formula will be used to calculate the future value of a 6 month ordinary annuity that offers an annual interest rate of 4.8%, weekly payments of $100, and weekly compounding.

Define each of the variables but do not calculate the future value.

Solution

A = amount of annuity (Future Value) = unknown

P = periodic payment amount = $100

r = annual interest rate = 4.8% = 0.048

n = number of compounding periods per year = 52

t = time (in years) = 0.5 years

Note that although the term of the annuity is 1/2 year, the interest calculation involves weekly compounding so n = 52 since there are 52 compounding periods in a year.

TRY IT 2

Assume that the formula will be used to calculate the future value of a 9 month ordinary annuity that offers an annual interest rate of 5.5%, monthly payments of $200, and monthly compounding.

Define each of the variables but do not calculate the future value.

Show answer

A = unknown

P = $200

r = 5.5%
n = 12
t = 0.75 years

We will now use the formula to calculate the future value of a six month ordinary annuity that offers an annual interest rate of 6%, monthly payments, and monthly compounding.

EXAMPLE 3

Use the annuity formula to find the annuity amount in 6 months if $1000 is deposited monthly at 6% compounded monthly. Compare this answer to the answer obtained in the table in Figure 4.

Solution

$A = ?$ $P = 100$ $r = 6\% = 0.06$ $n = 12$ $t = 6 months = 0.5 years$	$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$	
	$A = \frac{12(100)\left[\left(1+\frac{0.06}{12}\right)^{12(0.5)}-1\right]}{0.06}$	Replace variables
	$A = \frac{1200\left[(1+0.005)^{6}-1\right]}{0.06}$	Divide and multiply
	$A = \frac{1200\left[(1.005)^{6}-1\right]}{0.06}$	Add
	$A = \frac{1200(1.0303775-1)}{0.06}$	Calculate the power
	$A = \frac{1200(0.0303775)}{0.06}$	Substract
	$A = \frac{36.45301}{0.06} = 607.55$	Multiply and divide

The annuity is worth **$6075.55**. This answer is $0.04 different than in the table in Figure 4 due to rounding off .

TRY IT 3

Use the formula to calculate the future value of a 9 month ordinary annuity at an annual interest rate of 3%, monthly payments of $50, and monthly compounding.

Show answer
Future value = $454.53

EXAMPLE 4

Use the annuity formula to find the annuity amount in 4 years if $500 is deposited semiannually at 3.6% compounded semiannually.

Solution

$A = ?$ $P = 500$ $r = 3.6\% = 0.036$ $n = 2$ $t = 4 years$	$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$	
	$A = \frac{2(500)\left[\left(1+\frac{0.036}{2}\right)^{2(4)}-1\right]}{0.036}$	Replace variables
	$A = \frac{1000\left[(1+0.018)^{8}-1\right]}{0.036}$	Divide and multiply
	$A = \frac{1000\left[(1.018)^{8}-1\right]}{0.036}$	Add
	$A = \frac{1000(1.153406-1)}{0.036}$	Calculate the power
	$A = \frac{1000(0.153406)}{0.036}$	Subtract
	$A = \frac{153.41}{0.036} = 4261.28$	Multiply and divide

The annuity is worth **$4261.28**.

TRY IT 4

Use the formula to calculate the future value of a 5 year ordinary annuity that offers an annual interest rate of 4.8%, semiannual payments of $4000, and semiannual compounding.

Show answer
Future value = $44,608.43

The future value or annuity amount includes all payments and compound interest. To determine the total interest we must subtract the total value of all annuity payments from the future value of the annuity.

Interest on an Annuity

Interest on an Annuity =

Future Value of the Annuity – Total Value of the Payments =

Future Value of the Annuity – (Payment amount × number of payments per year × number of years)

EXAMPLE 5

Consider Example 3. For a 6-month annuity where $1000 is deposited monthly the value of the annuity at the end of 6 months is 6075.51. Determine the total interest earned on the annuity.

Solution

Interest Earned = Future Value of the Annuity – Total Value of the Payments (Deposits) = $6075.51 – ($1000 x 6 payments) = $75.51.

Refer to the table in Figure 4 to confirm that the column “interest earned” adds to this identical amount.

TRY IT 5

Refer to Try It 3. For a 9 month ordinary annuity with monthly payments of $50, determine the total interest earned on the annuity.

Show answer
Future value = $4.53

EXAMPLE 6

a) How much would an annuity be worth in 2 years at 2.4% compounded monthly if the periodic payments are $40 per month?

b) Determine the total interest earned on the annuity.

Solution:

a)

$A = ?$ $P = \$40$ $r = 2.4\% = 0.024$ $n = 12$ $t = 2$	$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$
	$A = \frac{12(40)\left[\left(1+\frac{0.024}{12}\right)^{12\times 2}-1\right]}{0.024}$
	$A = \frac{480\left[(1+0.002)^{24}-1\right]}{0.024}$
	$A = \frac{480\left[(1.002)^{24}-1\right]}{0.024}$
	$A = \frac{480[(1.04912-1]}{0.024}$
	$A = \frac{480[(0.04912]}{0.024}$
	$A = \frac{23.58}{0.024} = 982.41$

The annuity is worth **$982.41** after 2 years.

b) The total interest earned is: $982.41 – ($40 × 12payments/yr × 2yr) = $982.41 – $960 = $22.41

TRY IT 6

Consider a ten-year ordinary annuity that offers an annual interest rate of 4.5%, semiannual payments of $1000, and semiannual compounding

a) How much would the annuity be worth in 10 years?

b) Determine the total interest earned on the annuity.

Show answer
Future value = $24911.52
Interest = $24911.52 – ($1000)(10 years)(2 payments/year) = $24911.52 – $20000 = $4911.52

EXAMPLE 7

Tish plans to go back to university and opens an account into which she will deposit $300 at the end of every month for 4 years. The account offers an annual interest rate of 4.8% compounded monthly. Assuming a fixed interest rate and no additional deposits or withdrawals, how much will be in the account at the end of 4 years? How much interest will Tish earn in the 4 years?

Solution

$A=?$ $P=\$300$ $r=4.8\%=0.048$ $n=12$ $t=4$	$A=\frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$
	$A=\frac{12(300)\left[\left(1+\frac{0.048}{12}\right)^{12(4)}-1\right]}{0.048}$
	$A=\frac{3600\left[(1+0.004)^{48}-1\right]}{0.048}$
	$A=\frac{3600\left[(1.004)^{48}-1\right]}{0.048}$
	$A=\frac{3600[(1.21121-1]}{0.048}$
	$A=\frac{3600[(0.21121]}{0.048}$
	$A=\$15,840.75$

Interest	$=\$15,840.75-(\$300)\,(4\,years)\,(12\,payments/year)$
	$=\$15,840.75-\$14,400$
	$=\$1440.75$ in interest

TRY IT 7

A credit union is offering 6.8% compounded monthly on a savings account. If you deposit $100 at the end of every month for two years (assume no withdrawals) how much will be in the account at the end of two years? How much interest will you earn?

Show answer
Amount in 2 years = $2563.10
Interest = $2563.10 – ($100)(2 years)(12 payments/year) = $163.10

Determining the Annuity Payment

Businesses and individuals often wish to accumulate a certain amount of money by making regular deposits (payments) into an annuity. Perhaps an individual or business wishes to purchase a larger ticket item such as an appliance or a piece of equipment in one year's time. Rather than taking out a loan they could choose to deposit a specific amount every month so as to accumulate the required funds by the end of the one year. The amount that needs to be deposited is represented by the **payment** in the annuity formula. The formula must be solved for the payment (P).

Periodic Payment Formula

The **periodic payment formula** is:

$$P = \frac{A(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}$$

P = periodic payment amount

A = annuity amount

r = annual interest rate

n = number of times interest is calculated in a year

t = time (in years)

EXAMPLE 8

A home bakery wants to purchase a new oven in one year's time. The oven is estimated to cost $5000. The baker has found an account that offers 3.2% monthly compounding.

a) How much must be deposited at the end of each month to accumulate to the $5000?

b) What is the total amount that the baker deposits over the one year?

c) How much of the $5000 is interest?

Solution

P = ? A = \$5000 r = 3.2% = 0.032 t = 12 n = 1	$P = \frac{A(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}$
	$P = \frac{5{,}000(\frac{0.032}{1})}{(1+\frac{0.032}{1})^{1(12)}-1}$
	$P = \frac{160}{(1.032)^{12}-1}$
	$P = \frac{160}{0.4593}$ = \$348.33

a) The baker must deposit **\$348.33** per month.

b) The baker deposits \$348.33/mth × 12 months = \$4179.96 in one year.

c) Since there is \$5000 in the account at the end of the year, the interest component will be:

\$5000 – \$4179.96 = \$820.04

TRY IT 8

Cara is saving to start college in three years and hopes to have saved \$12,000 in three years. She opens an account offering 4.8% compounded monthly.

a) How much must Cara deposit at the end of each month to accumulate to the \$12000?

b) What is the total amount that Cara deposits over the three years?

c) How much of the \$12,000 is interest?

Show answer
a) Monthly deposit must be \$310.57 b) Total deposited is \$11,180.67 c) \$819.33

EXAMPLE 9

Sara hopes to accumulate \$140,000 in 12 years. She has found an annuity that offers 8% annual compounding and requires that she make a deposit at the end of each year.

a) How much must Sara deposit at the end of each year to accumulate to the \$140,000?

b) What is the total amount that Sara deposits over the twelve years?

c) How much of the $140,000 is interest?

Solution:

a)

P = ? *A* = $140,000 *r* = 8% = 0.08 *t* = 12 *n* = 1	$P = \frac{A(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}$
	$P = \frac{140{,}000(\frac{0.08}{1})}{(1+\frac{0.08}{1})^{1(12)}-1}$
	$P = \frac{11{,}200}{(1.08)^{12}-1}$
	$P = \frac{11{,}200}{1.51817}$ = $7,377.30

Sara must deposit **$7,377.30** per year

b) 12 years × $7377.30/yr = $88527.60

c) Interest = $140000 – $88527.60 = $51472.40

TRY IT 9

Zach is saving to go on a trip in one year's time. He hopes to have $3200 at the end of one year so he makes monthly deposits into an account offering 2.4% compounded monthly.

a) How much must Zach deposit at the end of each month to accumulate to the $3200?

b) What is the total amount that Zach deposits over the twelve months?

c) How much of the $3200 is interest?

Show answer
a) Monthly deposit must be $263.75 b) $3165 c) $35

EXAMPLE 10

What monthly payment is necessary for an annuity to be worth $10,000 in 3 years at 7% compounded monthly?

Solution

P = ? A = $10,000 r = 7% = 0.07 t = 3	$P = \frac{A(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}$	
n = 12	$P = \frac{10000(\frac{0.07}{12})}{(1+\frac{0.07}{12})^{12\times 3}-1}$	Replace variables with their values
	$P = \frac{10000(0.0058333)}{(1.0058333)^{36}-1}$	Divide 0.07 by 12
	$P = \frac{58.333}{1.23292-1}$	Multiply and calculate the power
	$P = \frac{58.333}{0.23292}$ = 250.42	

The periodic payment is **$250.44**

TRY IT 10

Zach has become more ambitious and is saving to go on world cruise in four years. He anticipates that the cruise will cost $38,000. How much will he need to deposit each month in an account offering 3.6% compounded monthly to accumulate to $38, 000 in four years?

Show answer
$737.22/month

Key Concepts

- to determine the future value of an ordinary annuity (A):

$$A = \frac{nP\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{r}$$

- to determine the interest earned on an annuity:

Interest on an Annuity =

Future Value of the Annuity – Total Value of the Payments

- to determine the payment, given the future value for an ordinary annuity:

$$P = \frac{A(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}$$

Glossary

annuity

is a series of payments made at fixed intervals.

annuity due

is an annuity where the payment is due at the beginning of each payment period.

future value of the annuity

is the final amount of the annuity. It is the total of all annuity payments and the accumulated compound interest.

ordinary annuity

is an annuity where the payment is due at the end of each payment period.

payment interval

is the time between successive annuity payments.

term of the annuity

is the time from the beginning of the first payment interval to the end of the last payment interval.

Exercise Set 9.4

1.

a. Complete the table below for an ordinary annuity, where $2000 is deposited annually for 5 years at 5% compounded annually.

Year	Amount at start of the year	Interest earned	Annual deposit at end the year
1	—	—	$2000
2	$2000	$100	$2000
3	$4100		
4			
5			

b. Use the ordinary annuity formula to calculate the amount at the end of the 5-year term. Do your formula and table amounts agree?

2. Assume that the formula will be used to calculate the future value of an ordinary annuity for the information provided. For each of these state: the payment amount (P), the time in years (t), the number of compounding periods (n) and the interest rate (r). Do not actually calculate the future value.
    a. A three year ordinary annuity that offers an annual interest rate of 2.8%, with semiannual deposits of $1500 and semiannual compounding.
    b. An 8 month ordinary annuity that offers an annual interest rate of 4.6%, with monthly deposits of $180 and monthly compounding.
3. Find the future value of an ordinary annuity when
    a. A periodic payment of $1000 per year earns 8% compounded annually for 10 years
    b. A payment of $100 per month earns 4% compounded monthly for 5 years
    c. A payment of $200 quarterly earns 3.82% compounded quarterly for 7 years
4. Daniel contributes $100 per month into an investment that earns 6% compounded monthly. How much money would Daniel have in:
    a. 1 year?
    b. 2 years?
    c. 5 years?
    d. 10 years?
5. The Andersons plan to retire in 25 years and want to start saving for it now. They hope to be able to earn about 10% compounded annually. Determine the amount of their annuity if they make the following periodic payments.
    a. $500 per year
    b. $1000 per year
    c. $2000 per year
    d. $3500 per year
6. The Mitchells are choosing between two ordinary annuities. They have the choice of either contributing $1200 a year at 10% compounded annually for 25 years or contributing $100 per month at 10% compounded monthly for 25 years.
    a. How much would the Mitchells have in 25 years if they make annual contributions?
    b. How much would the Mitchells have in 25 years if they make monthly contributions?
    c. Which investment (yearly or monthly) would earn the greater amount and by how much?
7. The Gardners plan to save for their child's education by depositing $40 a month into a special savings plan which pays 8% compounded monthly.
    a. How much would the annuity be worth after 1 year?

   b. How much after 18 years?

8. Imagine you start saving for your retirement and contribute $1000 yearly and average 6.4% compounded annually. The amount of the annuity depends on the length of the annuity. Complete the table below.

Years	Annuity amount
10	
15	
20	
25	
30	

9. In question 8 above, what is the effect of saving for your retirement over a 30 year period as opposed to a 10 year period?
10. Find the periodic payment needed to accumulate to an annuity amount of:
    a. $1000 at 5% compounded monthly for 1 year
    b. $20,000 at 10% compounded yearly for 15 years
    c. $5000 at 8% compounded quarterly for 3 years
11. Mike wants to buy a $1500 stereo 9 months from now. How much will he have to deposit every month into a savings plan paying 6.5% compounded monthly?
12. You would like to save $3500 in two years. What monthly payment would you have to make if your investment can earn 5% compounded monthly?
13. The Wests need $60000 for their child's education 6 years from now. How much should they put aside every month if they hope to earn 4% compounded monthly?
14. Paul wants to save $20,000 in order to purchase a vehicle in 4 years time. He plans to make equal monthly contributions for 4 years. He found an annuity offering 2.6% compounded monthly and was about to commit but then found another option offering 3.4% compounded monthly.
    a. Determine the monthly payments for each of the two options.
    b. Determine the total amount of money that Paul saved for other uses by finding the account offering 3.4%.
15. Imagine you wanted to be a millionaire 30 years from now. How much would you have to contribute to an ordinary annuity every year if you think you could earn 12% compounded yearly?

Answers

1. a.

Start Date	Amount at start of the year	Interest earned	Annual deposit at end of the year	Amount at end year
1	—	—	$2000	$2000
2	$2000	$100	$2000	$4100
3	$4100	$205	$2000	$6305
4	$6305	$315.25	$2000	$8620.25
5	$8620.25	$431.0125	$2000	$11051.26

   b. $11,051.26 The answers should be the same.

2. a. P = $1500 t = 3 years n = 2 r = 2.8% = 0.028
   b. P = $180 t = 2/3 years n = 12 r = 4.6% = 0.046
3. a. $14 486.56
   b. $6629.90
   c. $6385.47
4. a. $1233.56
   b. $2543.20
   c. $6977.00
   d. $16 387.93
5. a. $49 173.53
   b. $98 347.06
   c. $196 694.12
   d. $344 214.71
6. a. $118 016.47
   b. $132 683.33
   c. Monthly by $14 666.87
7. a. $498.00
   b. $19 203.45

8.

Years	Annuity Amount
10	$13431.03
15	$23997.73
20	$38407.19
25	$58056.88
30	$84852.51

9. The annuity is worth $(84852 \div 13431)$ over 6 times more
10.
    a. $81.44/month
    b. $629.48 per year
    c. $372.80 per quarter
11. $163.09 per month
12. $138.97 per month
13. $738.71 per month
14.
    a. For the 2.6% account payments of $395.83/mth; For the 3.4% account payments of $389.56/mth
    b. 48 months of saving a difference of $6.27 provided $300.96 extra for Paul
15. $4143.65 per year

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9.5 Loans and Mortgages

Learning Objectives

By the end of this section it is expected that you will be able to:

- Determine the periodic payments on an installment loan
- Determine the amount financed and the finance charge on an installment loan
- Determine the payments and finance charge on a mortgage

Installment Loans

A loan is something that is borrowed. In the case where this is a sum of money the amount that will be paid by the borrower will include the original amount plus interest.

Some loans require full payment on the **maturity date** of the loan. The maturity date is when all principal and/or interest must be repaid to the the lender. Consider a one year loan of $1000 at a simple interest rate of 5%. At the end of one year (the maturity date) the borrower will pay back the original $1000 plus the interest of $50 for a total of $1005.

For major purchases such as vehicles or furniture there is a different type of loan, called the **installment loan**. The average consumer cannot afford to pay $25000 or more for a new vehicle and they may not want to wait three or four years until they have saved enough money to do so. The qualifying consumer has the option of paying for the item with an **installment loan.**

Installment loans do not require full repayment of the loan on a specific date. With an installment loan the borrower is required to make regular (installment) payments until the loan is paid off. Each **installment payment** will include an interest charge. An installment loan can vary in length from a few years to perhaps twenty years or more (in the case of real estate).

Consider an installment loan for a $4000 television. The purchaser takes out a $4000 loan with a four-year term at an interest rate of 4.5%. The monthly installment payments will be $91.21. Although the television has a purchase price of $4000, the total cost to the purchaser will be more than $4000. The total of the installment payments will be:

Total Installment Payments = Number of Installment Payments x Payment Amount =

4 years x 12 payments/year x $91.21/mth = $4378.08

The $4000 television ends up costing $4378.08 because the consumer is charged interest. Each payment includes an interest component that adds to the overall cost of the item. The total of the interest charges is referred to as the **finance charge** on the loan.

Finance Charge

The **finance charge** is the sum of the interest charges on a loan. These interest charges are embedded in the installment payments. To calculate the finance charge:

Finance Charge = Total Installment Payments – Loan Amount

= (Number of Installment Payments x Payment Amount) – Loan Amount

For the $4000 television the finance charge will be calculated as follows:

Finance charge = Total Installment Payments – Loan Amount =

(4 years x 12 payments/year x $91.21/payment) – $4000 = $4378.08 – $4000 = $378.08

Over the 4-year term of the loan the purchaser will have paid the $4000 loan amount plus an additional $378.08 in interest (the finance charge).

Sometimes the borrower will make an **initial payment** at the time of purchase. This is called a **down payment**. When a down payment is made the remaining amount is the **amount financed** or the **loan amount.**

Amount Financed

The **amount financed** or **loan amount** is the purchase price of the item less any down payment:

Amount Financed = Purchase Price – Down Payment

Consider the $4000 television. Assume the purchaser makes a down payment of $1500.

The **amount financed** is: Purchase Price – Down Payment = $4000 – $1500 = $2500.

In this case the purchaser borrows $2500 rather than $4000. The amount financed is therefore $2500. Assuming the same 4-year term and an interest rate of 4.5%, the installment payments on the $2500 will be reduced to $57.01 per month. In this case the finance charge will be calculated as follows:

Finance charge = Total Installment Payments – Loan Amount =

(4 years x 12 payments/year x $57.01/payment) – $2500 = $2736.48 – $2500 = $236.48

With the down payment of $2500 the total finance charges will be reduced to $236.48 from $378.08.

The total cost of the television to the purchaser will be:

Purchase Price + Finance Charge

= $4000 + $236.48 = $4236.48

Alternatively we can calculate:

Total Installment Payment + Down Payment

= $2736.48 + $1500 = $4236.48

As one can see, the finance charges are a hidden but added cost. This cost will become more pronounced with more expensive purchases such as with real estate.

Installment Loan Terminology

Total Installment Payments = Number of payments x Payment Amount

Finance Charge = Total Installment Payments – Loan Amount

Amount Financed or **Loan Amount** = Purchase Price of Item – Down payment

EXAMPLE 1

Paul purchased a home entertainment system at a total cost of $6000. He obtained a 3 year loan at an interest rate of 7.5%. His monthly payments will be $186.64 over three years.

a) State the amount financed.

b) Determine the total installment payments.

c) Determine the finance charge.

Solution

a) Since there was no down payment the amount financed (or loan amount) will be $6000.

b) The total installment payments will be:

Number of payments x Payment Amount

= 3 years x 12 payments/year x $186.64

= $6719.04

c) Finance Charge = Total installment payments – Loan Amount

= $6719.04 – $6000

= $719.04

TRY IT 1

Cassie purchased a new washer and dryer at a total cost of $3800. She obtained a 4 year loan at an interest rate of 6.2%. Her monthly payments will be $89.59 over four years.

a) State the amount financed.

b) Determine the the total installment payments.

c) Determine the finance charge.

Show answer
a) $3800.00 b) $4300.32 c) $500.32

EXAMPLE 2

Mike purchased a home entertainment system at a total cost of $6000. He made a down payment of $1800 and to pay the balance he obtained a 3 year loan at an interest rate of 7.5%. His monthly payments will be $130.65 over three years.

a) State the amount financed.

b) Determine the total installment payments.

c) Determine the finance charge.

d) Determine the total amount that Mike paid for the home entertainment system

Solution

a) Amount Financed = Cost of Item – Down Payment

= $6000 – $1800 = $4200

b) The total installment payments will be:

Number of payments x Payment Amount = 3 years x 12 payments/year x $130.65

= $4703.40

c) Finance Charge = Total installment payments – Loan Amount

= $4703.40 – $4200

= $503.40

d) Total paid = Purchase Price + Finance Charge = $6000 + $503.40 = $6503.40

TRY IT 2

Carl purchased a new washer and dryer at a total cost of $3800. He made a down payment of $1500 and obtained a 2 year loan for the remaining amount at an interest rate of 6.2%. His monthly payments will be $102.14 over two years.

a) State the amount financed.

b) Determine the total installment payments.

c) Determine the finance charge.

d) Determine the total amount that Carl paid for the washer and dryer.

Show answer

a) $2300.00 b) $2451.36

c) $151.36 d) $3951.36

Loan Payments

When consumers obtain installment loans they often just trust the lender to determine the installment (periodic) loan payments. In Example1 Paul purchased a home entertainment system at a total cost of $6000. He obtained a three year loan at an interest rate of 7.5%. If Paul attempts to calculate his monthly payment by simply dividing the loan amount by the number of payments he will underestimate his monthly payment as he has ignored the interest component:

$$\$6000 \div 36 = \$166.67$$

Paul's actual monthly payment of $186.64 is slightly higher than Paul's estimate because of the interest component.

The actual amount of a periodic loan payment can be determined using a formula, a table or technology. In this section we will illustrate the use of a formula.

Periodic Payment on a Loan

The **periodic payment on a loan formula** is:

$$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$$

P = periodic payment amount
A = amount of loan
r = annual interest rate (in decimal form)
n = number of payments made in one year
t = time (in years)

EXAMPLE 3

Refer back to the purchase of a television for $4000. The purchaser agrees to a 4 year term at an interest rate of 4.5%. a) Use the formula to determine the monthly installment payment b) Determine the total installment payments

Solution

a)

$$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$$

where P = payment (unknown), A = $4000, r = 4.5%, n = 12, t = 4 years

$$P = \frac{4000(\frac{0.045}{12})}{1-(1+\frac{0.045}{12})^{-12(4)}} = \frac{15}{1-(1.00375)^{-48}} = \frac{15}{0.16445} = 91.21$$

The monthly payment is confirmed to be $91.21.

b) Total installment payments = monthly payment amount x no. of payments

$91.21 x 48= $4378.08

TRY IT 3

A dining room table set is purchased for $5600. The purchase is financed with a 3 year loan at an interest rate of 12.5%. a) Use the formula to determine the monthly installment payment b) Determine the total installment payments.

Show answer

Monthly payment is $187.34; Total Installment payments $=187.34 \times 36 = 6744.24$

EXAMPLE 4

Paul purchased a home entertainment system at a total cost of $6000. He obtained a 3 year loan at an interest rate of 7.5%. Use the formula to determine his monthly payments. Confirm that this matches the amount in Example 1.

Solution

$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$		P = payment (unknown)
	where	A = $6000 r = 7.5%
		n = 12 t = 3 years

$$P = \frac{6000(\frac{0.075}{12})}{1-(1+\frac{0.075}{12})^{-12(3)}} = \frac{37.5}{1-(1.00625)^{-36}} = \frac{37.5}{0.20092} = 186.64$$

The monthly payment is confirmed to be $186.64

TRY IT 4

Cassie purchased a new washer and dryer at a total cost of $3800. She obtained a 4 year loan at an interest rate of 6.2%. Use the formula to determine her monthly payments. Confirm that this matches the amount in Try It 1.

Show answer

Monthly payment of $89.59 is confirmed

EXAMPLE 5

Determine a) the annual payments b)the total installment payments and c) the finance charge on a 5 year loan of $5000 where payments are made annually and the interest rate is 6%.

Solution

a)

$$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$$

P = payment

A = $5000 r = 6%

n = 1 t = 5 years

$$P = \frac{5000(\frac{0.06}{1})}{1-(1+\frac{0.06}{1})^{-1(5)}} = \frac{300}{1-(1.06)^{-5}} = \frac{300}{0.25274} = \$1186.98$$

The annual payment will be $1186.98.

b) Total installment payments = $1186.98 x 5 = $5934.90

c) Finance charge = $5934.90 – $5000 = $934.90

TRY IT 5

Determine a) the annual payments b) the total installment payments and c) the finance charge on a 5 year loan of $5000 where payments are made monthly and the interest rate is 6%.

Show answer

a) Monthly payment is $96.67

b) Total Installment payments = $5800.20

c) Finance charge $800.20

Recall that interest is calculated only on the loan amount and not on any downpayment. When determining the periodic payment on an installment loan be sure to exclude the downpayment when calculating the periodic payment.

EXAMPLE 6

Mike purchased a home entertainment system at a total cost of $6000. He made a down payment of $1800 and to pay the balance he obtained a 3 year loan at an interest rate of 7.5%. Use the formula to determine his monthly payments. Confirm that this matches the amount provided in Example 2.

Solution

$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$		P = payment (unknown)
	where	A = $4200 r = 7.5%
		n = 12 t = 3 years

$$P = \frac{4200(\frac{0.075}{12})}{1-(1+\frac{0.075}{12})^{-12(3)}} = \frac{26.25}{1-(1.00625)^{-36}} = \frac{26.25}{0.20092} = 130.65$$

The monthly payment is confirmed to be $130.65

TRY IT 6

Carl purchased a new washer and dryer at a total cost of $3800. He made a down payment of $1500 and obtained a 2 year loan for the remaining amount at an interest rate of 6.2%. Use the formula to determine his monthly payments. Confirm that this matches the amount provided in Try It 2.

Show answer
Monthly payment of $102.14 is confirmed

EXAMPLE 7

Pat has decided to purchase a used vehicle that costs $12,500. He considers two options. For each option, determine a) the monthly payment b) total installment payments c) the finance charge for each option. What is the difference in the finance charge with the down payment?

Option 1) Paying the full amount with a 4 year loan, monthly payments, and an interest rate of 6.8%.

Option 2) He will cancel a planned trip and and instead make a $3500 down payment on the purchase. He will pay the remaining amount with a 4 year loan, monthly payments, and an interest rate of 6.8%.

Solution

Option 1)

a) P = unknown A = $12,500

r = 0.068 n = 12 t = 4

$P = \frac{12500(\frac{0.068}{12})}{1-(1+\frac{0.068}{12})^{-(12)(4)}}$	$= \frac{70.8333}{1-(1.005667)^{-48}}$
	$= \frac{70.8333}{1-0.76244}$
	$= \frac{70.8333}{0.23756}$
	$= \$298.17$ payment

b) Total Installment payments $= \$298.17 \times 4 \times 12 = \$14,312.16$

c) Finance charge = Total Installment Payments – Loan Amount = $14312.16 – $12,500 = $1812.16

Option 2)

a) P = unknown   A = $12,500 – $3500 =$9000

r = 0.068   n = 12   t = 4

$P = \frac{9000(\frac{0.068}{12})}{1-(1+\frac{0.068}{12})^{-(12)(4)}}$	$= \frac{51}{1-(1.005667)^{-48}}$
	$= \frac{51}{0.23756}$
	$= \$214.68$ payment

b) Total Installment payments $= (\$214.68 \times 4 \times 12) = \$10,304.64$

c) Finance charge = Total Installment Payments – Loan Amount = $10,304.64 – $9000 = $1304.64

With a down payment there will be a savings of **$507.52** on the finance charges.

TRY IT 7

Mick has decided to purchase a home entertainment system at a cost of $9200. He considers two options. For each option determine a) the monthly payment b) total installment payments c) the finance charge for each option. What is the difference in the finance charge with the down payment?

1) Paying the full amount with a 3 year loan that offers an interest rate of 8.4%.

2) Forgoing the purchase of a new electric bike and instead makinga $2000 down payment on the bike purchase. He will pay the remaining amount with a 3 year loan at an interest rate of 8.4%.

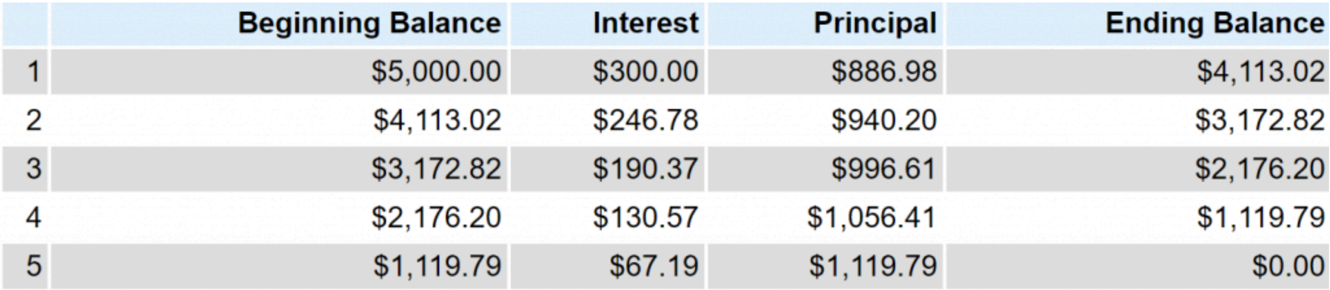
Show answer
With no down payment: a) $290 b) $10440 c) $1239.83

With a down payment a) $226.95 b) $10170.20 c) $970.30; With the down payment the finance charge is $269.53 less

Amortization

Amortization is the process of spreading out a loan into a series of fixed payments. A portion of each payment will be applied to the interest charge and a portion will be applied to the principal amount of the loan. Although each payment is equal, the amount that applies to the interest versus the prinipal will change with each payment period. We can get a better sense of the impact that a loan payment has by examining the amortization schedule for a loan.

Consider the amortization table for the installment loan in Example 5. Recall that the loan amount is $5000 at 6% for 5 years and annual payments are $1186.98. Note then that for each year the sum of the interest and principal is equivalent to the payment of $1186.98. Refer to Figure 1 for the amortization schedule of this loan.



	Beginning Balance	Interest	Principal	Ending Balance
1	$5,000.00	$300.00	$886.98	$4,113.02
2	$4,113.02	$246.78	$940.20	$3,172.82
3	$3,172.82	$190.37	$996.61	$2,176.20
4	$2,176.20	$130.57	$1,056.41	$1,119.79
5	$1,119.79	$67.19	$1,119.79	$0.00

Fig. 1

To calculate the interest (I) we use the simple interest formula I = P*rt* . The principal (P) will be the beginning balance for each year. The time in years is the portion of the year for which interest is being calculated. In this example the time (t) is one year and the interest rate is 6%.

In **Year 1** the interest on the loan of $5000 will be:

I = Prt = $5000 x 0.06 x 1yr = $300.

The periodic payment amount is $1186.98 and the portion that will go towards interest is $300.

The portion that will go towards paying down the principal will be:

periodic payment amount – interest =

$1186.98 – $300 = $886.98.

Although the payment was $1186.98, only $886.98 will be applied to the outstanding loan amount. At the end of year 1 the remaining balance on the loan will be:

beginnining balance – portion applied to the principal

= $5000 – $886.98 = $4113.02

In **Year 2** the beginning balance on the loan is $4113.02. The interest on the loan will be:

I = Prt = $4113.02 x 0.06 x 1yr = $246.78.

Note that interest is calculated on the remaining balance of the loan, not on the original $5000. For the periodic payment of $1186.98, the portion that will go towards interest is $246.78.

The portion that will go towards paying down the principal will be:

periodic payment amount – interest

= $1186.98 – $246.78 = $940.20.

At the end of year 2 the remaining balance on the loan will be:

beginnining balance – portion applied to the principal

= $4113.02 – $940.20 = $3172.82

In **Year 3** the beginning balance on the loan is $3172.82. The interest on the loan will be:

I = Prt = $3172.82 x 0.06 x 1yr = $190.37

The portion that will go towards paying down the principal will be:

periodic payment amount – interest
= $1186.98 – $190.37 = $996.61

At the end of year 3 the remaining balance on the loan will be:

beginnining balance – portion applied to the principal
= $3172.82 – $996.61 = $2176.21

The cycle repeats for five years until the loan is paid off. If we add the interest charges in the table they will total to $934.91. This is the same as the finance charge (ignoring the 1¢ difference due to rounding) that was calculated in Example 5.

The amortization table illustrates that in the early periods of the loan a larger portion of the payment goes towards interest and a smaller portion contributes to paying down the principal (loan) amount. Over time a larger portion of the payment will be applied towards paying down the balance on the loan. For large purchases it can take several payment periods before the payment contributes substantailly to the principal balance of the loan. A down payment is beneficial as it will reduce the total finance charge.

Mortgages

A long term loan that is used for the purchase of a house is called a **mortgage.** It is called a mortgage because the lending agency requires that the house be used as **collateral** for the loan. This means that if the mortgage holder is unable to make the payments the lender can take possession of the house.

Mortgages generally tend to be for longer time periods than an installment loan and the terms of the mortgage will often change over the course of the mortgage. Take for example the purchase of a house with a twenty year mortgage. The purchaser might sign a mortgage agreement for a five year term. The mortgage agreement will include the interest rate, the frequency of payments and additional rules which may allow the mortgage holder to make lump sum payments or change the payment amount. At the end of the five year term a new agreement will be required and the conditions of the mortgage usually change.

Although it is possible to do the calculations manually, that is beyond the scope of this book. We will use technology to calculate the periodic payments and interest charges and to generate an amortization schedule.

Example 8 will illustrate that amortizing a mortgage is similar to amortizing other loans except that the mortgage amortization generally involves many more payment periods.

EXAMPLE 8

A $400,000 home is purchased with a 20% down payment on a 20-year mortgage at a fixed interest rate of 3.4%.

a) Determine the down payment.

b) Use an online mortgage calculator to determine the monthly payment and the total interest paid.

c) Generate an **annual** amortization schedule.

d) Determine the total payments for one year

e) Use the table to determine how much of the first year's payments will go towards interest and how much will go towards the principal.

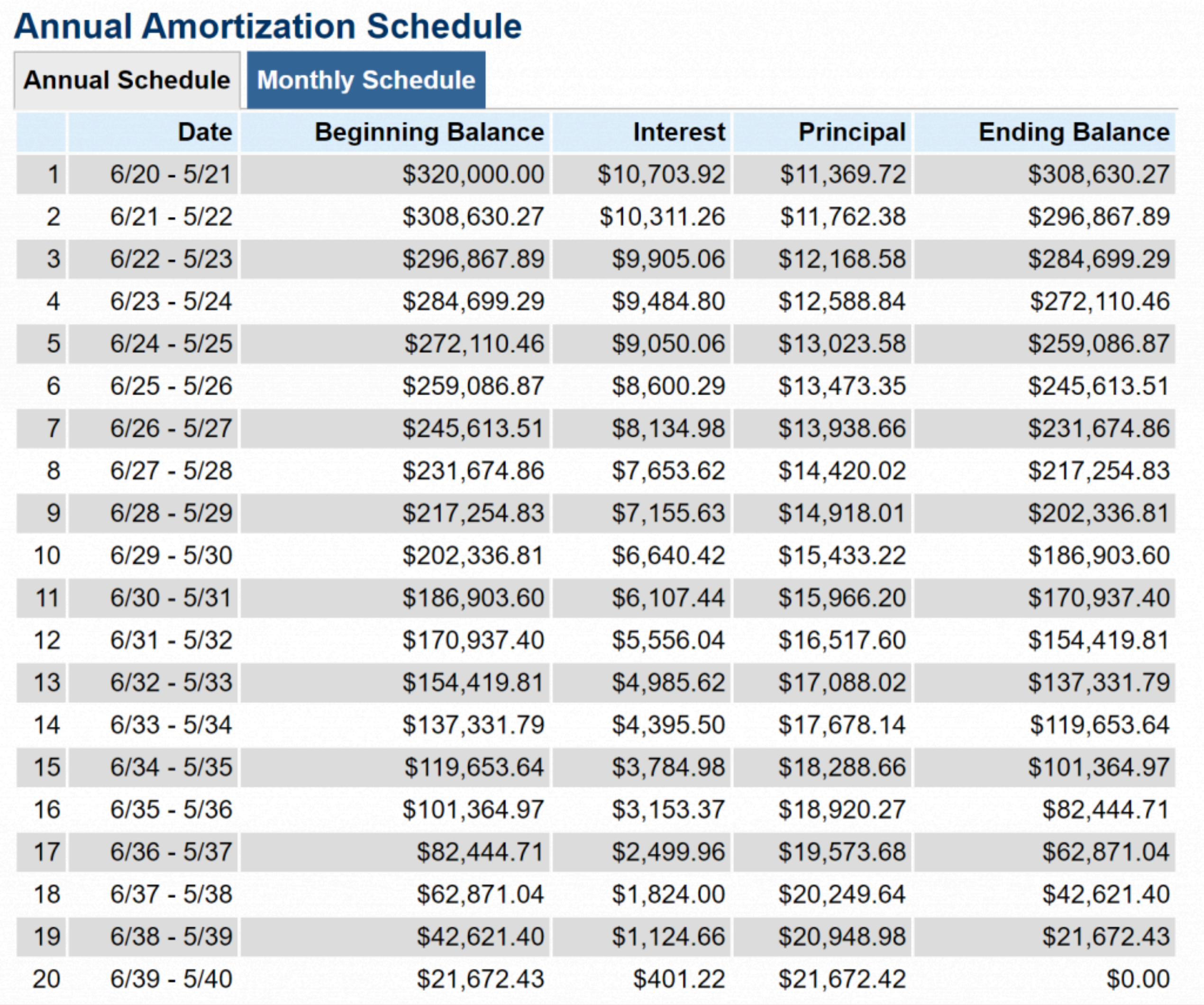
f) Use the table to determine how much of the final year's payments will go towards interest and how much will go towards the principal.

Solution:

a) The down payment will be 20% x $400,000 = $80,000.

b) The monthly payment will be $1839.47 and the total interest will be $121, 472.75.

c)



	Date	Beginning Balance	Interest	Principal	Ending Balance
1	6/20 - 5/21	$320,000.00	$10,703.92	$11,369.72	$308,630.27
2	6/21 - 5/22	$308,630.27	$10,311.26	$11,762.38	$296,867.89
3	6/22 - 5/23	$296,867.89	$9,905.06	$12,168.58	$284,699.29
4	6/23 - 5/24	$284,699.29	$9,484.80	$12,588.84	$272,110.46
5	6/24 - 5/25	$272,110.46	$9,050.06	$13,023.58	$259,086.87
6	6/25 - 5/26	$259,086.87	$8,600.29	$13,473.35	$245,613.51
7	6/26 - 5/27	$245,613.51	$8,134.98	$13,938.66	$231,674.86
8	6/27 - 5/28	$231,674.86	$7,653.62	$14,420.02	$217,254.83
9	6/28 - 5/29	$217,254.83	$7,155.63	$14,918.01	$202,336.81
10	6/29 - 5/30	$202,336.81	$6,640.42	$15,433.22	$186,903.60
11	6/30 - 5/31	$186,903.60	$6,107.44	$15,966.20	$170,937.40
12	6/31 - 5/32	$170,937.40	$5,556.04	$16,517.60	$154,419.81
13	6/32 - 5/33	$154,419.81	$4,985.62	$17,088.02	$137,331.79
14	6/33 - 5/34	$137,331.79	$4,395.50	$17,678.14	$119,653.64
15	6/34 - 5/35	$119,653.64	$3,784.98	$18,288.66	$101,364.97
16	6/35 - 5/36	$101,364.97	$3,153.37	$18,920.27	$82,444.71
17	6/36 - 5/37	$82,444.71	$2,499.96	$19,573.68	$62,871.04
18	6/37 - 5/38	$62,871.04	$1,824.00	$20,249.64	$42,621.40
19	6/38 - 5/39	$42,621.40	$1,124.66	$20,948.98	$21,672.43
20	6/39 - 5/40	$21,672.43	$401.22	$21,672.42	$0.00

d) In one year the total payments will be 12 x $1839.47 = $22,073.64.

e) Of the first year's payments, almost half, $10,703.92, will go towards interest. $11,369.72 will go towards paying down the principal.

f) Of the final year's payments, $401.22 will go towards interest. $21, 672.42 will go towards the principal.

TRY IT 8

A 20-year mortgage is obtained to purchase a $550,000 home with a 15% down payment at a fixed interest rate of 4.6%.

a) Determine the down payment.

b) Use an online mortgage calculator to determine the monthly payment and the total interest paid.

c) Generate an **annual** amortization schedule.

d) Determine the total payments for one year

e) Use the table to determine how much of the first year's payments will go towards interest and how much will go towards the principal.

f) Use the table to determine how much of the final year's payments will go towards interest and how much will go towards the principal.

Show answer

a) The down payment will be $82,500

b) the monthly payment will be $2982.93 and the total interest will be $248,403.36

d) In the first year the total payments will be $35,795.16.

e) In the first year $21,199.84, will go to interest. $14,595.32 will go towards paying down the principal.

f) In the final year $876.17 will go to interest. $34,918.99 will go towards paying down the principal.

EXAMPLE 9

A young couple have received an inheritance and they now have enough money for a down payment on their first home. They plan to take out a 25 year mortgage at an interest rate of 3.8%. They are considering a new house for $750,000 or a smaller older home for $380,000. If they purchase the larger house they plan to make a 20% down payment. With the less expensive smaller house they can afford a 35% down payment.

a) Use an online mortgage calculator to determine the down payment, the monthly payment and the total interest paid for each of the two houses.

b) For each of the houses, what is the principal balance owing after 5 years?

Solution

a) $750,000 house: $150,000 down payment; $3101.14 monthly payment; Total interest $330,341.81

$380,000 house: $133,000 down payment; $1276.64 monthly payment; Total interest $135,990.71

b) $750,000 house: After 5 years the balance owing is $520,767.80

$380, 000 house: After 5 years the balance owing is $214,382.74

TRY IT 9

A couple has won $50,000 in the lottery and they decide to put this towards the purchase of a vacation cottage or a house. They plan to make a 10% down payment and are considering a 25 year mortgage at a rate of 2.9%. They are deciding between the purchase of a cottage for $500,000 or a house for $880,000.

a) Use an online mortgage calculator to determine the down payment, the monthly payment and the total interest paid for the cottage and for the house.

b) For each of the cottage and the house, what is the principal balance owing after 5 years?

Show answer

a) Cottage: The down payment will be $50,000, the monthly payment will be $2110.62 and the total interest will be $183,185.76

House: The down payment will be $88,000, the monthly payment will be $3714.69 and the total interest will be $322,406.93

b) Cottage: After 5 years the balance owing is $384,024.74

House: After 5 years the balance owing is $675,883.55

Key Concepts

- For an Installment Loan:
  - to determine the **total installment payments**:

    Number of Payments x Payment Amount
  - to determine the **finance (interest) charge**:

    Total Installment Payments – Loan Amount
  - to determine the **amount financed:**

    Purchase Price – Down Payment
  - to determine the **total amount paid** for the item:

    Purchase Price + Finance Charge

    or

Total Installment Payments + Down Payment

- to determine the **periodic payment P:**

$$P = \frac{A(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}$$

Glossary

amortization

is the process of spreading out a loan into a series of fixed payments.

amount financed

is the purchase price of the item less any down payment.

finance charge

is the total of the interest charges on a loan.

installment loan

is a type of loan that is repaid over time with a set number of scheduled payments (installments). The term of loan may be vary and could be few months or many years.

maturity date

is when all principal and/or interest must be repaid to the lender.

9.5 Exercise Set

1. Bette purchased a new appliance package at a total cost of $7500. She obtained a 3 year loan at an interest rate of 5.75%. Her monthly payments will be $227.32 over three years.
    a. State the amount financed.
    b. Determine the total installment payments.
    c. Determine the finance charge.
2. Paul purchased a new vehicle at a total cost of $21,300. He obtained a 5 year loan at an interest rate of 4.2%. His monthly payments will be $394.20 over five years.
    a. State the amount financed.
    b. Determine the total installment payments.
    c. Determine the finance charge.
3. Theresa purchased a home entertainment system at a total cost of $4300. She made a down payment of $1000 and to pay the balance she obtained a 2 year loan at an interest rate of 5.5%. Her monthly payments will be $145.52 over two years.

   a. State the amount financed.
   b. Determine the total installment payments.
   c. Determine the finance charge.
   d. Determine the total amount that Theresa paid for the home entertainment system.

4. The Johnsons purchased a new vehicle at a total cost of $32,500. They made a down payment of $5000 and to pay the balance they obtained a 4 year loan at an interest rate of 3.6%. The monthly payments will be $616.01 over four years.
   a. State the amount financed.
   b. Determine the total installment payments..
   c. Determine the finance charge.
   d. Determine the total amount that the Johnsons paid for the vehicle.
5. Determine the monthly (periodic) payment and finance charge for each of the following installment loans.

Annual Interest Rate	Number of Years	Loan Amount	Monthly Payment	Finance Charge
2.8%	1	$2000		
4%	2	$4200		
5%	3	$5200		
4.5%	3	$8000		
6.5%	4	$11,000		

6. A dining room set is purchased for $2300. The purchase is financed with a 2 year loan at an interest rate of 6.4%
   a. Use the formula to determine the monthly payment
   b. Determine the total installment payments.
   c. Determine the finance charge.
7. A new vehicle is purchased for $32, 000. The purchase is financed with a 5 year loan at an interest rate of 4.8%.
   a. Use the formula to determine the monthly payment
   b. Determine the total installment payments.
   c. Determine the finance charge.
8. The Connors purchase a hot tub for a total price of $8500. They make a downpayment of $2300 and finance the remainder with a 3 year loan at an interest rate of 2.6%.
   a. Determine the loan amount

b. Use the formula to determine the monthly payment

c. Determine the total installment payments.

d. Determine the finance charge.

e. How much in total did the Connors actually pay for the hot tub?

9. The Tanners purchase a small RV for a total price of $48,000. They make a downpayment of $8000 and finance the remainder with a 4 year loan at an interest rate of 3%.

   a. Determine the loan amount

   b. Use the formula to determine the monthly payment

   c. Determine the total installment payments.

   d. Determine the finance charge.

   e. How much in total did the Tanners actually pay for the RV?

10. Matt borrows $4000 for 4 years at an interest rate of 5%. He will make 4 annual payments.

    a. Determine the annual payment and the finance charge.

    b. Complete the following amortization table for the loan.

Year	Beginning Balance	Interest	Payment towards the Principal = Payment – Interest	End Balance
1	$4000	$200		
2				
3				
4				

    c. Confirm the finance charge by totalling the interest column.

11. Kate purchases an electric bike for $4800 and she makes a down payment of $2200. She takes out a one year loan at 3.2% to pay the balance owing in monthly payments.

    a. Determine the amount of the loan

    b. Determine the monthly payment on the loan.

    c. Determine the total installment payments

    d. Determine the finance charge.

    e. Complete the following amortization table for the first four months of the loan. (Hint: When calculating simple interest the time (t) will be 1/12 of a year).

Year	Beginning Balance	Interest	Payment towards the Principal = Payment – Interest	End Balance
1				
2				
3				
4				

f. How much did Kate actually pay for the bike?

12. You purchase a kayak for $4800 and take out a 3 year loan with monthly payments at an annual interest rate of 3.5%. You are pondering whether to put $2000 down or go on a holiday with that $2000.
    a. Assuming no down payment, determine the monthly payment, total installment payments, and finance charge.
    b. Assuming a down payment of $2000, determine the monthly payment, total installment payments, and finance charge.
    c. What is the difference in finance charges between the two options?
13. Nick purchases a used motorbike for $12,000 and takes out a 4 year loan with monthly payments at an annual interest rate of 5%.
    a. Determine the payment, total installment payments, and finance charge with no down payment.
    b. Determine the payment, total installment payments, and finance charge with a down payment of $4000.
    c. What is the difference in finance charges between the two options?
14. A $350,000 home is purchased with a 20 year mortgage at a fixed interest rate of 3.4% and a down payment of 10%.
    a. Use an online mortgage calculator to determine the down payment, the monthly payment and the total interest paid.
    b. Determine the total payments for one year.
    c. Generate an amortization schedule and determine how much of the first year's payments will go towards principle and how much will go towards interest.
    d. Generate an amortization schedule and determine how much of the final year's payments will go towards principle and how much will go towards interest.
15. A $350,000 home is purchased with a 20 year mortgage at a fixed interest rate of 3.4% and a 20% down payment.
    a. Use an online mortgage calculator to determine the down payment, the monthly

payment and the total interest paid

b. Compare your answers for #14 and #15 Part a). What was the impact on the monthly payment and the total interest charges when the down payment was doubled?

c. Determine the total payments for one year.

16. a. A $650,000 home is purchased with a 10% down payment on a 25 year mortgage at a fixed interest rate of 4.2%. Use an online mortgage calculator to determine the down payment, the monthly payment and the total interest paid.

b. A $650,000 home is purchased with a 10% down payment on a 25 year mortgage at a fixed interest rate of 2.2%. Use an online mortgage calculator to determine the down payment, the monthly payment and the total interest paid

c. Compare your answers for parts a) and b). How does the lower interest rate impact the total interest paid?

Answers

1. a. $7500

b. $8183.52

c. $683.52

2. a. $21,300

b. $23,652

c. $2352

3. a. $3300

b. $3492.48

c. $192.48

d. $4492.48

4. a. $27,500

b. $29,568.48

c. $2068.48

d. $34,568.48

5.

Annual Interest Rate	Number of Years	Loan Amount	Monthly Payment	Finance Charge
2.8%	1	$2000	**$169.21**	$30.52
4%	2	$4200	**$182.38**	$177.12
5%	3	$5200	**$155.85**	$410.60
4.5%	3	$8000	**$237.98**	$567.28
6.5%	4	$11,000	**$260.86**	$1521.28

6. a. $102.35
   b. $2456.40
   c. $156.40
7. a. $600.95
   b. $36,057
   c. $4057
8. a. $6200
   b. $179.21
   c. $6451.56
   d. $251.56
   e. $8751.56
9. a. $40,000
   b. $885.37
   c. $42,497.76
   d. $2497.76
   e. $50,497.76
10. a. Annual Payment = $1128.05. Finance Charge = $(1128.05 \times 4) - 4000 = \$512.20$

b.

Year	Beginning Balance	Interest	Payment towards the Principal (Balance) = Payment – Interest	End Balance
1	4000	200	1128.05 – 200 = 928.05	4000 – 928.05 = 3071.95
2	3071.95	153.60	1128.05 – 153.60 = 974.45	3071.95 – 974.45 = 2097.50
3	2097.50	104.87	1128.05 – 104.87 = 1023.18	2097.50 – 1023.28 = 1074.32
4	1074.32	53.72	1128.05 – 53.72 = 1074.33	1074.32 – 1074.33 = 0

c. Interest = $512.19 (rounding difference of 1¢)

11. a. Loan Amount = $2600

b. Monthly Payment = $220.44

c. Total Installment payments = $2645.28

d. Finance Charge = $45.28

e.

Year	Beginning Balance	Interest (t = 1/12 year)	Payment towards the Principal (Balance) = Payment – Interest	End Balance
1	2600	6.93	220.44 – 6.93 = 213.51	2600 – 213.51 = 2386.49
2	2386.49	6.36	220.44 – 6.36 = 214.08	2386.49 – 214.08 = 2172.41
3	2172.41	5.79	220.44 – 5.79 = 214.65	2172.41 – 214.65 = 1957.76
4	1957.76	5.22	220.44 – 5.22 = 215.22	1957.76 – 215.22 = 1742.54

f. $4845.28

12. a. Payment = $140.65; Total Installment Payments = $5063.40; Finance Charge = $263.40

b. Payment = $82.05; Total Installment Payments = $2953.65 Finance Charge = $153.65

c. $109.75 less with a down payment

13. a. Payment = $276.35; Total Installment Payments = $13264.80; Finance Charge = $1264.80

b. Payment = $184.23; Total Installment Payments = $8843.04; Finance Charge =

$843.04

c. $421.76 less with a down payment

14.

a. down payment of $35,000; monthly payment of $1810.73; total interest paid $119,574.74

b. total payments for one year $21,728.76

c. $10,536.66 towards interest and $11,192.10 towards principal.

d. $394.95 towards interest and $21,333.81 towards principal.

15.

a. down payment of $70,000; monthly payment of $1609.54; total interest paid $106,288.66

b. With the downpayment being doubled, the monthly payment was reduced by close to $200 and the total interest paid was reduced by more than $13,000.

c. $19,314.48

16.

a. down payment of $65,000; monthly payment of $3152.81; total interest paid $360,843.77

b. down payment of $65,000; monthly payment of $2536.90; total interest paid $176,070.84

c. the total interest is almost $200,000 less!

Versioning History

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.01. If the edits involve substantial updates, the version number increases to the next full number.

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Version	**Date**	**Change**	**Details**
1.00	September 8, 2021	Book published.	